# Discrete Time Signal Processing

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## Lecture – 16
## Inverse z-transform

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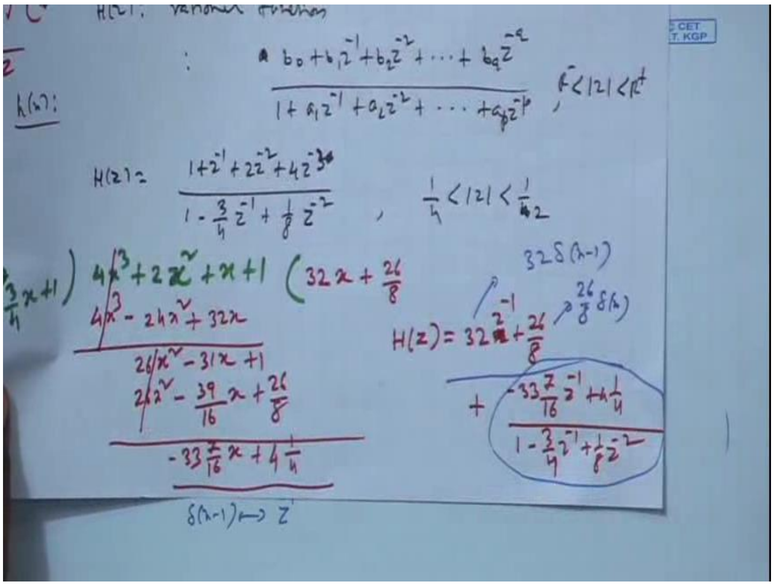

We will discussing inverse z transform calculations, but for these we will take H z to be rational formation not any arbitrary capital H z, but typical H z rational function which be specifically transfer function of a liner simply behavior, we have seen there is typically of the form some b 0 constant plus b 1 z inverse b 2 z inverse 2 dot, dot, dot may be b q divide by 1 plus a 1 z inverse plus a 2 z inverse 2 dot, dot, dot a p z inverse p there will be some r o c given there will some mortgage is given for this some r o c r plus can be infinity since it is the outside the circle causal r plus can be finite, but r minus can be 0. So, since have a circle or otherwise both are finite. So, it is both sided all those possibilities are there, but this is what with your H z.

How to find out H n how to find out there is the question that we will get by inverse z transform for that first thing you should ensure that numerators polynomial is a polynomials z inverse right z inverse you can replace by say some constant called x. So, it is b 0 plus b 1 x plus b 2 x square up to b q x to the power q and if the denominator 1 plus a 1 x into a 2 x, a 2 x square dot, dot, dot a p x 2 to the power p, you first make sure that the numerator polynomial power that is q is less than p if it is not less than we will divide the numerator by the denominator first like an algebraic division and take the remainder out constant will come out separately and remainder by this denominator will be the remaining rational part and we carry out inverse z transform back.

Let me explain through an example suppose H z is given as 1 plus z inverse plus 2 z inverse 2 plus 4 z inverse 3 and here it is a just a minute and it is given mode z, sorry there is a r o c that outer circle radiance up inner circle radiance 1 by 4. So, the region between them is having zs with mode z between these 2. So, that is the region of convergence given.

Now, here we see numerator is having power 3 in terms of x if you call z inverse x in terms a denominator has power 2 so obviously, here numerator has higher power than 2 that denominator. So, what will do we divide the numerator by denominator. So, I write for your convenience I write z inverse replacing z replace z inverse by x. So, I have got 4 and I divide. So, 4 x cube plus 2 x square plus x plus 1 I am writing for the way highest power then the lowest power 4 x cube plus 2 x square plus x plus 1 and I dividing it by 1 by 8 x plus, sorry minus 3 by 4 sorry 1 by H x square minus 3 by 4 x plus 1 alright if I divide what will I get.

