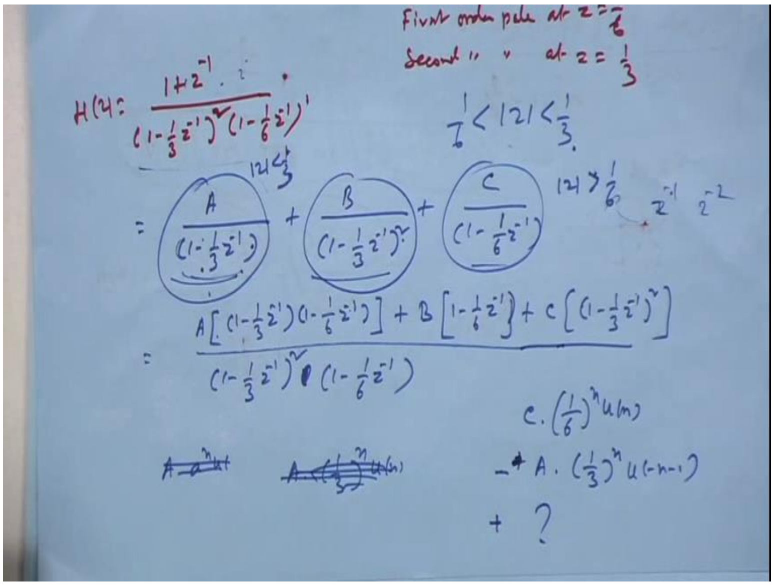
Discrete Time Signal Processing

Prof. Mrityunjoy Chakraborty
Department of Electronics and Electrical Communication Engineering
Indian Institute of Technology, Kharagpur

Lecture - 17
Introduction to DFT

I show one more example. If suppose, we have got things like this, some numerical polynomial, but denominator if you factorized, we have one factors say, which is occurring twice.

(Refer Slide Time: 00:29)



So, it is square is coming, another is say, this. Then we say, this has pole at z equal to 1 by 3, but pole order of the pole is 2. It has a pole at z equal to 1 by 6, whatever is the pole is 1, because this factor has a power 1, this factor has a power 2. So, pole at z equal to 1 by 3, but that has order 2. Second order pole, at z equal to 3. That makes a factor like this, 1 minus 1 by 3, z inverse. Pole at z not at z equal to 3, z equal to 1 by 1 by 3 because, if inputs z equals 1 by 3 it inverse of that, cancels with 1 by 3, you get 1, 1 minus 0, division by 0, it is a pole, this is second part of pole, because the power factor is getting repeated. So, there is a power of 2, we call, we say there is a pole of h z, h z has a pole at z equal to 1 by 3, of order 2. Second order pole, it has a first order pole z equals to 1 by 6.

First order, second order, this 2 factorized, you see, which have a by, as before 1 minus 3, z inverse, then b by, we have this again, why I will explain. This factor here, the factor is squared, and then c by the other factor. Why? Because if you do not have this, then denominator will be the product of 2, but here, we have got power of 2, this must be present. How about this? This denominator will be product of this, you will get this kind of form but, along with this 2, even if I have, a by the first order factor, still, denominator will be this product because, it is of the lcm of the 3 factors, 3 terms, so this, and square of this, this square of this, and then this. So, denominator will be this, into this, which is this. So, we have to have a provision for this also, this kind of term also, because these not enough. This, this kind of term of fixed possible to exist, because if we did just (Refer Time: 03:18) first order factor, here second order factor, the same factor, at this. When you are doing algebraic addition, denominator will be product of this 2 only.

