**Discrete Time Signal Processing**

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**Lecture – 19**
**Introduction to Circular Convolution**

So, what did we do is we are now continuing with this same problem.

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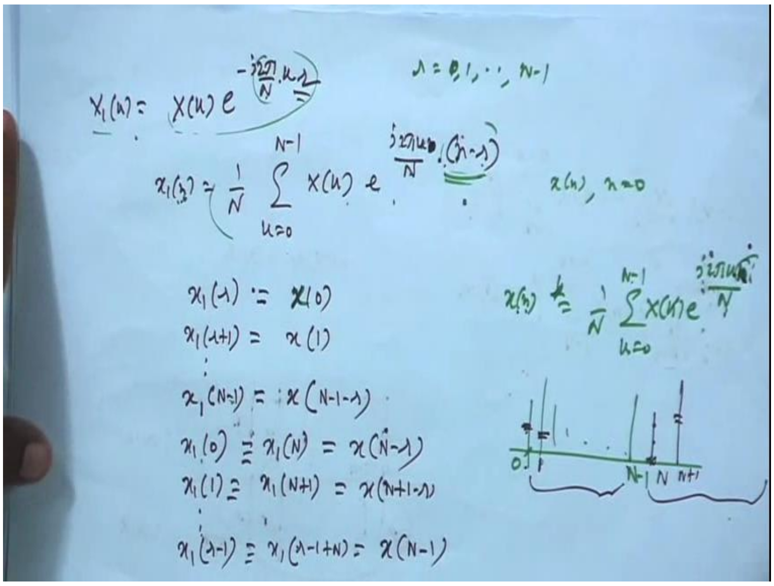

And that time I was considering capital X k giving, so we multiplied by e to the power minus j 2 pi k by n into r. Optimize that originally I took 2 pi k by N and now optimize that. Or, actually can be 0, 1, but the same range to make life simple. Then, what is x 1 n? So, I carry out that inverse d f t of this capital X 1 k. So, this I put back here and I get this. You know capital X k e to the power minus j 2 pi k r by N; that is coming from this second term j 2 pi k by N into minus r. First term comes from that inverse d f t formula. And, this is what. So, now, I start with this one r; r means n equal to r; that means this is 0. So, it is this inverse d f t of x n at n is equal to 0, if it is 0. If it is 0 and capital X k is the inverse d f t of x n sequence with this index being 0. So, you should get back x 0.

You remember what is x n that was X k e to the power j 2 pi k n by N; n summation over this. So, x 1 r means small n is r here; r minus r 0. So, this summation is what? 1 by N (Refer Time: 01:59) 1 by N summation summation k equals 0 to N minus 1 0 to N minus 1; capital X k capital X k; e to the power j 2 pi k by N; same here; all right. But, this summation here, this index is becoming 0; that is, as though 2 pi k into 0 by N. If this is 0; it is x 0 because for x 1 it is r x 1 r; that means, n is r here; but, r minus r is 0. So, e to the power j 2 pi k into 0 by n. If you have e to the power j 2 pi k into 0 by N; this 0 means x 0; that is why x 1 r equal to x 0. In the same way, if you put x 1 r plus 1; that means this n is r plus 1. So, r plus 1 minus r is 1. So, as though this summation with index here is 1; that means x 1 dot dot dot. I go up to capital N minus 1. So, it is capital N minus 1 means N minus 1 minus r. As though this index is small n in index is capital N minus 1 minus r. So, it is x N minus 1 minus r.

And then, I go to x 1 0. But, x 1 0 from the property of d f t we have seen that, if you take the inverse d f t and suppose this is the sequence; if you go beyond this, it becomes periodic. So, if you are going to n-th point; whatever you observe, that will be same as this 0 0 plus N; this block will be repeating periodic. So, if I take the inverse d f t; but, allow the index n to go beyond this capital N minus 1 say to capital N; so, that value x 1 at capital N will be same as x 1 at 0, because of the periodicity. So, x 1 0 is same as x 1 capital N. But, x 1 capital N in this formula; if you put capital N minus r as though here this index n is capital N minus r. So, it will be x N minus r.