So, you have to multiply by how much 32 x, 32 x means it will be your 4 x cube minus sum x square 8, 24 x square plus 32 x. So, this cancels this is 26 x square minus 31 x and plus 1. So, next time you multiply by 26 by 8. So, you get 26 x square minus 3 by 4 into 26 by 8, 26 by 8 that is 39 by 16 x plus 26 by 8, this is canceled, this will be something this I do not calculate here it is how much I believe minus 33, 7 by 16 that much x 31 and 39 by 16 means what 2 16 into 2, 32 and 7 n by 16. So, 2 and 31 33 and this will be your 4 this.

So, this entire polynomial I will write as H z is where constant will come out. So, 32 is what is this after all that is this into this plus this is equal to this right. So, this by this is. So, much of constant coming out and then the same numerator there is I write in terms of z inverse now same numerator sorry numerator is this part remainder part that is minus 33 z inverse plus 4, 1 by 4 divided by denominator is same that is 1 minus 3 by 4 z inverse plus 1 by 8, constant and remainder by divisor that is a standard thing that is this into this plus this is equal to this alright this.

We know that if you are dividing if you are doing this you know if I dividing y by x if constant is r and remainder is z then y is r times y is what y by y is r times x plus z r times x plus z is y which means y by r is x plus z by r y by y by x is r plus constant comes up and remainder by same x that is what. So, constant comes up remainder by the divisor.

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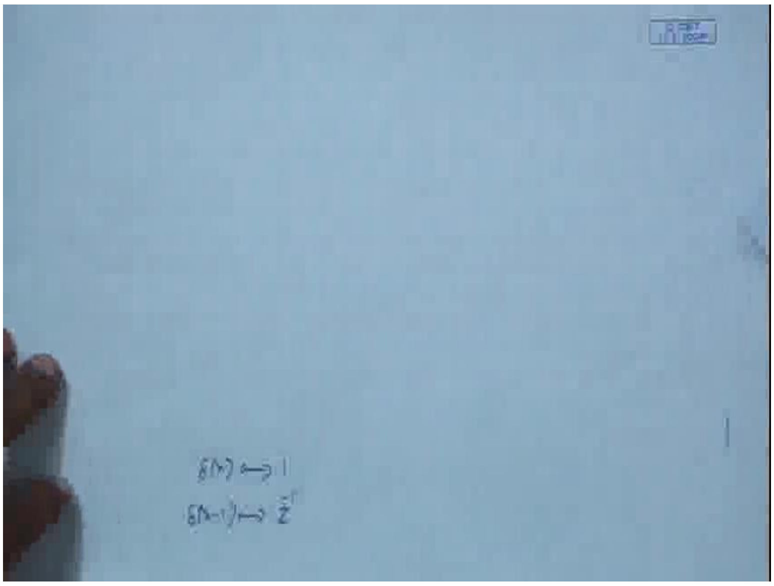

That is what comes now this part if you take the inverse z transform z inverse that could have been z inverse 2 then z inverse z inverse means I mean we have seen delta n has got z transform 1. So, delta n minus 1 will be having z transform z inverse times 1. So, this will nothing, but this will give rise to 3 if you take the inverse z transform of that it will give rise to just these 26 by 8 it will give rise to 26 by 8 into delta n because delta n has z transform one. So, 26 by 8 will be the z transform of 26 by 8 delta n. So, that that will be z transform will be 26 by 8 into 1.

So, this part is easily doable. So, will be then targeting I mean rational function of these form for numerator is a polynomial denominator is a polynomial both in terms of z inverse would you can the x, but numerator degree power is less than the power of the denominator how to take the inverse transform of those part that is what we will be focusing on the constant part will be having terms z inverse z inverse 2 z inverse 3 z inverse give rise to delta n minus 1 z inverse 2 gives rise delta n minus 2 dot, dot, dot, dot that you can get directly polymerize here. So, I will now consider this kind of expressions, but before you do that some basic facts will be of use.