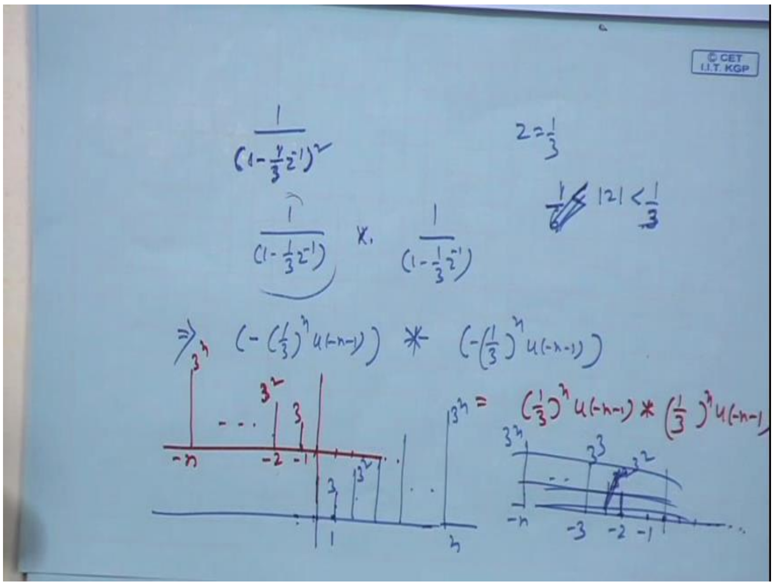
So, you can not rule out the possibility of a term, of this kind also, in this expansion. This is called partial factor expansion. If you add them, you know you will have, 1 by square. So, what will happen, 1 minus 1 minus 3, z inverse, you will take that out, you will be getting a into, then b, if you take this out, it is just, and c, if you take out, c into, now if you simplify, you will see, this will have a second order term z inverse 2, this first order z inverse, this will have z inverse 2, z inverse, here also z inverse. So, what are the 2 will be say, polynomial, where there will be z inverse (Refer Time: 04:43), z inverse 2 (Refer Time: 04:43), then constant (Refer Time: 04:44), that you, can equate with 1 plus z inverse.

So, here the z inverse 2 terms is absent. So, corresponding coefficient is equal to 0, coefficient corresponding to z inverse, you will equate with 1, the constant you equate with 1. You solve for ABC, you will get the coefficient, and we have this expansion. That I am not doing, that I am not doing, but that is 1 we are doing it. We will expand it, you write this, numerator expression collect the terms for z inverse. So, the corresponding coefficient you equate with 1, collect the terms for z inverse 2, corresponding coefficient is equal to 0, correct the terms for the constant, equate with 1, solve, you get ABC. So, you will get these factors; let them be as it is. The question is I have got the same roc as before. So, as per this term is constant, this term is constant, I know what they will be. Here the roc is, pole is z equal to 1 by 3, either mod greater than

1 by 3, or less than 1 by 3. I am giving less than 1 by 3. So, I will take roc to be mod z, less than 1 by 3 here.

So, this will give raise to capital A, a to the power, a is 1 by 3. So, capital A, 1 by 3 (Refer Time: 06:13) sequence, sorry. Let us start with this, c by 1 minus 6 z inverse, here, roc will be, because roc can not contain in the pole, z equals 1 by 6. So, either mod z greater than 1 by 6, or mod z 1 by 6, we have to take mod z greater than 1 by 6. So, this will give raise to c into, a to the power n, u n, because it is a right sided sequence, we have seen already. c constant, 1 by 1 minus z inverse, mod z greater than mod a, mod a is same as mod 1 by 6, there is, on by 6 itself. So, from our previous exercise, you get this term. In the case, of this a into, it will be minus a to the power n, that is 1 by 3 to the power n minus u minus n, minus 1. ABC you already found out, but there is 1 more term, this, this will be what? This is the question.

(Refer Slide Time: 07:31)



So this, I will give you answer now. I will come back to that. If you had an expression like this, or; obviously, this has a pole, secondary pole, z equal to 1 by 3. So, this cannot be part of the roc, of this part, this fraction. So, either mod z less than 1 by 3, or mod z greater than 1 by 3, we are giving mod z less than 1 by 3.

So, this 1 this is what was given, mod z less than 1 by 3, greater than 1 by 6, we have to take mod z, less than 1 by 3, for the roc. That means, you can write this also as, 1 system in to another. So, if one gets transformed into another, this has roc, mod z less than 1 by

3, because total was giving to me mod z less than 1 by 3 greater than 1 by 6. So, as I told you, roc for each of them, is either mod z greater than 1 by 3, or less than 1 by 3, but we are giving mod z less than 1 by 3, greater than 1 by 6 for the overall. So, we have to take mod z less than 1 by 3 only, to conform to that.

