Discrete Time Signal Processing
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Lecture – 18
Properties of DFT

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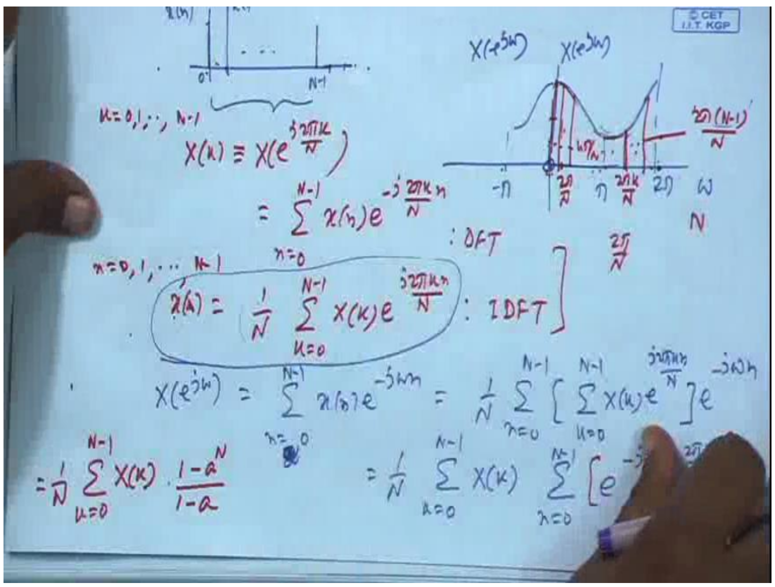

So, yesterday we started with DFT. So, if the sequence is finite, x n, 0 1, x 1, dot, dot, dot, up to say n minus 1, this is the duration, over which the sequence is given. So, this is a finite length sequence, outside it is 0, sometimes this range is called as support, support is finite, from 0 to capital N minus 1, what this is defined, this is FIR sequence, then, you have seen, that if you take the DTFT, and we plot the DTFT, it is periodic, right? Suppose it is like this, from minus pi to pi, you can as well have, another period here, it was 2 pi. So, instead of taking period from minus pi to pi, here we take from 0 to 2 pi, any segment of length 2 pi is 1 period.

Because if you repeat this, here again, here again, you get the interval form, either take minus pi to pi, I repeat that here, or 0 to 2 pi, whatever you have, you go on repeating, you get the same thing. Anyway, so suppose if I have the FIR sequence, and this is the DTFT. I made a point yesterday that, if the sequence is finite length of capital N number of points, then there is no need to store the entire envelop, DTFT, because then you will need infinite points on this curve, because 0 to 2 pi, plus Omega, is a continuous patch. So, there are infinite frequency points, It will give you infinite values, infinite number of values, you cannot store them, but I told you, that is not required, if the length is capital N, you should take n number of samples, of this, that is sampling period 2 pi by n.

So, 0 1 sample, then 2 pi by n, another sample then 4 pi by n, another sample, dot, dot, dot, dot, at 2 pi. So, 0th sample at 0, first sample at 2 pi by n, second sample at 2 into 2 pi by n that is 4 pi by n kth sample 2 pi k by n, dot, dot, dot, it will go up to 2 pi in to n minus 1, by n, n minus 1th sample. So, 0, 2 pi by n, into 1, 2 pi by n into 2, 2 pi by n into k, 2 pi by n into n minus 1. Next sample will be 2 pi into n, by n, n, n cancels. So, this is 2 pi, which is this, because of period, this is as same as this. So, we will not go up to that we will take here. So, how many samples? 0 to n minus 1, so total n sample, we take n samples, period is 2 pi by n.

This kth sample, we denote by k, which is nothing, but DTFT, at this frequency, e to the power j, 2 pi, this much Omega 2 pi k by n, that was, if you it was the DTFT formula, a to the power minus j, Omega n, Omega is this 2 pi k by n, into small n. And summation over, 0 to, not minus infinity to infinity, because sequence is from 0 to n minus 1, outside that is 0, this is called DFT. Then using this samples, that is the samples, x k and k can be 0, 1 up to n minus 1. So, using this sample, you can get back your x n, that is you do not need the entire envelope, and then compute inverse DTFT. You can take this samples only, how many? Capital N of them, and get back your x n, by this formula, that x k, is sum over k, k equal to 0, k is equal to 1, like that. So, k, k also ranges from 0 to n minus 1, at this formula, e to the power j, 2 pi k small n, here, here it was minus, it is plus, here it was summed over n, here it is summed over k, here small n is your choice, fixed from here, here k was your choice, fixed from outside, n also is from 0, 1, as given here, n minus 1. This called inverse DTFT, inverse DFT, and IDFT.