So, 0, 1, N minus 1, minus r; then, N minus r; then, x 1; 1 will be if it is 1, this value will be same as whatever you get at N plus 1 because of periodicity. So, x 1 capital N plus 1; that is, capital N plus 1. So, as though index is n plus capital N plus 1 minus r dot dot dot dot. Finally, r minus 1; there is same as again; because of periodicity, you get capital N to that; you will get the same thing or minus 1 or r minus 1 plus N because of the periodicity, we will get the same thing. But, if you put r minus 1 plus N, r r cancels; we are left with capital N minus 1. As though this is capital N minus 1, this is these. So, graphically what it means?

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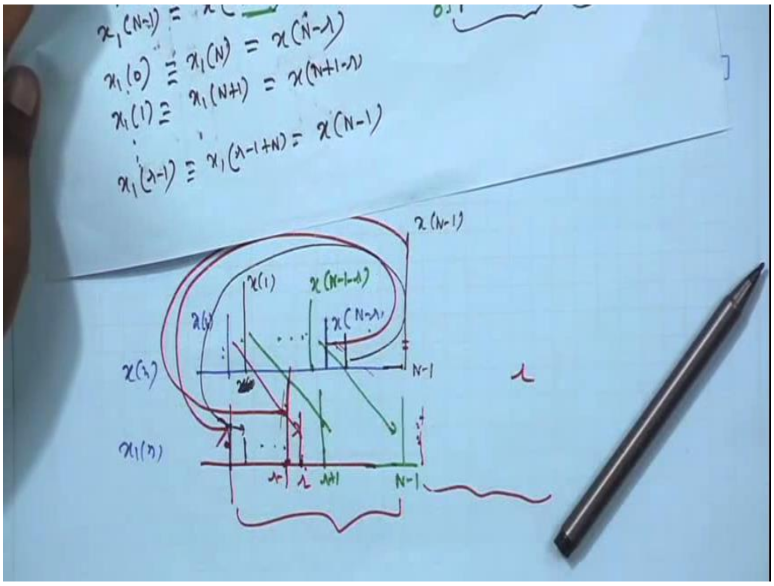


But, this was the x n sequence; this is x 1 n sequence. Here I had x 0; it will go to x 1 at r-th point. So, it will go to r-th point. Then, I had x 1; it will go to r plus 1-th point dot dot dot x 1 n minus 1. Will come N minus of 1 minus r. So, somewhere here is x n minus 1 minus r; that will go to n minus 1. So, you see this is getting shifted by r. This is getting shifted by r. This is getting shifted by r, because N minus 1 minus r plus r is N minus 1. So, we are getting shifted by r. But, the next guy because next guy is this; that is, at x, next is N minus r, because increasing by 1, 0, 1, 2, 3. So, after this, it is N minus 1 0minus r. So, next will be N minus r; N minus r goes to x 1 0. So, x 1 0; that is, it does not go out to the right; you can say it goes to the right; you can say it goes to the right. But I am because of periodicity whatever you have within this block, same way you repeat. So, this wills same as this since I am interested only one block. So, basically, this same, this will come here. So, you can say this is coming back here.

Then, the next guy; if the next guy is this much; this will come back to the next dot dot dot. And finally, at N minus 1, if it is x N minus 1; this comes at x 1 r minus 1; so, here. So, x 0 shifted by r; x 1 shifted by r to the right dot dot dot. This shifted to the r and it goes to last point capital N minus 1. Next guy will be shifted back to the 0-th point. Next, will be the first point. Next, will be the second point dot dot dot. Last guy will come to r minus 1-th because x 0 came to r-th. So, I can go only up to the right before this r minus

1-th. This is called surplus shift. It is a right surplus shift. By how many index at a time? R; everybody is getting shifted r times. They are getting shifted in the forward direction. And, from this point onwards, it is shifted circularly. This is a very important property.