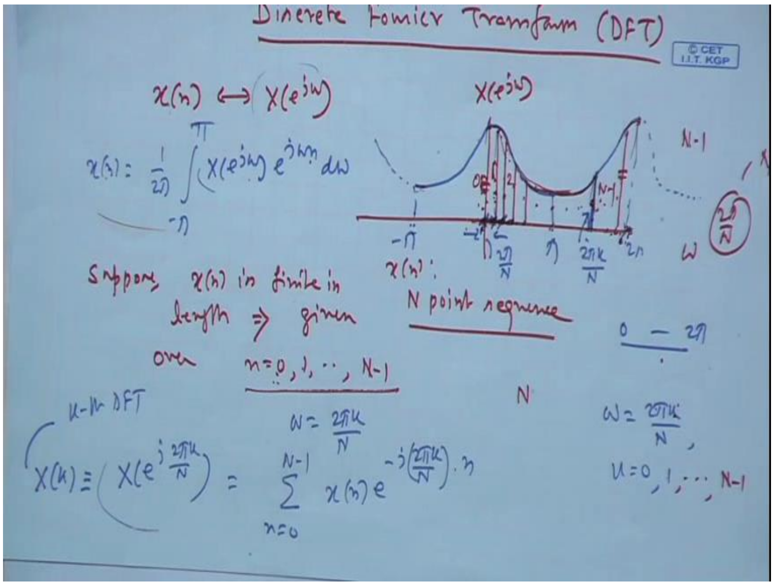
So, therefore, this has mod z less than 1 by 3, this has mod z less than 1 by 3. So, this will rise to, a sequence minus, 1 by 3 to the power n, u minus n, minus 1. And this will give raise to same thing, minus 1 by 3 n, convolution, because z transforms if you, if you are multiplying with z domain, it is called convolution in time domain. This convolution you work out, these convolution if you work out, you will get the result. What is this function? This function starts like this, 1, because minus and minus plus, you can forget the minus, it is same as, 1 by 3 whole to the power n, this minus is for all samples, minus is for all samples, minus, minus plus, forget all minus, same thing.

So, it is starts at n equals minus 1 onwards, n equal to minus 1 it will be, 1 by 3 to the power minus 1, that is 3. 1 by 3 to the power minus 1, then, 1 by 3 to the power minus 2, to 3 square, dot, dot, dot at minus nth, it will be 3 to the power n. This sequence then this side is 0, 0, 0, 0, 0, 0, this have to convolute with itself. If you want to convolute with itself, you have to flip it first, if you flip it, this will go here, then here, then here, like this, dot, dot, dot at 1, it will be 3, then 3 square dot, dot, dot, 3 to the power n, at n. You will flip it, then if you do not shift just multiply term wise, you get 0, because 0 into 0, this into 0, this into 0. So, at convolution result, at origin will be 0. If you shift it further to the right, there will be no overlap, this will. So, you continue to get 0 values.

If you shift into the left, this 0 comes here, this comes here, and still you get 0. So, if you shift into the left, up to minus 1, if you shift by minus 1, I will shift it to the left. So, this 0 comes here, and this comes here, still there is no overlap. So, you get 0, if you shift further, from minus 2, then 3 comes below this, say about 3 square, 3 cube, dot, dot, dot 3 square 3, cube 3. This 3 comes here, so 3 square, how many shift? 2 shift, where is it minus 2, you get 3 square, at minus 3, you get 3 cube dot, dot, dot minus n, you get 3 to the power n, you sum them, you get a value. Next time when you move, next time when you move, this 3 comes below 3 squares, c square comes below 3, just a minute. So, I will leave this exercise to you. You carry out this convolution and you get the result, convolution between that 2.

Let this be an exercise. May be I will put it in the assignment. So, that is all for inverse z transform calculation.

