Discrete Time Signal Processing
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Lecture - 20
Graphical Interpretation of Circular Convolution

Quickly what we did yesturday.

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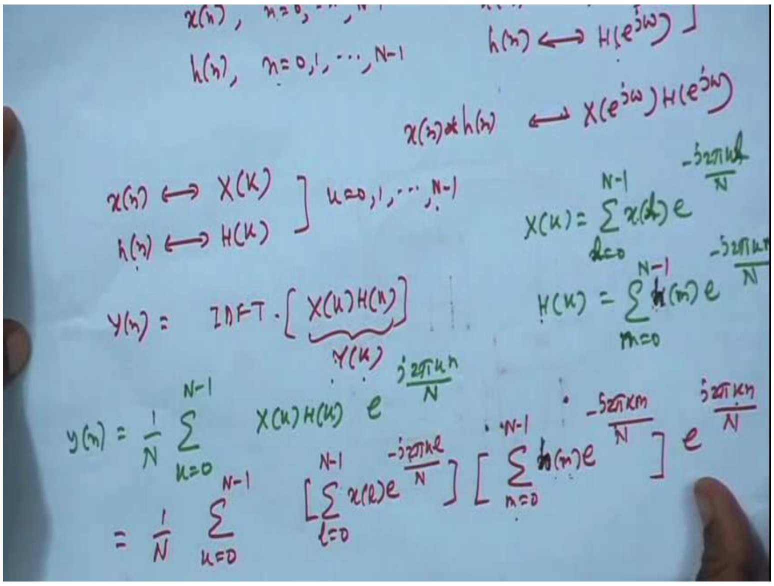

In the last class was this x n was a sequence finite length capital length point h n was a finite length sequence. If you take DTFTs and if you multiply the two DTFTs then they correspond to a sequence in the time duet which is a convolution between linear convolution between x n and h n that is multiplication in frequency domain it is convolution in time domain they are equivalent. Suppose, if the DFT domain I do a same thing I will take the samples of this DFT kth sample is kth DFT capital X k we have already seen.

I take kth sample of these to DTFT that will be h k, so again multiply them. Then if I take and how many I get if I multiply them I call it capital Y k, how many for k equal to 0 to N mius 1. So, I take n point inverse DFT and I get a length n sequence now y n; y n be related to x n and h n. We will let be linear convolution answer is no. Then what it is? For that we carried out this job capital X k was that DFT of x n. So, this is x l where we using the index l here l 0 to N minus 1, x l e to the power of minus j 2 pi k l by n.

Similarly capital H k is small m 0 to capital N minus 1 h small m e to the power of j 2 pi k m by n there was standard. Y n is the inverse DFT of y k y k is the product. So, we run the inverse DFT at an index small n which is fixed here from outside. So, this k equal to 0 to n minus 1 this product e to the power j 2 pi k small n coming from outside then capital N by capital N and the sum towards small k this is the standard inverse DFT formula. Then capital X k was replaced by these DFT formulam, h k I replaced by this DFT formula I get a triple sum, I intouched with the order of summation I bring the outer most summation the inside. If I do that what I get is this.

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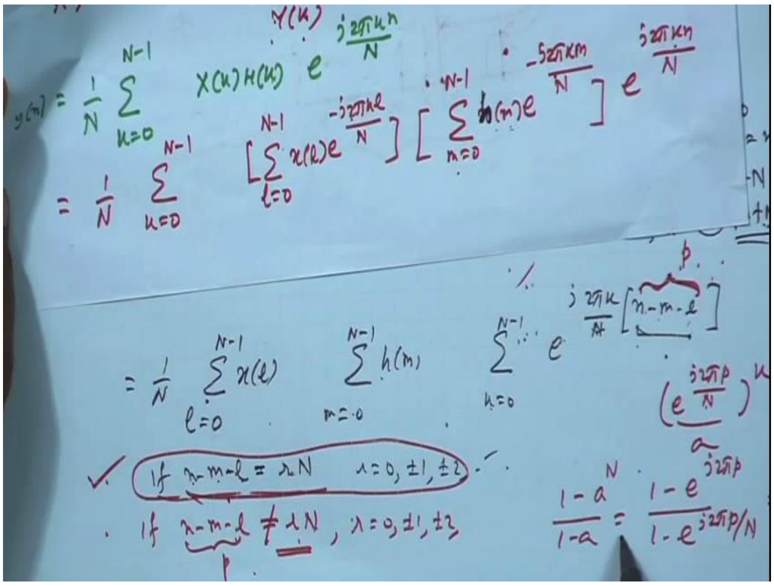

One by n x l remains outside with this respect to l h m remains here in the next summation with should be m, but the three exponentials contain k at the inner summation is the now with respect to k so all of them come together. So, you have got these n minus m minus l; n is fixed from outside but n is also in this range within that range only, m from this summation from the same range, l from the summation from the same range. So, n integer m integer l numerator so their difference n minus m minus l is an integer I call it p.