That means, I can get back my original x n, for the DFT only, all right? Now there are several properties, which we will follow, which we will discuss. Number one, if I take this x n, I got the x n, from x n, I can get back the full DTFT, by this formula, n is equals to 0 outputs, 0 to n minus 1, full DTFT, given any x n, I know full DTFT, but x n, I can write in terms of this DTFT samples, only capital X k. So, if you bring that back here, 1 by n, outer summation with respect to small n, inner summation, that is capital, this small x n, you will replace by, this entire summation. So, n is equal to 0, to n minus 1, then x n will be replaced by this entire summation, 1 by capital N already taken out. So, I have got x k, this much is x n, into.

Now, this DTFT, summation was about n. So, that is prevails, x n is given, x n, every time you pick up 1 small n, that n is fixed, x of that n. So, x of that n is this summation, 1 by capital N is outside, this summation, writes that, you put that small n, and run the summation over k; this will give you x n, x n into this. And now I told you, that in DSP, whenever you come across double summation, you interchange the 2 summations. So, it will be 1 by n, outside is capital X k, it is depends only on k, it comes here, n equal to 0 to n minus 1, e to the power this summation, minus j Omega minus, 2 pi k by n into small n, all right? This is a GP series, because if you consider this to be small a. So, a to the power n, that is a to the power 0, a to the power 1, up to a to the power capital N minus 1. So, this summation will be, into 1 minus, a to the power capital N, that is GP series, what is the series? a to the power small n first 0, that is a to the power 0, then a to the power, small n equal to 1, that is a to the power 1, a to the power 0, plus a to the power 1, plus a to the power 2 dot, dot, dot, dot, up to a to the power capital N minus 1.

So, summation will be this. This elementary GP series formula, and this a value, if you substitute, this is x k, 1 minus, this is you are a, a function of this kind, so; that means, DTFT, can be obtained from it is own samples, after all, what is capital X k? k is equal to 0, k is equal to 1, up to k is equal to n minus 1, means you are basically taking the whole DTFT itself, and sampling it.

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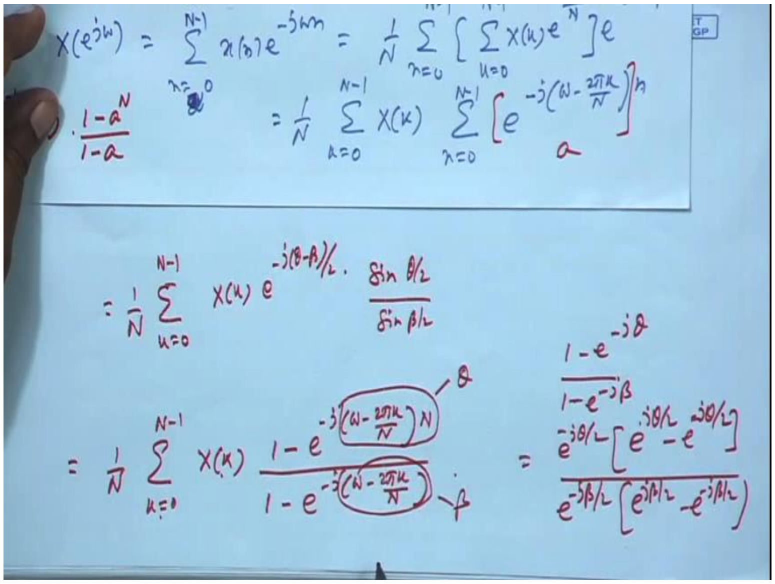

So, if you know this samples, that is the capital X ks, using them, we can get back x n, and then therefore, you can get back the DTFT envelop itself. So, DTFT, for DTFT, I do not need store the entire envelope, if I store the capital N number of samples, where is the length of the original sequence, then using the samples in this formula, by putting the samples in this formula, I can get back my DTFT. So, it is another interpolation formula. So, this is one property, this shows, the DTFT itself can be obtained, by an interpolation formula of this kind, using it is own samples, every sample multiplied by function like this. This function, in fact, you can do some manipulation, you know, it is of the form e to the power minus say j theta, you call this angle theta, divide by e to the power minus j, say beta, you call this angle beta.