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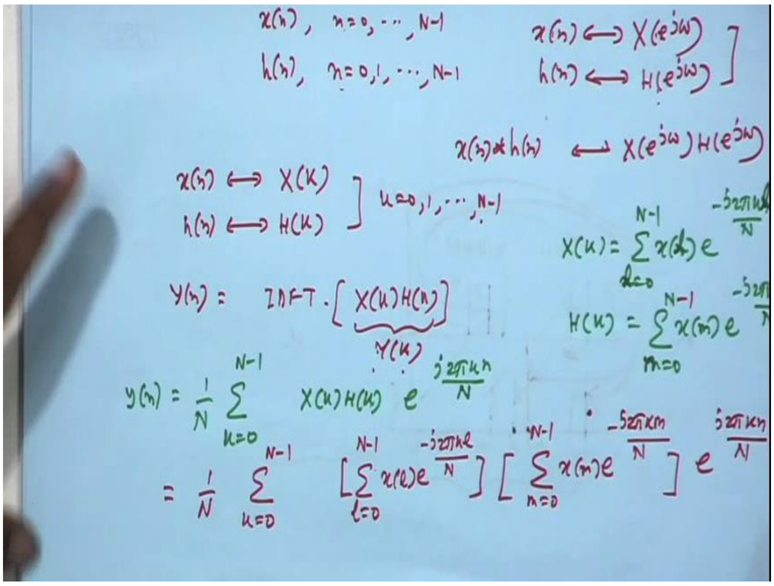


Another thing relates to convolution. And, that is another very very important thing. Earlier we had seen. But, suppose I give you a sequence x n from 0 to n equal to 0 to say N minus 1; x n has DTFT, and another sequence h n. Now, there we saw that, if we multiply in frequency domain, that is, this; then, the inverse DTFT of this is the convolution of this. There is convolution in time domain; gives the sequence whose DTFT is the (Refer Time: 09:47) of these two d t DTFT in frequency domain.

Now, suppose x n it has got d f t, that is, the DTFT sampled capital N number of times n point. We say n point d f t. And, in both cases, k equal to 0, 1 dot dot dot dot n minus 1; where, this n is same as this n the length of the sequence. So, here also I multiplied the two DTFT's. Here instead of the DTFT as such, I am taking k-th sample of this as X k; k-th sample of this as H k multiplying them. X k H k. And then, I take the inverse DTFT of this inverse to get Y n. That time when you took inverse DTFT of this, I got convolution between the two.

Now, if I take inverse d f t of this; you can call this product as capital Y k. So, you have got capital Y 0 capital Y 1 up to capital Y n minus 1 because k takes those values. So, on that sequence, if you carry out i d f t you get again a length n sequence say y n. What will be y n in terms of x n and h n? Will it be linear convolution? It will be no, because linear convolution as such and if you convolve the two sequences, its length increases. But, here when you take i d f t, y n also is length n. So, obviously, it cannot be a convolution; would be x n and h n. Then, what is y n? So, that we can work out; and, that is a very important thing.

Work that out; we carry out this we know that x k is instead of m, I give another symbol here m. Now, I D F T; that means y n is 1 by capital N I D F T means summation over k y k; there is this product e to the power j. Since small n is used here; let me change this index also from here; let me call it l; l l, because n is used up here. So, I cannot use another n here; all right. Now capital X k I replace by this; H k I replace by this. So, what I have then, the summation; this is capital X k; there again capital H k, because N minus 1. So, capital X k capital H k into e to the power j; now, there is (Refer Time: 14:01) summation. So, I can interchange the two summations; I will bring the outermost summation; I will make it the innermost summation.