Suppose p is 0 then e to the power 1 1 1 if you add you get capital N, capital N and one n by n cancells you get these summation. Or suppose p is capital N or minus capital N or 2 capital N minus 2 capital N every time N and N will cancels so we do the e to the power j 2 pi k into an integer, so there is 1 and we get 1 1 1 added that is capital N that will cancel with these.

So, under this case when this is integer I get some value, but suppose it is not n minus m minus l is neither 0 nor an integer times or times capital N. So, p if I divide by capital N I dont get an integer r. In that case this summation becomes what is dp series e to the power j 2 pi k p by n and is in terms of k, so bring e to the power j 2 pi p by n inside and call it small x so e to power k k from 0 to n minus if you run the gp series it is 1 minus a to the power n capital N by 1 minus a. By substitution it is 1 minus e to the power of j 2 pi p n n cancels and 1 minus e to the power of j 2 pi p by n.

Now, this is one e to the power j 2 pi p because p is an integer so 1 minus 1 is 0. In the denominator p by capital N is cannot be an integer probably here this is not under these case p by n here was integer r, but here is not integer. So, p by n if you start at integer e to the power j 2 pi into some fraction that is cannot be 1 which is denominator cannot be 1 minus 1 0, so numerator at 0 the denominator non zero you get 0. So, this cases we will go out because the reality is 0, only these case is we left to be consider. In that case what is the summation summation N and N cancelled summation is y n is this outer summation x l inner summation h m; and that is all N and N cancelled. But not all l all m will be considered only this case will be considered that is only that m and that l will be considered for which this is satisfied.

So, that then we grieve two lines one for m one for l over the same range 0 to n minus 1 0 to n minus 1.

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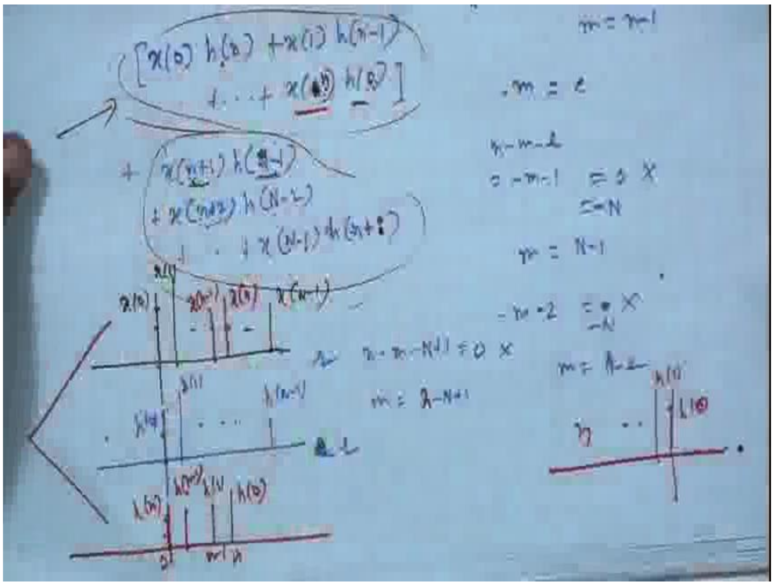

And suppose we start like this. And you have to find about y gets small n that is somewhere here in this range suppose it is here. So start with l equal to 0 x 0, then if we put 0 here n minus m must be equal to either 0 or plus N minus N dot, dot, dot, dot. Suppose you start with 0, so m will be small n which is fine because small n within the range of m, but if you put capital N here then if small m will be m minus N; this is outside this range for small m. Because if you give small n is here capital N minus 1 if you substract n from it you have here outside. If you put minus capital N here then n will be n plus capital N. So, again it is outside the range even if n this small 0 if you add a capital N to it you are outside.

So, those values we are not considering. So therefore, m is in this case just n. That is why this summation why you have got x 0 we have got to h n then x 1 l is 1, if you put 1 here n minus 1 if it is 0 m will be n minus 1. By the same logic you cannot have capital N minus N all those here because you will be out of the range you can veryfy. So, x 1 we have done in the yesterday. So, x 1 h n minus 1 because if m is l is one n minus 1 n mius m minus 1 if we take it to be 0 m is n minus 1. If you take this to be capital N m will be n minus 1 minus capital N which is further outside the range. If you take to be plus capital N minus capital N it will be these again it will be futher outside the range. You cannot do that, so this is a range.