So, then you take, e to the power j, minus j theta by 2. So, it will be e to the power j theta by 2 minus e to the power minus j theta by 2 divided by again you take e to the power j beta by 2. So, it will be j beta by 2 minus e the power minus j beta by 2. So, this is twice j, sin theta by 2, this is twice j, sin beta by 2. So, it is basically becomes twice j, twice j, cancels e to the power minus j theta minus plus theta minus beta by 2. So, this much was theta, and this much is beta that by 2, into sin theta by 2 by sin, beta by 2, we had theta is this much, this is theta is much is beta. Anyway, this shows that, DTFT envelop can itself we obtained from it is own samples, if the sequence is finite length of n points, and I take n number of samples. So, this is shows that, no need to store the entire DTFT envelop, just most important of the n samples, using them I can do anything I can give back the original x n, I can give back to the original DTFT, this is property number one.

Then, another question, that is suppose some DTFT is given, this DTFT is given, I take this n number of samples, capital X k, there is capital X 0, capital X 1, and capital X 2, and put there in the formula, in right hand side, and I get the sequence x n.

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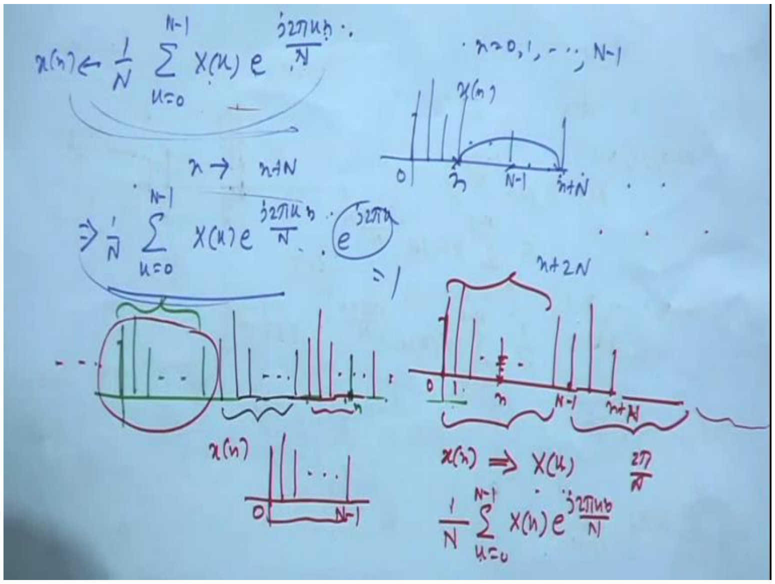

Now suppose, I allow this small n to go beyond the range, that is if I take this IDFT inverse, DTFT formula. So, I got this samples from the DTFT, I am putting them here, and I get some sequence. Now, suppose, this entire thing is a sequence of duration, originally n from 0, 1, dot, dot, dot, dot, n minus 1; I get some, if you plot this, this entire thing it will be a sequence, right, what this duration, we have 2 n minus 1, but suppose just for fun, just for mathematical fun or game in this formula I allow small n, to go beyond this, to take value of capital N, and n plus 1 n plus 2 and all that, then what happens to this formula? Just if we have substitute small n equal to something, for here say, first time, suppose originally n was here, and you are going to the right, to n plus capital N, what will happen if you bring n plus capital? If you replace n here by n plus capital N, you see what will happen is, one, this formula will give rise to small n plus capital N, small n plus capital N, capital N and capital N will cancel.

So, it will, e to the power j, 2 pi k, n, n cancels, and this is equal to 1. So, what we have left with this, which is same as this; that means, if this sample is suppose, some x n, x n, and now, in this formula if I allows small n to go beyond, jump by capital N, same x n will come here. If I go to n plus 2 n, it will be n plus 2 n means, n plus 2 n, 2 n and n will cancel. It will be j, e to the power j, 4 pi k, and n is 1. So, can you will get this summation, which is same as this. So, x n will come at n plus 2 n also. So, this way, if that original sequence is here, if nth point is here, you took the DTFT, then sampled it, you got capital X k, and then put in the, your using this capital X k, and forming this summation, if is small n is, from 0 to capital N minus 1, you get back this, this sequence right? But if we this summation, I allow small n, this right hand side, if I allow this small n to go beyond this range, then what will happen? nth point will come back here, at n plus 2 n, n plus capital N, same value will come.