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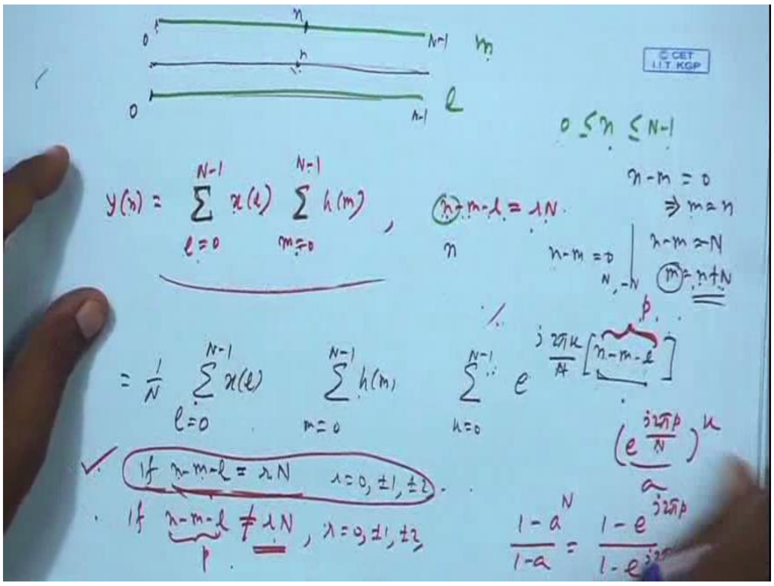

In that case, you will have 1 by N outside; so, l equal to 0 to N minus 1. Then, m equal to 0 to N minus 1 and inside is over k. So, all functions of k should be here, x l will be here; we come as common only in the context of summation over l, because it depends on l. x m one is this; this is not x, this is h; I made this mistake capital X k came from x; capital H k should come from h. So, this is now small h because there are two sequences; it is not x; this is h. So, this is h; all right. So, this x l came here; h m because it depends on m; this comes in this. But, this exponential, this exponential, this was exponential; they all are functions of k. So, they go inside. If we have got e to the power j 2 pi k by N into small n minus m minus l; all right.

Now, if it is 0 or if it is capital N or if it is 2 n 3 n minus capital N minus what we will have, e to the power for instance, suppose it is 0; e to the power 0 1 1 plus 1 plus 1; you will get capital N; that capital N and this 1 by N will cancel. If this is not 0; now, this is capital N. Again N N will cancel; e to the power j 2 pi k is 1 1 1 1 1 you sum; you get capital N; capital N capital N cancels. Same if it is 2 capital N, 3 capital N, 4 capital N. If it is 2 capital N, N N cancels; e to the power j 4 pi k; which is again 1. So, you sum over capital N; 1 by N cancels. If this is minus capital N; even then it is not (Refer Time: 16:31) Minus capital N means minus capital N and N; so, N N cancels e to the power so, it becomes e to the power minus j 2 pi k, which is again 1; so, same thing.

So, when the summation; if n minus m minus l is some r to the power N or can be 0, plus minus 1, plus minus 2 dot dot dot; if it is 0, n minus m minus l is 0. If it is 1, it is capital N, if it is minus 1, minus capital N, and so on and so forth. In that case, what happens this will give you as to capital N; capital N capital N cancels; you have carry out this summation; that we will do. But, if it is not; that will come true. But, suppose it is not; first, let me do this case and then I will come to this. Suppose it is not equal to this; in that case, it will be just an integer; it will be an integer. And then, this depiction if you sum; this will turn out to be 0. If you call this integer to suppose you can give a name; n has been used; m has been used; l has been used; r has been used. So, what is still there? Suppose you call it p. n is integer, m is integer, l is integer. So, n minus m; l you are choosing from here; m you are choosing from here. So, n minus m minus l and n is fixed from outside; you want to find out y n at a particular n within the range 0 to capital N minus 1. So, integer minus integer minus integer; it is an integer. Can be positive or negative; it does not matter.

And, the summation e to the power j 2 pi k by capital N into p. So, if you take e to the power j 2 pi into p by N; then, whole to the power k; if you call it a; so, e to the power k; k from 0 to capital N minus 1. So, summation of this will be 1 minus e to the power N by 1 minus a. So, a to the power N (Refer Time: 18:59) have 1 minus this to the power N N will cancel. So, e to the power j 2 pi into integer p minus 2 pi p by N; all right? Now, look at this m n minus m minus l; the numerator part is 0, because this is 1; 1 minus 1 is 0; denominator we have seen earlier; we can explain again; denominator cannot be 0. Why it cannot be 0? Because it cannot be 0, because for it to be 0, this must equal to 1, in that case, p must be either 0 or capital N or 2 N and like that. But, that case we are not considering here. For this to be denominator to be 0, p must be either equal to 0; this is your p. p must be either equal to 0 or N, 2 N or minus N, minus 2 N dot dot dot; that is of this form. But, I am assuming this case first; that is, this is not equal to of this form; which means denominator is this is never equal to 1; and therefore, this is never equal to 0. But, numerator is 0. So, this will be 0.