And this goes on x l h 0 when l it is not x l it is x n sorry, when l takes the value n here. So, this is at n 0 this is 1 n minus 1 or it 2 n minus 2, dot, dot, dot when it is n it becomes 0. So h is 0 when you say n is 0, 0 n 1 N minus N 2 n minus 2 n 0. Then when l becomes n plus 1 what should be m, now n plus 1, we have seen yesterday n plus 1 if l is one plus N it is n. So, you have got minus m minus 1, if we equate that to be 0 m is minus 1 m is here minus 1, but that is not allowed. But, in this case if instead of taking this to be 0 we take it to be n say minus N because r n 2 n 0 is possible plus N possible minus N possible. So, if I put minus N then m is capital N minus 1 which is within this range. So, we will take this, no other value is possible. So, n minus 1 that is why it is becoming n minus 1 what is small n plus 1.

Then if it is small l plus 2 then n and minus m minus N minus 2, so m is it will become minus m minus 2 and you this only choice so you take m becomes capital N minus 2 that is why n minus 2 and, dot, dot, dot, dot. Finally when it is becoming capital N minus 1 it is becoming small n plus 1. If you add the two indices you see small n plus capital N 1 and one cancels small n plus capital N 2 minus 2 cancels, like that.

Now, this whole summation is that we define from linear convolution. We will see then this component is actually linear convolution between x n and h n, but not these because of the presence of this the whole thing is different. So, from y n I cannot get linear convolution. Why this is linear convolution if you have two sequences say x 0 x 1, dot, dot, dot, dot, so x n minus 1 in terms of say r and we have another sequence h 0 h 1, dot, dot, dot, dot say h n minus 1 in terms of l in terms of again r, how do you convolve graphically we hold this sequence as it is as a function of r we flip it and if you start giving shift.

So, if you flip it and suppose I have to find out the convolution that an index n n is from 0 to capital N minus 1, so I have to shift it by n. So, h 0 if you flip it h 0 h 1 h 2, dot, dot, dot of this side and this side becomes 0. There is 0s from left come over here, is like these h 0 h 1, dot, dot, dot, dot and 0s on this side. Then we shift to the right by small n. So, h of 0 comes here, then h of 1 at n minus 1, dot, dot, dot, dot here it will be h n at 0, nth point h 0, n minus 1th, n minus 2 h 2, 0 means n minus h it is h n. So, if I now convolve that means, I have to multiply this with this so nth point this sequence as x n.

So x n into h 0; you see x n into h 0. Then we will be x n minus 1 into h 1 previous time will be x n minus 1 h 1 this is x n minus 1 h 1, dot, dot, dot, dot. This here x 0 h, n x 0 h n here h 1 sorry; here x 1 h n minus 1 right h 0 h 1, dot, dot, dot h n minus 1 h n.

So we see. So you get linear convolution, evaluated it an index n this part, but this part is not but finally I will be coming as a summation of the two within the summation this is hidden I want to extract this linear convolution from y n, but I cannot because this is coming as the summation of the two. So how to, what should we do so that these I can extract; that is the next question. But before doing that we should understand the graphical procedural of carrying out this convolution. This overall is thing is call circular convolution. Let us give a graphical interpretation of these then we will proceed for attending by which the linear convolution part can be extracted.

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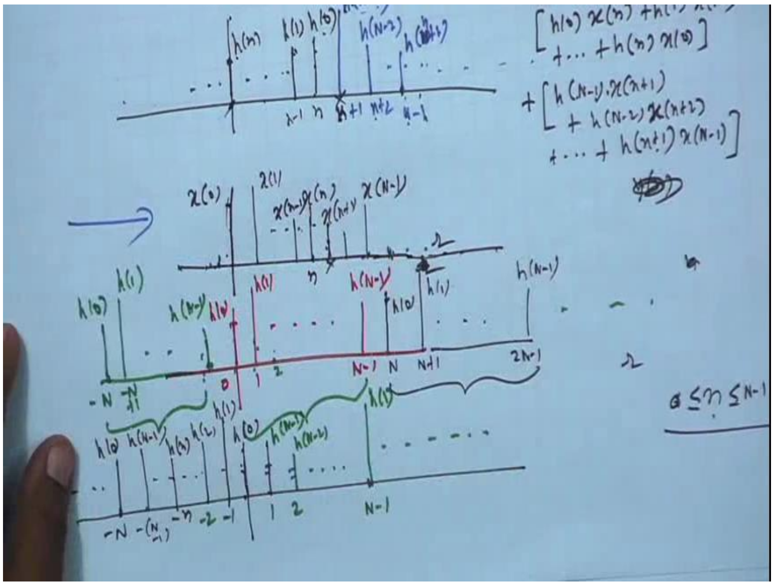