(Refer Slide Time: 13:35)



I now move to the next topic, which is very, very powerful topic, very important topic. Discrete Fourier Transform, DFT. Discrete Fourier Transform, DFT it is not discrete time fourier transform, but lets start with discrete time fourier transform, originally I have x n, one sequence may be, infinitely long, from minus infinity to plus infinity, it has a DTFT, x to the power j Omega which is a complex function. So, as I told you if you have to really plot it, you should plot it magnitude versus Omega, and phase versus Omega, and it will be both, it will be periodic in both cases over Omega, or a period of 2 pi. But for our purpose, I just have 1 plot, and that will be enough to explain the points, but actually it is not a wrong, correct thing, because I should have a mod plot, and case plot, because this x is, is real. So, mod is real, phase is real, right, but this is a complex number, complex number cannot be shown by 1 axis, nevertheless, just to make my point, I am, just showing x, e to the power j Omega.

The whole logic will be the same, if you separate the plots, 1 the magnitude mod x, e to the power j Omega, versus Omega, and at the phase, this angle of this, what is Omega. Same logic will hold good. Just keep, take it that way. Now, this I know it will be a periodic function, right? It might be like this, from minus pi to pi. So, if you go to 2 pi, same period will continue, same thing will continue here, like this, and again here. Now you see, x n is a sequence, I have got the DTFT, either you take from the here to here, 1 period from minus pi to pi, or you take from here to here, 0 to 2 pi. For this topic chapter, we will consider 0 to 2 pi, because it is a periodic function, any period of duration 2 pi is fine. So, either you take from minus pi to pi, or you take from 0 to 2 pi, whatever you get, repeat it, again, again, again, you get the (Refer Time: 15:57) form, right? This part will go on repeating, or this [part/particular] part will go on repeating, if this is a period, you go on repeating it, if this is a period, you go on repeating; you get the same thing. Does not matter how you chose your period, period is 2 pi.

So, I may chose this, but whether you chose this, or this, one thing, this entire wave form envelope is required, for instance, inverse DFT, x n is there, to get x n back, what was the DTFT? That is, x n was, if you from minus pi to pi, if you follow this minus pi to pi, integrate n is your choice, you want to find out sample at particular n, carry out the integration overall Omega, from minus pi to pi. So, you need this entire wave form full wave form, from minus pi to pi, then only 3 integral will be, you can carry out, you can get x n, is it not? That is what we have studied so far. Now I will make a claim, that suppose, x n is finite in length, that is, is given over, say, n equal to 0, 1, up to this. We call it capital N point, capital N point sequence, x n; That means, this given over this zone, outside this zone, it is all 0. We also call it f i r sequence, because you know, I mean, if you, if it is an impulse response, of any system linear (Refer Time: 17:46) system impulse response (Refer Time: 17:47) sample response. There is finite in length, there is called f i r sequence, finite impulse response, f i r sequence, n point sequence, f i r sequence, it all means, that is length is finite, outside that window, from 0 to capital N minus 1, outside that is all 0. Using this window only, you have got non 0 values of x n.

Suppose x n is given to me of that kind, this is not infinitely long; it is only finitely long over this region 0 to capital N minus 1, for some capital N. So, length is capital N, you are counting from 0, 0, 1, up to n minus 1. So, total length is capital N points. We call it n point sequence. For this also, we have the same DTFT expression, only thing is that summation will go from n equal to 0 to capital N minus 1, not from minus infinity to infinity, because outside that zone all samples are 0, but still it will be DTFT, to get that back, we should have this impulse DTFT.

So the entire wave form of DTFT is required, apparently, but now I will make a claim, that if this is given, it will not require this entire wave form. Just some samples of this, will be enough. Not all the entire wave form. That is I do not, I mean, in order store the entire wave form, I need infinite memory, because all values how many values, from here to here, if you go continuously, infinite values, right? Omega, infinite values of Omega, from this ranges 0 to 2 pi, but if I say that, if the sequence is given to finite in length, then, I am saying, that I will not require this entire wave form, I will require only it is samples, some samples, because you taken appropriately and my claim is this samples are good enough, if I have capital N number of samples we will see, that, those samples are good enough for me, to get that back, get that x n. I do not need the entire wave form. So, I need only to store some samples, actually capital N number of samples that is a claim I am making.

