Discrete Time Signal Processing
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Lecture – 14
Regions of Convergence of z-transform

So, now I consider a sequence which is right sided; that is it starts at any n 1 and any n 1 greater than equal to 0 and goes up to infinity x of n.

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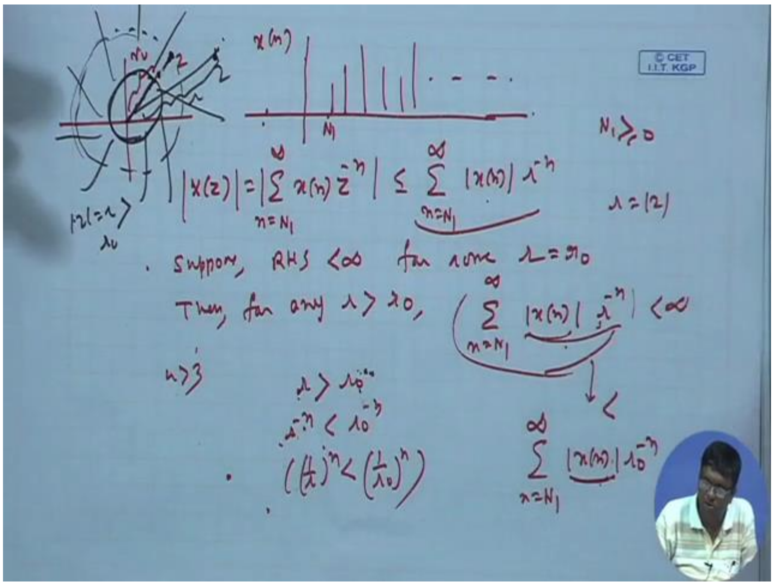

To start with n 1 greater than equal to 0, then we will consider the case when we starts at a finite negative point then goes to the infinity, but that later. So, first is as before x z, we will talk take from n 1 to infinity only, because values to the left of n 1 the sample values of x n as 0. So, those products are 0 no point in considering. Then as you before you take the mod, mod of a summation less than equal to sum of mod it continues mod of x n and mod of z to the power of minus n will give raise to r to the power minus n. Now, suppose this r h s, r h s means right hand side this quantity, is finite for some r, r is nothing but mod of z magnitude part. Then for some r equal to say suppose r naught.

If you put r naught here in the summation, this summation is finite. Then for any r greater than r naught, what happens to the summation. It is r to the power minus n in this summation you see, r to the power r is greater than r naught, which means r to the power minus n is less than r naught to the power minus n, r is greater than r naught. So, r to the power minus n means what 1 by r to the power n; that will be less than 1 by r 0 to the power n is, is not very simple, because 4 is greater than 3. So, 1 by 4 less than 1 by 3, any positive power of 4, because the powers are positive powers, small n is starting from N 1 to infinity this all positive numbers. So, the positive power of 1 by r; that is 1 by 4 here will be less than any positive power of 1 by 3 that is 1 by r 0. So, this means actually 1 by r to the power n, n is positive, because this summation considers n in the positive zone right; that means, this summation is, for sure any small nth entry mod x n was present here also here also, but that time I assume that this summation is finite for r equal to r naught; that is if you have r naught, r naught to the power minus n. this is finite this is what I have assumed that it exist.

Now, in this summation consider any sample nth mod x n present here mod x n present here, but that time I was multiplying by r 0 to the power minus n, and I am now multiplying by a smaller number r to the power minus n. So, in the previous summation this positive number, this has mod x n multiplied by 1 positive number r 0 to the power minus n. Here again the same particular number mod x n multiplied by a smaller positive number r to the power minus n. You see r to the power minus n less. So; obviously, this is lesser than this. So, this whole sum is less than this summation if this summation is finite, this is also finite. This summation is less than this summation, because for mod x n mod x n common r to the power minus n is a positive number less than r 0 to the power minus n, as you have seen here, because r is greater than r naught I am taking r greater than n naught. So, r to the power minus n is less than r to the minus n.

Which means this will be finite. Therefore, it gives what if this convergence for some z with magnitude r naught. Then there is suppose this much is your n naught and this is z, and this much is your r naught, you draw a circle. So, these are the radius r naught. Therefore, any z outside the circle magnitude will be r; mod z r will be greater than r naught right for any z here, say z here. Now this magnitude is r is greater than r naught. So, for that r this summation is finite, if r is greater than r naught, this summation is finite. Therefore, if z is here then for that z transform will exist. It means that entire region outside the circle up to infinity; entire z plain outside the circle will be your region of convergence.

Now, I arbitrarily started saying that suppose say this is finite that is say z transform exist here, for some r equal to r naught. Now you start exploring if I go below r naught, still does it converge. If yes still I make it lesser, finally, a point it will come when you will be in the bordering point, if I make r naught lesser, than these then summation will not converge.

So, I will stop here, and from previous discussion the entire region outside the circle will remain region of convergence. So, if it is purely right sided sequence starting from a index n 1 which is either 0 or positive, r o c is region of convergence that is r o c is outside the circle, up to infinity. No restriction, it can go up to z equal to infinity.