So, in these cases, there is s l is moving and m is moving; if this summation is not equal to either 0 or capital N or 2 N; then, this sum is (Refer Time: 20:44). So, I have to only consider these cases. And, in these cases, this summation will give you as to capital N; capital N and 1 by N will cancel. So, we are left with this; that means, y n is given by these; this part with this condition r into N either r equal to either it is 0 or N or 2 N or minus N, minus 2 N. So, in this summation, you choose l and m accordingly, but for a giving small n; but, only that l, that m, so that n minus m minus l is at a 0 or capital N or 2 N or minus N. Only those choices of l and m from these two summations will be allowed.

Then only, because this is the condition; if it is not so, summation was 0; we have found out. So, if it is not so, that case is gone. So, we are considering this case; where, n minus m minus l is r into N; that is, either 0 or 2 N or 2 N or minus N, minus 2 N like that. And, in that case, if this is, that is, 0; N, 2 N minus N, minus 2 N, whatever; this summation is capital N; N N cancels; we are left with this – this summation. But, considering this, m and l should be chosen from this range, so that n minus m minus l is of this form either 0 or N, 2 N, 3 N or minus N, minus 2 N, minus 3 N, like that. Then, whatever you get that will be a y n.

So, let us try to see what are the choices of l and m, we can pick up under this constant; and, what will be the summation then. Suppose I have got one axis n m; this small n is your choice. You are trying to find out y at a small n; but, small n is within this range (Refer Time: 23:44) you multiply the two D F T‘s and taking the n point I D F T. That is how you got. Remember these two D F T’s we multiplied; called it y k; you take n point I D F T. So, this is a length n sequence length capital N sequence. So, I am finding out why small n for small n within this range; because small n is your choice you have fixed it within that range.

And now, you have got two summations: one for m; another one l. So, start with m equal to 0 because m equal to 0. Or, start with l equal to 0; l equal to 0. If l equal to 0, m is taking all values; but, all values are not allowed. This value will be allowed, so that this is either 0 or capital N and m again varies within this range. So, l is 0. So, n minus m either 0; there is m equal to n or it is m capital N or minus N, like that. But, you see if n minus I repeat again l I am taking 0; if l is 0, what is the possibility of m? n minus m minus this 0; n minus m minus 0 that should be either equal to 0; which means m is minus n; or, if it is capital N, then small m will be n minus capital N, but that is not allowed, because if n is within this range; if you subtract capital N, you are going outside the range to the left.

But, small m cannot take this value, because small n is within this range 0 to N minus 1. If you have this scale, small n can be anywhere here from 0 to n minus 1, because for small m if you go back by capital N, you are going behind. So, that is small m cannot take any value from that range, because small m range is from 0 to capital N minus 1. So, that is ruled out. If it is minus N; if it is n plus N; so, we are going to the right from small area, going to the right; we are going beyond the range. That is also not allowed. And similarly, plus 2 N, minus 2 N; so only they are not allowed. So, only n minus m equal to 0; which means small m equal to n. So, if l is 0, small m will be n and the product will be x 0.