So let, I will have to justify the claim, but the claim I am making is this, this total zone, 0 to 2 pi right? This I will define in to, some sampling points, this period of sampling point, period, will be 2 pi by n. So, 0 to pi by n, point pi by n, 6 pi by n, dot, dot, dot. kth sample, kth will be a 2 pi k by n. So, this is a kth sample. This will be Omega equal to 2 pi k by n. So, if k is 0, Omega equal to 2 pi k by n. And k can be 0, if k is 0, Omega is here. If k is 1, 2 pi by n, I am here, if k is 2, 4 pi by n, I am here, and dot, dot, dot, for any k, it is here. So, I am just sampling, like this how many samples can I get? n minus, n samples, 0 samples, first sample, second sample like that, but if the last sample here. This has the 2 pi k, it is the last sample. Last sample will be, n minus 1s sample, 0 sample, first sample, because I have divided the range 2 pi into n.

So, 2 pi by n, 2 pi by n, 2 pi by n, how many 2 pi by ns? This period's n times right. So, this is 2 pi by n, before the 0th sample. Next 2 pi by n, first sample, next 2 pi by n, second sample. So, capital Nth, 2 pi by n period, before that will be n minus 1th sample. 0th sample 1, 2 pi by n, first sample, another 2 pi by n, 2 second, sample 3 times 2 pi by n. So, n minus 1 sample, n times to pi by n. Understand? I repeat it again, 0th sample, you go ahead 2 pi by n, first sample, you go ahead further, 2 into 2 pi by n, second sample, you go ahead further, 3 into 2 pi by n, then n minus 1th sample, you go ahead further, you cover this entire zone, n into 2 pi by n, that is 2 pi. That is why if you count, it will be n minus 1, 0th, first, 0th sample, first sample, second sample, dot, dot, dot, n minus 1th sample.

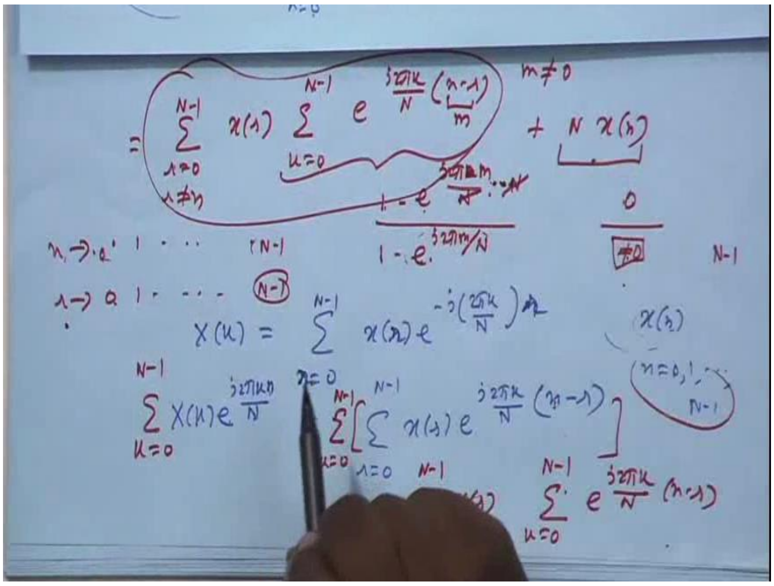
k will be 0, for zero th sample, k will be 1, for one th sample, dot, dot, dot, up to n minus 1. If you make k equal to n, it will be n, n canceling, Omega into 2 pi, and 2 pi means this sample is same as original sample, because a periodicity, this is same as this. This is will not go out nth; it will go to n minus 1th sample. So, total will be 0 to 1 minus 1, which will be n sample. So, here generate sample, at Omega equal to 2 pi, k by n, how much is the value that is DTFT? At this, this much Omega there is? That DTFT will be, x, e to the power j instead of Omega, it is 2 pi a by n.

So, by DTFT formula it will be x n, e to the power minus j Omega n, j Omega n, but Omega is this. So, 2 pi k by, this much is Omega, into small n, and small n will be what the range, but range is not from minus infinity to infinity, range is only for this, the instead of writing it as capital X, to the power j 2 pi k by n, in short, just write it as a function of k. So, in the kth sample, that is called kth DFT of x n, DFT means, Discrete Fourier Transform, that is x n is give sequence, a finite length sequence, of length capital N. I am sampling the DTFT at capital N number of point, not less than capital N, what is capital N? It was a length of original time sequence.