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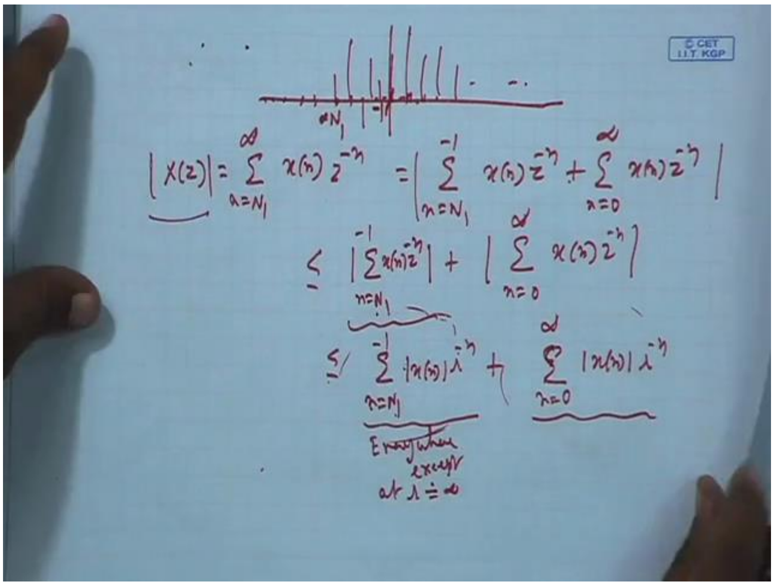

Now, I have taken more general that suppose the sequence is right sided, but it starts at a finite negative value here n 1, n 1 is a negative number dot, dot, dot, dot, and then this that is, and then to the left all zeros. Then z transform as before n will start at n 1 negative value up to infinity, and this summation I can write as a summation of two part; one will go from n 1 the negative value, up to minus 1 here, and other will go from n equal to 0 up to infinity, from here to here up to infinity. Then mod x z is mod for mod of a summation, is less than equal to summation of the mod. So, this is less than equal to summation of mod of this. This part is a finite duration sequence; you have to only consider this, from some negative index n 1, to another negative minus 1.

So, this will always exist, except for z equal to infinity, because minus n and n is taking negative value. So, this will be plus n. So, when you write z equal to, you know I mean basically if you make it further it becomes mod of summation less then equal to summation of mod, as purely as before, and summation of mod means product of the mod, mod of the product is same as product of 2 mod, mod x n and mod of z to the power minus 1 will be raise to this, and on this side again mod of a summation less than equal to summation of mod. The summation of the mod on the product, mod on the product is product of the mods, where mod of z to the power minus n is r to the product minus n this alright.

This summation is always finite we have seen is finite duration, it is like a finite duration sequence from n 1 to minus 1, only those finite number or points in this summation otherwise no point. So, this will always exist for any r except for r equal to infinity, because n is taking negative values here. So, here it is positive r infinity means it will diverge. So, this is converging everywhere, except at r infinity. And this summation is for a sequence that starts at n equal to 0 up to infinity. So, it is a purely right sided sequence. So, it is convergence will be outside a circle up to infinity. Now for z transform to exist this should be positive this should be finite, this should be finite, because this is a positive number this is a positive number.

So, they cannot cancel each other, both have to be finite is not that this is infinity this is infinity, but 1 is plus 1 is minus infinity minus infinity is 0 that does not happen, this is positive, this is positive, and this non negative non negative. So, this must be finite this must be finite, but this summation is finite means this is for a right hand sided sequence, because n is from 0 to infinity that is this part, this sequence where left hand side is zeros. So, we have already considered this case, for this kind of z transform that is right sided sequences, z transform exist, the region of convergence include I means is what outside a circle, which we have like this outside the circle up to infinity, and if this part is present infinity is not allowed, overall. Yes, outside a circle, but excluding z equal to infinity r equal to infinity or z equal to infinity.

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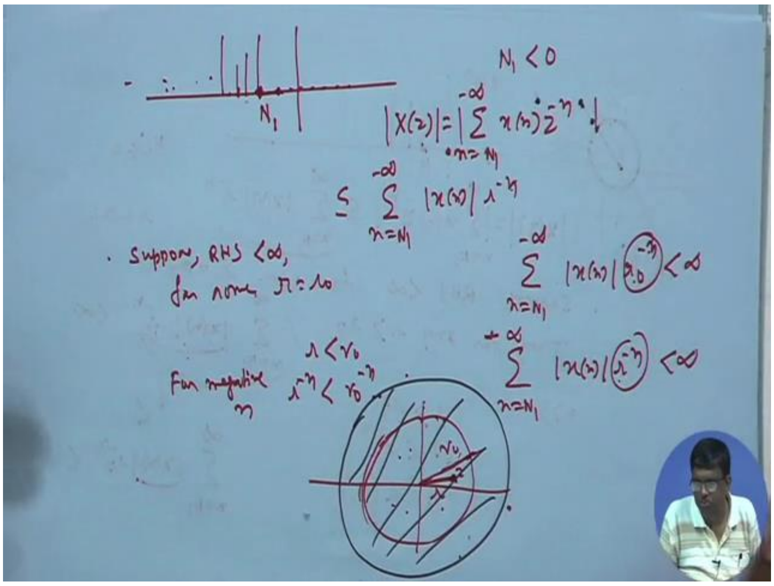