Suppose, remember this is my expression these plus these. How to calculate graphically I am showing you, earlier I have shown graphical way of doing linear convolution now I will show graphical way of doing circular convolution there is this whole summation. Suppose you got two sequences then the rule is hold number one sequence as it is x 0 same links then you have a function of r and other sequence which you have you make it periodic, then if it is give it like these h 0 then h 1, dot, dot, dot, dot h n minus 1 you make this periodic that is you add further repetition, again same h 0, again same h 1; so h 0 comes at nth point this is at n plus Nth point, dot, dot, dot, dot up to here. N minus 1 so this will be 2 n minus 1 this is another period. Similar of this side, this side again you add another period. If it is 0 0 into plus N now 0 will go to minus N again here the same h 0 same h 1 this is at minus N 1 plus 1, dot, dot, dot here same h n minus 1. So, this is one period this is one period this is one period, dot, dot, dot you make it periodic.

So, I am telling you the rules if you are doing circular convolution between two sequences graphically I am suggesting a rule and then will verify that will give you this expression. So, what is the rule? One of the sequence is you hold as it is as a function of r not n because you want to find out the convolution at a fixed small n so make it of a function of variable r. At the other sequence also with making function of r, but if it is one period is given from 0 to n minus 1 you repeat it again, again, again. So, one block same block, same block again, same block again so you get a periodic sequence, this is step number two. Step number one hold first sequence of function a r. Step number two make the other sequence as a periodic sequence this is periodic repetition of that block as a function of r. Then third step is flip this periodic sequence that is from right side it goes to left side and left side goes to right side.

So your flipping, flipping means h 0 will remain here this h 1 will go here, dot, dot, dot, dot. N minus 1 so at minus N at minus N minus 1 this term will go, then at minus N this h 0 will go, dot, dot, dot, dot. And on this side on to this side h capital N minus 1 from left it will go here, this goes here, this goes here like that. Point number one goes to minus 1 minus 1th thing goes to plus 1. At plus whatever it goes to minus 2 minus 2 whatever it goes to plus 2 this is called flipping; right side gets about to the left hand side and left hand side gets about to right hand side. So, here these were minus 1 minus 2 like that so at 1 h n minus 1 will come then at 2 h n minus 2 will come, dot, dot, dot, dot. At n minus 1 we will come which were was at minus N plus 1 that is this h 1. Capital N minus 1 will go to this place minus N plus N and from minus N plus 1 h 1 will come and then, dot, dot, dot, dot and I do not need them. You can find out. What is the flip sequence?

So, first step hold one sequence as it is of the function of r. Second step other sequence you make it periodic as a function of r. Third step flip this periodic sequence right side goes to left, left side goes to right. Fourth step, now shift this periodic sequence to the right by n, n is what? N is the point where you want to find out y n, n can be in the same range. In the previous summation we found y n equal to this for n our choice in this range. So, if you want to find out y n for a particular in this range then they shift this flipped periodic sequence to the right by n. And if you add them you multiply sample wise this sequence with this sequence and obviously outside this window no need to consider because you have got 0 values. So, from the other sequence whatever be the samples that we will get multiplied from here by 0 so they do not contribute to anything. So, you will considerable thing which falls directly under this window from 0 to n minus 1.

So, if you now flip it I am writing here after flipping shifting by n it is h 0 and h 0 will go to h n, nth point. Here h 0 will move after n shift one shift h 0 will come here, two shift it will come here n shift will come here then you will be followed by h 1, dot, dot, dot, dot then h 2 dot, dot, dot. So, at nth index h 0 n minus 1th index h 1 n minus 2th index h 2, at 0th index there is N minus N, h small n will come of this side h 0 h 1 h 2 h n is here this side minus N it was a first on this side plus in a (Refer Time: 19:21) went and minus N. So, everybody is getting shifted to the right by n h 0 move to nth point h 1 moved to n minus 1th point from here from minus 1 to n minus 1, h n from minus N it will shift to the right by plus N so plus N and minus N 0 so h n will sit here come here, dot, dot, dot, dot.