So, we will get capital Nth point 0, whatever will be the value, that will come here, because 0 plus capital N, at 1 whatever the value, that will come here, and like that. So, this, this, whatever we have, that same thing will repeat, and it will go on repeating. So, it will be a periodic sequence; that means, if you take that inverse DTFT, inverse DFT, you get the sequence from 0 to capital N minus 1 first, but if the inverse DFT, I know, allow small n, to take all values from minus infinity to infinity, what you will get is the repetition of this basic period, again, again, again, which means, if I give you periodic sequence, if I give you in the opposite direction, if I give you a periodic sequence, this one period, and same thing is repeating say, same thing is repeating, is another period, then again same thing is repeating, like that.

So, this is one period, this is another period, and it goes on, dot, dot, dot, dot, dot, dot, dot, dot, dot, then what I do, I take out only this part, one period only, this part, this one period, from 0 to n minus 1, you call it x n, then from x n, you find out the DTFT, DFT, capital X k, which is nothing, but, if you take the DTFT of this, you sample, at period of 2 pi n, there is a 0, 2 pi by n, 4 pi by n, like that as we done in the previous case.

So, we get capital N number of samples, capital X k, capital X 0, capital X 1, dot, dot, dot, and then, you construct, you put them in that inverse formula. If you restrict small n in this formula, from 0 to capital N minus 1, you will get this, but if you now allow small n to go, from minus infinity to infinity, what we have seen, you will get back, periodic repetition this, that is why, this will, this followed by this, followed by this, this periodic sequence will come out, that means giving a periodic sequence, I can repeat it by a formula like this. How to get the formula? Take 1 period, from 0 to capital N minus 1, take its DFT, use the DFT in the inverse formula, and then allow small n to take all values, in that case, what we will get is periodic repetition of the basic sequence, basic finite length sequence.

So, this is one you have prepare thing, infinite I mean, periodic, but infinite length of sequence. You take one period, that is finite length, capital N point, take capital N point DFT, use them in the inverse DFT formula, if you allows small n to be only from 0 capital N minus 1, you will get back the original finite length sequence, but if you allow, you know, small n go anywhere, then you get an infinitely long sequence from this formula, but there, what you have seen, we have done, that this basic period will keep getting repeated. So, you will get back to your original periodic sequence that we will be given by this formula, where small n goes from minus infinity to infinity. That is, periodic sequence you can take, small n anywhere, here, here, here, here anywhere, put that small n in this formula, you will get the corresponding sample, if n is here, and if this much is the sample, take that n here, and from the summation, you will get back that, this is another property. Another very important property is circular shift.

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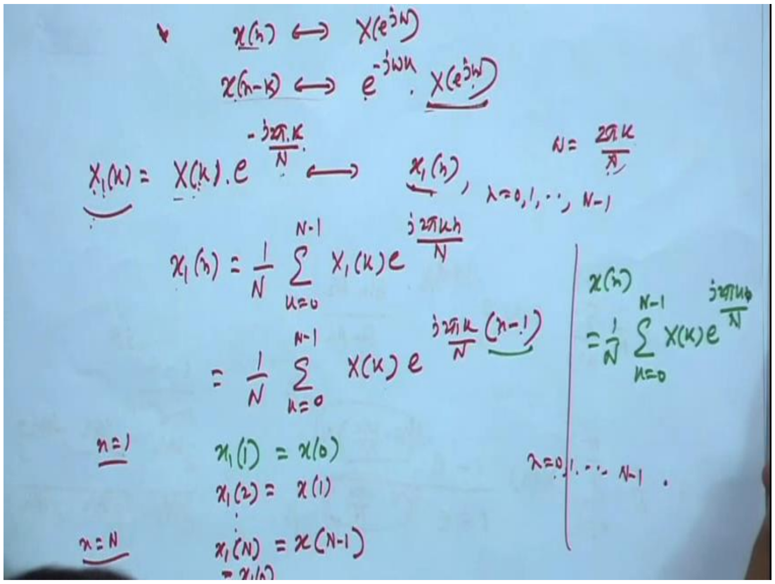

We have seen, that suppose, x n, has got a DTFT this. If I shifted by 1 to the right, like k to the right, you get e to the power minus j, Omega k, times this.

Now, I ask you a question, suppose capital X k, which is nothing, but, sample of this DTFT at, 2 pi into k by n, at that much Omega, 2 pi by n is the basic period, k sample mean k times a basic period. So, at that frequency, if you find out the DTFT, that is DFT. Discrete Fourier Transform, DFT, that I call capital X k. Capital X k is given to you, for k is equal to 0, 1, 2, up to n minus 1, but suppose I know multiplied by, e to the power say minus j, 2 pi, into k by n, and call it, x 1 k, if I take the inverse DFT of x 1 k, I will get some sequence x 1 n, I am taking n from 0 1 to this, there is not I am making a periodic from n minus 1 infinity to infinity, this basic period, finite length block is enough. So, I repeat capital X k, came from the original x n, DTFT was sampled, which gives rise to n point DFT. So, those DFT values capital X k, k could be 0, k could be 1 up to k n minus 1, so for any k, I multiplied by a complex factor.