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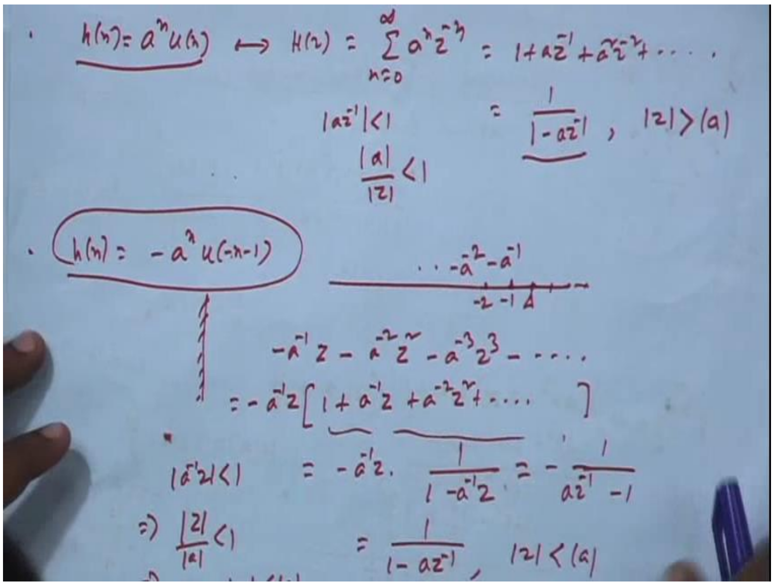

Suppose, I give you a to the power n u n that is the right sided sequence it is z transform is if you call it H n it is z transform will be what outside the circle and we know that summation a to the power n z to the power minus n u n. So, it will be from n equal to 0 to infinity, a to the power 0 z to the power 1 that a z inverse a square z inverse 2 dot, dot, dot, dot. So, this will be converging if mode a z inverse is less than 1 they need converge these 2 we have all done it in school 1 plus x plus x square plus x cube dot, dot, dot it converges to 1 by 1 minus x if mode x is less than 1 instead of x I have a z inverse seem that is true it will converge to 1 minus a z inverse or if it is true this will be in a z inverse plus mode a by z and mode of a by z is mode a by mode of z this you have seen earlier is less than 1 which means r o c will be outside the circle of radius module.

See, if I give you this sequence z transform of this, but r o c is this mode z greater than mode a that is 1, on the other hand suppose I have H n is equal to minus a to the power n what is this sequence if you started n equal to 0 u minus 1 that is 0 n equal to 1 u minus 2 0 n equal 2 u minus 3 0. So, n equal to 0 n equal to 1 n equal to 2 n equal to 3 as you go right ward this is equal 0, but if you start at n equal to minus 1. So, minus 1 has plus 1 minus 1 0 u 0 that is 1.

So, it will be minus a to the power minus 1. So, if you have a time axis and suppose this is your origin then this values will be all 0s from here it will be that is a minus 1 it is value will be minus a to the power minus 1 whereas, minus 2 it is value will be minus right if you put n equal to minus 1 here minus, minus plus, plus 1 minus 1 0. So, u 0 we all know one. So, minus a to the power minus 1 because n is minus 1. So, H minus 1 is minus a to the power minus 1 then you put n equal to minus 2. So, minus, minus, plus u 1, which is again plus 1, minus a to the power minus 2 dot, dot, dot, dot. So, if z transform if you write the expression, now it will be minus a inverse we have to start from here z to the power minus n I the z transform minus n, but n is minus 1. So, it will be z to the power plus 1 then minus this z to the power minus n, n is minus 2, minus, minus, plus z square.

No you take minus out a inverse z out. So, it will be 1 plus now this is an infinite sum again this will converge if a inverse z it is mod is less than 1 which means mod of z by mod of a less than 1 because a inverse z beats z by a mod of z by a is same as mod of z by mod of a this less than 1 which means mod z less than mod a. So, here r o c will be inside the circle of the same radius mod a and in that case this summation will be here it is as it is and this will be 1 by 1 minus if you call it x 1 minus x a inverse z and now if you divide numerator and denominator by a inverse z if have minus 1 this because minus 1 this becomes a z inverse I divide numerator by a inverse z. So, that cancels you get 1 denominator I divide by a inverse z. So, these cancels which a inverse z 1 and here 1 by a inverse z which a z inverse minus sign and then you minus you observe in the denominator it becomes 1 minus a z inverse.