If the length is capital N, that many at least that many, if I could take more, but at this that many samples is required of this envelop. If I take more, that will be in that finite length sequence, has further, and has more length larger than capital N, with extra zeros brought in. Foe instance, if I, suppose, instead of taking capital N number of sample, I take capital N plus 1 number of samples, there is 2 pi by capital N plus 1, is the period. Then it means, I am considering the time sequence to be of length, not capital N, but from capital N plus 1, that is 0 to capital N, not n minus 1, with a last sample as a 0.

So, it will not change to the DTFT summation. So, DTFT will remain same, but it will be like length n plus 1 sequence. Anyway to start with assume, that given sequence is length capital N, and I am sampling the envelop, there is a DTFT, at capital N point, which means the gap, sampling period is 2 pi by n, and 0 is sample, first sample, second sample kth, up to n minus 1 nth, is back to this. So, kth sample is called the kth DFT, which is by the DTFT formula simply this fine.

(Refer Slide Time: 25:46)



But what is the big deal about it? So, my point is, giving this capital X k, I will be able to get back my x n, for n equal to 0, n equal to 1, upto capital N minus 1. Which means, this samples are good enough for me, to reconstruct x n, I do not need the entire envelop, and the integral calculation to get back x n. So, for that we see this, this is given, x k, I am rewriting the same expression.

So, now suppose instead of n, I make it r, it is local index, does not matter, r. I want to find out, x n, n can be 0, where n can be either 0, 1, dot, dot, dot, n minus 1, because that is the range of this sequence. So, sequence is located from n equal to 0, 1, up to capital N minus 1, outside this window, this fellow is 0. So, I am trying to find out x n, nth sample, where n will be any of them, but if you fix n, of your choice, from this range, once if you fix, that is fixed. My question is how to get that x n, fine. So, once n is known, left hand side, I multiply by, e to the power j, 2 pi, k, n by capital N. This k, same k I put, right hand side also, e to the power j, and this 2, e to the power combine, j 2 pi k by n, 2 pi k by n, it is small n, it is small n, it is minus r, minus r.

I am just multiplying both sides, capital X k, for a particular k, this is the summation it is j 2 pi that k, into small n, by capital N, what is small n? You want to find out, the nth sample, x n, n is in this range. So, that when you put on this. So, right hand side, also multiply by the same thing, take that inside the summation, combine that to exponential, you will get this. Then what you do, I sum this over k, keeping n fixed. So, here also, this after all, if you sum, it is sum over r. So, it is a function of n also, k also. So, I sum it over k. So, this is becomes a function of n only, because after summing over k, k goes, the function of n this, also function of n, because inner summation was with respect to r.

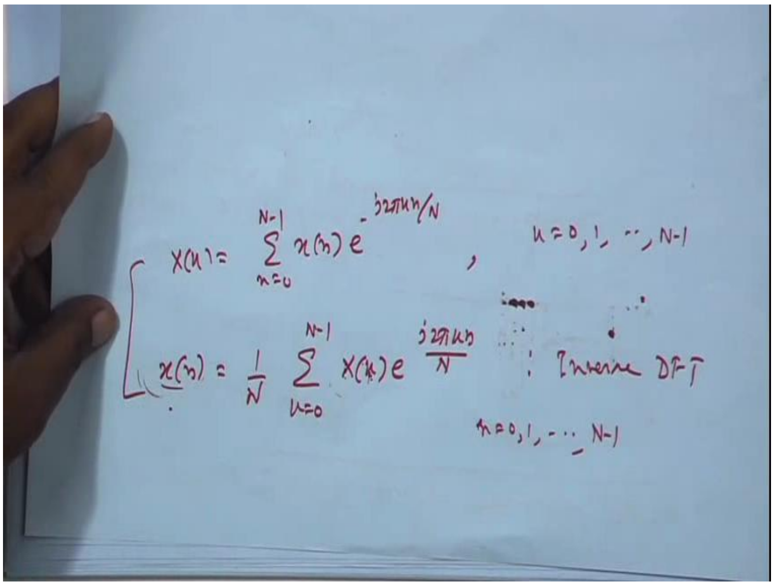
So, summation, this becomes function of a then k, then you are summing further over k. So, if you become a function of n. But I know whenever there is double summation, the rule of thumb is, you should interchange the 2 summation. So, r will go out, x of r, it depends only on r so, I can happily keep it outside, k equal to 0 to n minus 1, e to the power j, 2 pi k, depends on k, which comes here, n minus r, all right? Now small n is your choice, you have fixed n in this range. r is varying over the same range, 0 to n minus 1. So, as r is going from 0 to capital N minus 1, r will become equal to small n, once, on another occasion, it is not equal to small n. So, I can write that right hand side, as r equal to 0 to n minus 1, but r not equal to small n, or equal to small n, in that case I will write it separately. So, then you have got x r, summation, k equal to 0 to n minus 1, e to the power j, 2 pi k, by n, n minus r as it is, but n, r is changing, and it is varying, but it is not equal to n. So, n minus r will be integer.