So, the same k I will put here. So, this whole thing I call x 1 k. So, x 1 k means, I got x 1 0, x 1, capital X 1 0, capital X 1 1, capital X 1 2, dot, dot, dot. So, using them I can run the inverse DFT formula, and get a new sequence, that will be x 1 n. How is x 1 n related to x n, here when I took the DTFT, DTFT, multiplied by e to the power j, Omega k, I got x n minus k, that is delayed, shifted, here, if I take x k, instead of this DTFT, there is DTFT at Omega equal to 2 pi k by n, if I take that, and multiply by e to the power minus j, 2 pi k by n, that Omega. Then what will be, the corresponding sequence x 1 n, that time it was x n minus k, just shifted version by k, but this time x 1 n will be related to x n in which way. Fine, so x 1 n will be the inverse DFT of this, there is how I got x 1, inverse DFT of capital X 1 k, and there is a definition, I got from here, on here, but x 1 k, for any k, is this. So, as you run the summation, you bring that factor here, e to the power j, 2 pi k by n, into small n, minus 1.

Now, let us work it out. Suppose we start with n equal to 1, if you put n equal to 1, this is 0. So, it is like the original, what was the original IDFT? What was x n? That was 1 by n, summation. Now here, I will start with n is equal to 1 first, n is equal to 0 and be later, n is equal to 1, if n is equal to 1 is n minus 1 is 0, that is same as this formula, which n equal to 0, because x k is present, x k is present, e to the power j, 2 pi k, by capital N, e to the power j, 2 pi k, by capital N, all are taken this part is becoming 0; that means, here is n equal to 0, that is, this will be x 0. So, x 1, if n is 1 that is same as x 0. So, x 1, 1, so; that means, x 0 is getting shifted, then, take x 1 2, n equal to 2, if n is equal to 2, is 2 minus 1, 1, that is 1 here. So, x 1, dot, dot, dot, dot, take n up to, suppose n is capital N, I take n equal to capital N, then what happens? n equal to capital N, will be capital N minus 1.

So, x 1, capital N, will be, but point is capital N, x 1 n is also defined from n is equal to 0 1, dot, dot, dot, n minus 1, if I take capital N, it value that you will get that will be same as the value that n is equal to 0, because it is periodic. So, x 1 n is same as, is equivalent to x 1 0.

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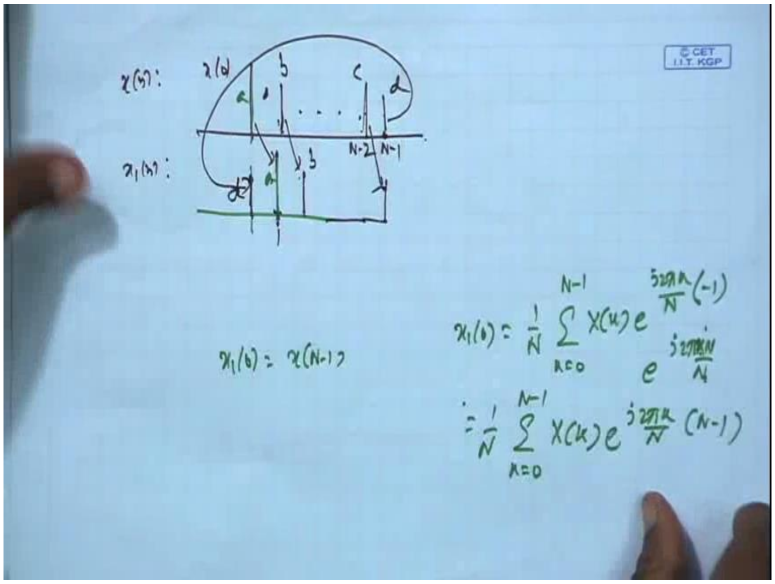

Let me explain, this is the formula. You started with n equal to 1, we got x 0; that means, that means originally you had x 0, this will go, x 0 will go to point number 1. So, now, this much will go here, if is a, a will go here, we have look at that 0. Now for the new sequence, this is x n sequence, this is x 1 n sequence, for x 1 n, at point number 1, the value will be same, as the original x at point 0. This a then if it is b, if it is b, b will go here, dot, dot, dot, dot, at n minus 1, and then n minus 2. Suppose at n minus 2, it is c, and then it is d, see here, if I take n equal to n minus 1, then; that means, capital N minus 2.