So, I work out this example earlier same expression you get, but r o c is different here r o c is inside the circle and as a result your time domain sequence is different from these this was right sided it is left sided you started at 0 it is started at minus 1 2 different sequences having the same expression for z transform, but different r o c, we will use this things in a general inverse transform calculation.

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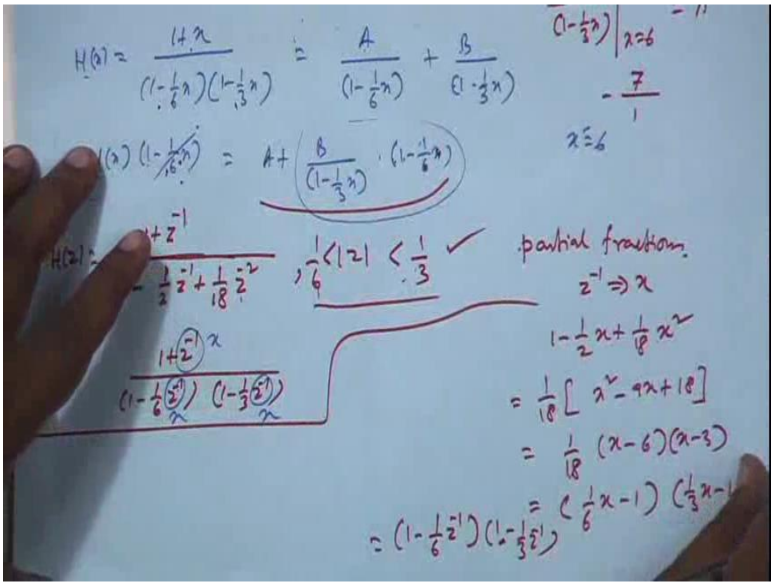


We say 1 minus just a minute this is given to you and r o c is given as that is a outer circle of radiance 1 by 3 inner circle of radiance 1 by 6, the intermediate region the region in between the 2 circles ring like region that is the r o c. Now, here you see numerator we are considering to have degree less than the degree of the order less than that of the denominator if you call z inverse to be x it is 1 plus x. So, x to the power 1 to power of the numerator polynomial is 1 degree 1 degree of the denominator is x square that is 2.

So that means, if earlier told you if the degree of the numerator is greater than or equal to degree of the denominator you divide the denominator by the numerator. So, that remainder comes out that you can handle separately easily then remainder part divide by the denominator remainder will have degree less than denominator because there was the divisor remainder cannot have the same degree as the divisor.

So, it will have degree less than that of this denominator this will be handling separately and we will follow u partial fractions for this you keep the numerator as it is factorize this polynomial into factors. This polynomial if you call z inverse x equal to x, if you say z inverse is x it is 1 minus half x plus 1 by 18 x square you can do factorization you can take 1 by 18 out x square and there it will be minus 9 x plus 8 if you want to do that way. So, it will be 1 by 18 x minus 6 x minus 3. So, it will be you can take 1 by 6 inside 1 by 3 here. So, it will be 1 by 6 x minus 1, 1 by 3 x minus 1 and then equivalently 1 minus 1 by c x take minus out take minus out and x is z inverse 1 minus like this we always prefer these kind of form 1 I mean like this constant and then something into z inverse not in terms of z, we always prefer in terms of z inverse.

So, we have factorization. So, you see this has got 2 poles at z equal to 1 by 6 1 by 6 in inverse 1 by 6 cancel 1, 1 minus 1 0. So, division by 0 this does not exists. So, H z does not exists at z equal to 1 by 6 we will have z equal to 1 by 3 that is why r o c could be either outside the circle of radius 1 by 3. So, it does not contain the pole is fine or it can be between the 2 circles not touching the circle, but between the 2 1 is of radius 1 by 3 another of radius 1 by 6. So, inter varying region they are z cannot be 1 by 6 z cannot be 1 by 3 or r o c could be mod z less than 1 by 6. So, there is a inner circle of radius 1 by 6 you go from the circle up to the origin that origin is z can be never become 1 by 6 or 1 by 3 that is again free of poles r o c cannot contain poles 3 possible r o cs out of these I was I consider these as an example for these r o c I have to calculate the inverse.