And h 0 before that I have h n minus 1 so h n minus 1 within here, this guy, then this guy at n plus 2, dot, dot, dot, dot, at n minus 1 we will go. Capital N minus 1 came at small n plus 1, capital N minus 2 that small n plus 2 you see if you add these two indices if it is small n plus capital N if it is small n plus 2 capital N minus 2 we get a small n to capital N 2 minus 2 cancels. So, if it is capital N minus 1 this would be small n plus 1. So if we add that two again 1 minus 1 cancels capital N plus N this have to remember. Small n plus 1 capital N minus 1 if you add the two small n plus capital N, here also n plus 2 capital minus 2 if you add the two small n plus capital N. Here also capital N minus 1 small n plus 1 if you add the two small n plus capital.

Now at to the right I have some value left, but this is a range of interest because now I will be multiply this with this sequence. This has non zero value only about 0 to n minus 1, that is why 0 to n minus 1 will matter. And what will you get consider these black ink black colors samples, h 0 will be set nth point. So, nth point I have got x n h 1 at n minus 1 this is earlier normal sequence it was x n minus 1, dot, dot, dot h n x 0. If you do this black sample versus black samples what you get is that linear convolution part. There is h sequence shifted I mean flip has no. This thing, x as it is h was shifted it and then re flipped h was flipped like this h 0 h 1 h 2 and then brought back here that is shifted to the right by m so h 0 comes at nth point h 0 come at nth point then h 1 up to h n.

So, this if I consider black samples versus black samples here only this part if you have to multiply there is h 0 into x n h 1 into x n minus 1, dot, dot, dot h n x 0 that is only this part remaining part I will do, but only this part I am doing separately. That is same as what we had here we will see if you are doing linear convolution x sensed sequence was brought it as r h sequence was brought it as r then flipped, flipped means h 0 here then h 1 h 2, dot, dot, dot then it is shifted to the right by one h 0 from zeroth location goes to nth location, h 1 from minus 1 location goes to n minus 1 location dot, dot, dot. H n from minus nth location goes to zeroth location then if you multiply these two you will get see h 0 x n same here h 0 x n then h 1 x n minus 1, h 1 x n minus 1, dot, dot, dot h n x 0, h n x 0.

So, that part you get the linear convolution part, that is h 0 x n plus h 1 dot, dot h n x 0 this part is the linear convolution. And what you get after that? After that you get h n minus 1 and at what point small n plus 1th point. So, sample value is x this is x n plus 1 because this index is n plus 1, so this into to this. So, at n plus N 1th index I have got x small n plus 1 and here I have got h capital N minus 1, so h capital N minus 1 into from here x n plus 1. Then next index I have got h n minus 2 that is small n plus 2 index that time I will have h at small n plus 2, so h small n plus 2 and dot, dot, dot.

Finally, at capital N minus 1 index h small n plus 1 and that time I have got the last sample x n minus 1. You see this is what we have in the circular convolution expressions. These are the linear convolution x 0 h 1 x 0 h 1, dot, dot, dot x n h 0 x n h 0 is that ready in a reverse way that part is the linear convolution. And then you have got h capital N minus 1 x small n plus 1, h capital N minus 1 x small n plus 1. Then h capital N minus 2 this is x sorry, x small n plus 2, dot, dot, dot h small n plus 1 h small n plus 1 x capital N minus 1 x capital N you see.

So, this is the graphical interpretation of circular convolution. We can use this technique. So, I repeat again you hold the origin one sequence as it is as a function of r, make the other sequence periodic by repeating that that block 0 to capital N minus 1 again, again. And then third step is flip it so right side goes to left, left side goes to right; I mean nth point goes to minus nth point minus nth comes to nth. There is sample at nth goes now to minus Nth point and minus Nth point whatever sample it was now it was comes to nth point, so flipped that now you want to carry out the convolution circular convolution at an index n within that range. So, within that range of that sequence know 0 to capital N minus this is my final range.

So, shift this flip periodic sequence to the right by the small n and if you do that and then whatever comes under this block of that first sequence you hold that only and then multiply sample wise, other terms of this flip sequence are of no consideration they will get multiplied from 0 from this first sequence because outside this block this original first sequence as only 0 values. Even if you try to multiply all the samples of these shift at flipped periodic sequence by the corresponding samples of this original first sequence, outside this window there is no point in carrying out this multiplication. Because this first sequence has 0 values only outside this window that is why we will consider this product sample y is an addition all you from 0 to at capital N minus 1. And if you do that and sum you get two out of which you got the linear convolution part and we have got circular convolution part.

In the next class I solve a technique by which we can derive, we can find out linear convolution out of circular convolution using this graphical distance. So that is all for today.

Thank you very much.