So, the same formula as this, but small n equal to capital N minus 2, it will be x, capital N minus 2. So, x capital N minus 2, this will go here. So, this is getting shifted here, this is getting shifted here, this is getting shifted here, all right? And last guy, I took this way, x 1 n, and that was this, you can view it this way also, what is x 1 0, an alternative view, x 1 0, x 1 0 means in this formula, 1 by n, k equal to 0 to n minus 1, x k, e to the power j 2 pi k, by n, into small n is 0 right. So, 0 minus 1, minus 1, for this, I can add, if I have, e to the power j, 2 pi here, you do not change anything, because e to the power j 2 pi is 1, so I can bring it here, and e to the power j 2 pi, it can written as 2 pi n by n. In fact, instead of j 2 pi, I can say, e to the power j 2 pi into k, because k is the integer. So, e to the power j 2 pi k, is 1, e to the power j 2 pi, e to the power j 4 pi, e to the power j 6 pi, they all 1. e to the power j 2 pi k is 1. And then I can write e to the power j 2 pi k, into n, by n. So, e to the power of j 2 pi, into k by n, I keep outside. So, it becomes, e to the power j 2 pi k by n, into n minus 1, all right? There is a same formula as this, small n replaced by capital N minus 1. That is x 1 0, is x n minus 1; that means, this guy, that the x 1 0 here, the x 1 0 at this point, it will be d, x n minus 1.

So, this will get. So, it is a circular shift, a goes to right by 1, b goes to right by 1, c this goes to right by 1. Last guy does not move further, it comes back to the original position. So, it is called a circular shift by 1, how it came? If I multiplied the x k, DFT by e to the power minus j 2 pi into k, by n.

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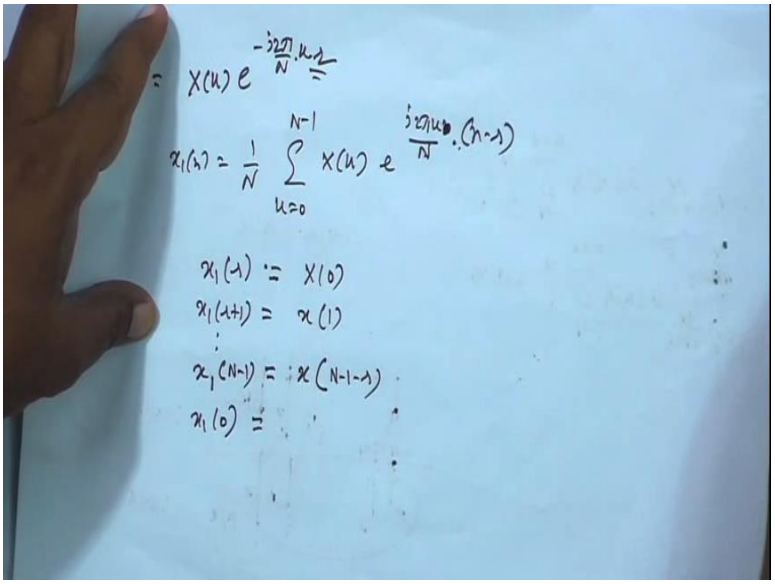

If now, I generalize further, x k, is obtained, you multiply by e to the power minus j 2 pi by n, into k, into r, and call it capital X 1 k, then what will be x 1 n? x 1 n will be, by the inverse DFT of this, e to the power of j 2 pi k by n, into small n, and in this case, minus r. So, n minus r outside right so; that means, you can start this way, x 1, if you x 1, r is given, r is fixed. So, suppose n is r. So, x 1 r, if you put n equal to r, r minus r that is 0, this formula it is give you x 0, x 1, r plus 1, this will give you, if you put n is equal to r plus 1, it will give 1.

So, this is the formula of x 1 then. If you have this e to the power j 2 pi k by n into 1, this will give you x 1, dot, dot, dot, x 1. So, n minus 1, if you have n minus 1, you should have n minus 1, minus r; n minus 1, minus r, it is, then x 1 0, x 1 0 means, if I put n is equal to 0, alright, if I put n equal to 0, then what happens? I ask you to think about this, and I will join here, in the next session.

Thank you.