These as before since we are not very expert in this you can continue to call it x x x if you have got 1 plus x divide by 1 minus 1 6 x 1 minus 1 by 3 x these, suppose I write as then I can explain how can I write like this as 1 factor another is why because if you add the 2 a and b are 2 constants to be evaluated why you can write like this because if you add the 2 denominator will be product of the 2 which is here and the numerator will be a into 1 plus b into 1 that will be a constant that will equate with 1 and then a into minus 1 by 3 x plus b into minus 1 by c x that I will equate to x.

So, numerator will be of this form only some constant and something x. So, what will be a to obtain a this is your H x right you multiply left hand side and right hand side by this factor that is H x left hand side if you multiply by this factor 1 by 1 minus 6 x right hand side this cancels a plus b into 1 minus 3 x into this are multiplying right hand side also left hand side also by this factor.

So, this factor when I multiplied this and this canceled it came here they canceled I got a back, but in case of b no cancellation. So, b by this 1 minus one third x variance and then multiplied by that 1 by 1 minus 1 by 6 x left hand side this. Now, if I take if I want to make this 0 that is if I take x is equal to 6 on the left hand side right hand side also it will be 0. So, I will left with a this part will be 0 because 1 minus 1 by 6 x at x is equal to 6 will be what 0 1 by 6 into 6 that is 1 one minus 1 is 0. So, for x equal to 6 this part goes I am left with a and a will be H 6 into 1 minus 1 by 6 x, but the z multiplied by H x by this you see this H x it will cancel this factor will cancel this denominator right.

So, what I will be left with is it this I will be left with if you multiply H x by this factor this factor will cancel the denominator. So, that was 1 plus x by 1 minus 1 by 3 x here I will be replacing x equal to 6 because of the right hand side I am replacing x equal to 6 here x is equal to 6 means 1 minus 1 6, 6, 6 cancel 1 minus 1 0 this part goes I am left with a. So, right hand side is a left hand side H x into 1 minus 1 by 6 x if you take this H x multiplied by 1 minus 1 by 6 x that cancels with this your left with 1 plus x by 1 minus 1 by 3 x you put x is equal to 6 here that will give you a you put x equal to 6 you have 6 plus 1 seven divide by 1 minus 1 by 3 into 6 that is 2 1 minus 2 that is minus 1 which means a is minus 7.

Similarly, find out b you find out b how the same way I multiply left hand side and right hand side by this factor now 1 minus 1 by 3 x. So, 1 minus 1 by 3 x 1 minus 1 by 3 x they cancel I am left with b, b and in case of a 1 minus 1 by 3 x here divide by 1 by 1 minus 6 x right hand left hand side H x into 1 minus 1 by 3 x. So, that cancels with this if I take H x multiplied by 1 minus 1 by 3 x that cancels with this guy. So, I am left with 1 plus x by 1 minus 1 minus 1 by 6 x now right hand side if x is 3, 3, 3 cancels 1 minus 1 0 this part becomes 0 left with b I am left with b. So, right hand side if I also have to put x is equal to 3. So, that will give me b how much is b if you put 3 here 3 plus 4 minus, 4 minus 1 minus half, minus 8.

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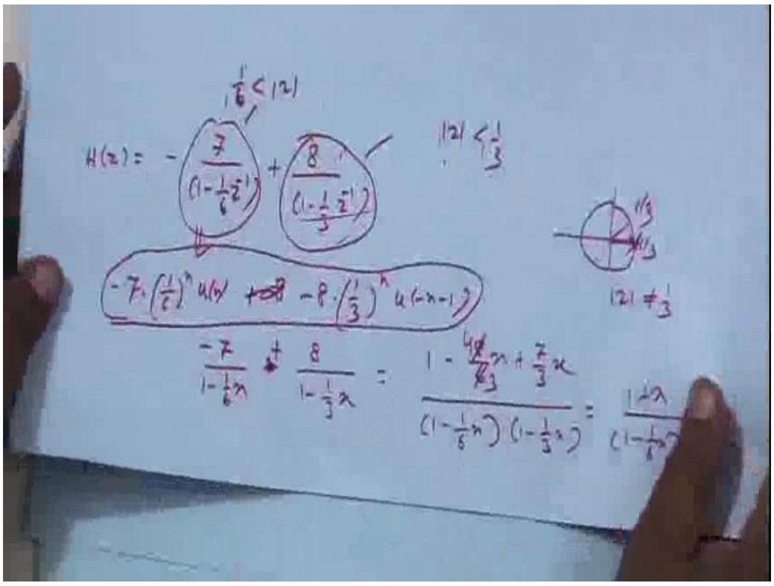