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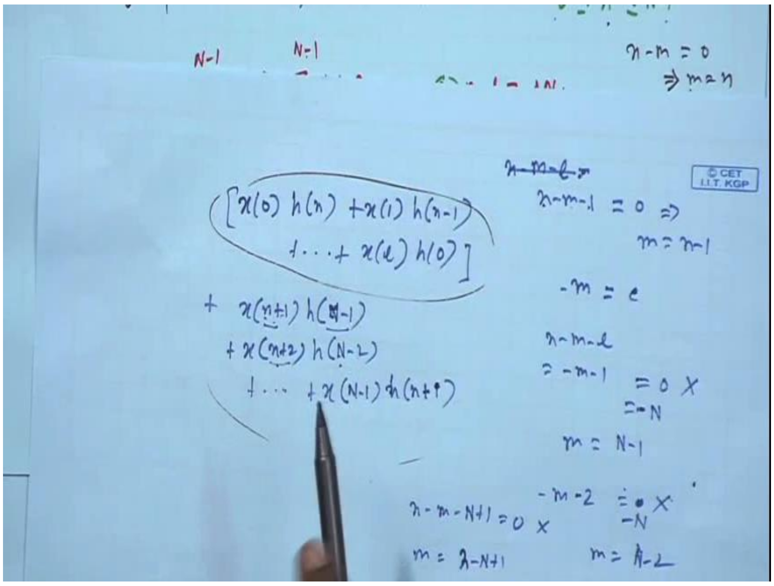

Product will be l 0; so, x 0 into h n. Then, suppose l takes 1; l is here – 1. If l is 1, what should m take, n minus m minus 1, if it is 0, m will be n minus 1; that is, this side. This is now n minus m minus 1; either this would be 0; which means m is n minus 1, which is fine; small n is here; so, n minus 1; that is fine. But, if it is capital N; that means small m will be n minus 1 minus capital N n minus 1 minus capital N; we are going beyond the range of m; that is not possible. If it is minus capital N, it will be small n minus 1 plus capital N small n minus 1 plus capital N.

So, even if you are here, small n minus 1; here 0; plus capital N will take you outside this range that is not allowed so on and so forth. So, which means when n is 1, it will be h; n minus 1 dot dot dot dot. When it is n when l is n; if l is n, then n minus m minus l and l is n. So, you have to get only minus m. So, this would be either 0 means m is 0, which is within range. But, if it is plus capital A dot minus capital N, m will be equal to e minus n or plus capital N like that, which is outside the range ruled out. So, m is 0 then. That is the only possibility, x l h 0; that is one part.

And then, then as I go further, l suppose is N plus 1; this is very interesting. x is l; then, n minus m minus l that will be l is again plus 1. So, if this can be 0; then, m is minus 1, that is, outside the range. So, this equal to 0 we will not consider. How about this equal to N?

This equal to N; is that possible within the given range of small m? It will mean small n equal to minus 1 minus n, not possible because of the negative direction. How about minus N? If it is minus N; then, small m will be N minus 1, which is fine here. So, there is one – that is the only possibility. So, it will be h N minus 1.

Then, x n plus 2; if it is n plus 2, l is n plus 2; it is minus m minus 2; that cannot be equal to 0, because then m is minus 2. So, this is ruled out; that cannot be equal to N, because that means, small m equal to minus 2 minus N going in the negative direction beyond the range not possible; minus n is, because then that will be m equal to n minus 2. So, here which is in the range? So, it will be h N minus 2. And, this way if you go along; finally, when we are here; that is, l is N minus 1; so, n minus m minus capital N plus 1. This will be – if it is 0, not possible, because then m will be n minus capital N plus 1; n minus that is, even if n is n minus capital N plus 1. That is not possible.

So, this is will be actually, this is outside the range. But, from here only you can see x n plus 1 h capital N minus 1; it is n plus 2 h N minus 2; n plus 3 h N minus 3. If you add these two indices; small n plus capital N; again small n plus capital N 2 2 cancels, so that means, if it is N minus 1, it will be h small n minus 1 small n plus 1. This is the summation. This part will explain; we will consider again in the next class. This is called circular convolution. In this, this part is a linear convolution between x and h, but there is an additional component that comes; and, together I mean it becomes a circular convolution. So, it does not give us the desired linear convolution result because of this additional term, which is we can say as an error term; error in the sense that I want this, but I am getting a mixture addition of the two.

We will discuss this later in the next class.

Thank you.