Positive, negative, does not matter, integer, it will not be 0. And r equal to n case, when r equal to n, it is x n, that case I am writing separately, if r equal to n, n minus n 0, it is got 0 n 1, you are adding capital N times, capital N times x n, this expression right? Now what happens to this series? It is a GP series, this is n minus r, or you can call it m, and it is, m is not 0, because r is not equal to n, in this summation. So, in this g p series, it will be 1 minus e to the power j, 2 pi k, not k, it is summed over k, 2 pi m by n, and if you are summing 0 to capital N minus 1. So, n will come; divide by 1 minus e to the power j 2 pi.

Now look at the numerator, this cancels e to the power j, 2 pi m, m is an integer, positive negative integer does not matter, e to the power j to pi to integer that is 1, 1 minus 1, 0. This is 0, but denominator, small m, by capital N, small m, is neither 0, and small n can not be equal to capital N, if it is capital N, then again e to the power j 2 pi, which is 1, 1 minus 1, 0, it will be 0 by 0, but that can not happen, because what is small n, small n was n minus r, right? n is varying from 0, 1, up to capital N minus 1, this n, and r also varying from 0, 1, dot, dot, dot, n minus 1.

So, n minus r, maximally can be, capital N minus 1. The difference can be, these minus 0, these minus 0, maximize. So, m, m can be maximally, capital N minus 1, or less than that, it can not be equal to capital N, it means, e to the power j 2 pi into this, this factor, can never be equal to 1, 1 minus 1, 0. 0 by 0 there will be case, does not exist, and arise. It will be something which will be non 0. Or it will be 0, it will require j, either m is 0, not possible, or m equal to capital N, that is not possible, because m is, n minus r, maximally, either maximum n or minimum r, or vice versa. Which is capital N minus 1 n minus 0, or 0 minus capital N minus 1, either case, it can not be equal to capital N, or minus capital N. So, therefore, this can not be 0. So, 0 by something non 0 is 0. So, this whole part disappears, which means I am left with n into x n. So, what is n into x n this summation.

(Refer Slide Time: 32:51)



This part, I started with this, this are the correct, this is equal to n into x n, which means x n will be 1 by n, into this summation. This is called inverse DFT, that is, if this k sample, this samples, n number of sample of the DTFT, are available with b, capital X k, capital X0, capital X1, capital X2, capital X minus 1, are available with b, I can write summation, where k is varying, so capital X0, capital X1, like that, and multiply by this factor, k is varying, n is your choice from outside.

So, this summation will give you a sample, original time data sample, at anything, next where n is, within the range. 0, 1, dot, dot, dot, n minus 1, which means, I can reconstruct my finite line sequence, n point sequence, from this samples, capital X0, capital X1, up to capital X minus 1. That is, from this fellows only, I do not need the entire wave form, entire envelop. So, just I need storage for only n points. This is the fantastic result. So, what is x k? x k is the called n point DFT. Here you are summing over n, k is fixed of your choice, k range was 0th sample, first sample, or up to n minus 1 sample. So, find this x ks, DFTs, then using them, you can get your x n this is inverse DFT. Please go through this, we will meet again in the class, but I need full clarity on this.

Thank you very much.