Now, if you put them back here, you can verify that is 7 by 1 minus x plus 8 by, sorry minus 8 by 1 minus 3 x give minus 7 b is plus 8 because if you put x is equal to 3 1 by 2 1 minus 1 by 2 is 1 by 2, no change of sign. So, 2 into 1 plus 4 it will be plus 8 this plus 8 see if you add the 2 8 into 1 and minus 7 into 1. So, you get 1 and then minus 8 by 6 x plus seven by 3 x 8 by 6 is 4 and 3.

So, minus 4 by 3 plus seven by 3, basically plus 1 divide by this which means you get back your 1 plus x divide by all right. So, now, you replace z by z inverse. So, your H z will be a 8 minus 7 by minus 7 by 1 minus 1 by 6 z inverse x you put back as z inverse plus 8 by 1 minus 1 by 3 z inverse this part has a pole at 1 by 3 if z is 1 by 3 it is 0. So, it is r o c cannot contain 1 by 3.

So, it will be either outside 1 by 3 or inside 1 by 3 there is also you draw a circle of radius 1 by 3 it will be either inside the circular radius 1 by 3 or outside it cannot be including the circle of radius 1 by 3 because on that circle z can be 1 by 3 if you have this circular radius 1 by 3 z here is 1 by 3 this point if you put that z this is 0. So, this pole this are it cannot be part of this r o c z mod z cannot be 1 by 3 this cannot be. So, either inside the circle outside the circle, but we are giving this 2 with r o c.

So, we have to confirm to this. So, this is inside the circle. So, for this we will take the r o c to be this part and this again has a pole at z is equal to 1 by 6. So, r o c for this will be either outside a circle of radius 1 by 6 inside a circle we will take out side because this is what is given to us we have to. So, for this it will be mod z greater than 1 by 6. So, for the overall it will be the intersection between the r o c of this and r o c of this and then you get this mod z less than 1 by 3 from here mod z greater than 1 by 6 that is the overlapping region intersection of the 2 r o cs.

Now, when you have this remember I consider this expression 1 minus a z inverse, but if mod z is outside a circle that mode z is greater than 1 by 6 it is outside the circle mod z is greater than it is of the form minus 7 by 1 minus a z inverse a is 1 by 6 here and roc is mod z greater than mod a mod a is same as a because a is that is real positive number 1 by 6. So, it is confirming to this form mod z greater than this a mode a in this case a is 1 by 6. So, mod of that is 1 by 6 itself.

So and mod of z greater than 1 by 6 is the r o c. So, r o c is outside the circle it is of this form. So, this will give rise to minus seven as it is and then a to the power n u n. So, a is 1 by 6, 1 by 6 to the power n u n because 1 by 1 minus a z inverse 1 by 1 minus a z inverse mode z greater than mod a mod z greater than mode a 1 by 6 is mode 1 by 6 itself. So, for this will be and in this case it is of the form 8 into 1, 1 by 1 minus a z inverse a is 1 by 3, but r o c is mod z less than a less than mod a this 1 same expression 1 by 1 a z inverse 1 by 1 minus a z inverse, but r o c is mod z less than mod a a is 1 by 3.

So, mod a is same as a because 1 by 3 is positive real. So, this will have this kind of sequence as inverse z transform it will give rise to minus 8 comes as it is a to the power n a is 1 by 3. So, 1 by 3 to the power n u minus n minus 1 all right in the early part of next class I will consider 1 more examples and then this topic will be over we will move to next topic that is discrete Fourier transform.

Thank you very much.