In this case it starts at n 1 to the right of origin or at origin, not at negative index, then it converges outside circle, outside some circle goes up to infinity. This for the right sided sequences it is absolutely similar manner. We can consider a left sided sequence; that is suppose to start with I assume that I am starting at some index; say n 1 and then I am going it is like this, and up to infinity, and then all zeros, and suppose n 1 is less than 0. In this case z transform n will be from n 1 which is a negative number up to minus infinity. So, you take mod x z, mod of a summation less than equal to all those things. Suppose r h s that is this summation right hand side, is finite for sum r equal to r naught. Remember here n has a minus sign, but n is taking negative indices, N 1 is negative and further you go to the negative. So, negative negative positive. So, basically positive powers of r are coming. now if this is true; that means, I am assuming n equal, this is to minus infinity mod x n r naught to the power minus n, if I replace r equal to i r naught, this is finite.

Now, if I take any r less than r naught, then how about this summation. Everything else is same mod x n mod x n n 1 2 N 1 2 minus infinity, but here it was r 0 to the power n and r to the power minus n. You see minus n is positive, because n is taking n number. So, r 0 to the power some positive power r to the, for the same positive power. Therefore, if r is less than r naught, then this is also less than this r to the power minus n for negative n, this is also less than, because if n is taken it means it is positive now, minus

minus plus minus minus plus. If r is less than r naught means r to the power of minus n for negative effect will be less than r r naught to the power minus n.

So, mod x n here multiplied by r naught to the power minus n, and here multiplied by a number r to the power minus n, but this is lesser than that. So; that means, summation is lesser than this summation. So, if this finite this is also finite. So, this means, if this converges for any r naught, any r naught you draw a circle (Refer Time: 14:52) same radius r naught. For any z on this circle you have got magnitude r naught, and this is satisfied means z transform is existing on the circle, in that case you consider any point z inside, say here the corresponding r is a magnitude that is less than r naught, if z is here for that z; obviously, z transform will exist, because this will be finite, because this is finite r is less than r naught, this much is r, this much is r naught.

Now, I will start 2 r naught arbitrarily. Now I will start expanding r naught if I expand the circle, make it bigger. Finally, a point will come that I cannot expand anymore if I expand any more it will not converge. So, suppose there is a limiting case. So, r o c will be full region inside the circle, outside it will diverge, or it is (Refer Time: 15:49) because I am considering negative power in this case. So, positive powers of r. So, r is 0 does not matter, 0 to the power positive r no problem. Now, I consider a sequence which will be left sided, but will start at a positive index that is a 0 0 0 0.

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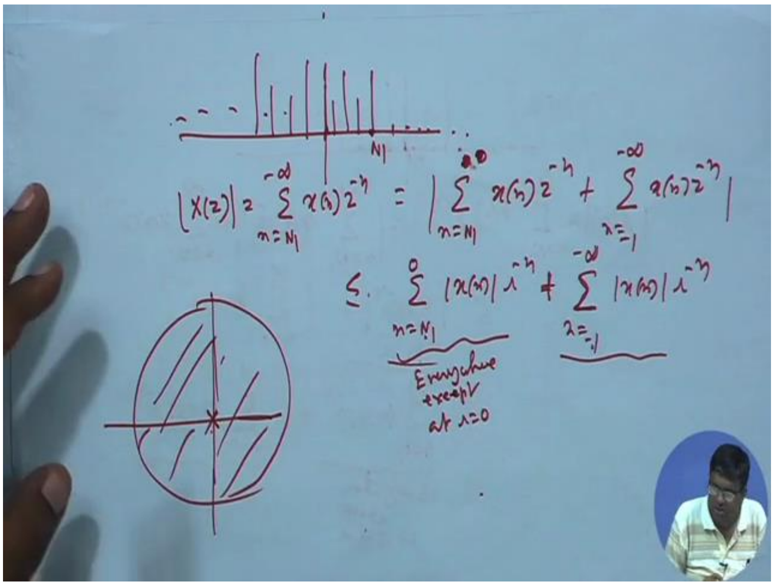

In this case this summation you write as summation of two component n equal to n 1 to 0. This is n 1 n 1 minus 1 up to 0. You can either take up to 0 or may be 1, or maybe 0, 0 will be better n 1 to 0, and then n from to minus infinity. Therefore, mod of x z will be this, which is less than equal to mod of this plus mod of this. Mod of this again less than equal to, mod of a summation less than equal to summation of mod of this and r to the power minus n, and again you get the same way. If this is finite, this is finite, this finite, none of this should be infinity, and therefore, positive. So, it cannot happen that this is infinity this is infinity, but infinity minus infinity, is 0 that cannot happen, the positive positive positive thing.

So, there is a positive sign here alright. It cannot be that this is positive plus infinity and this is minus infinity and they cancel it is not possible. Now this is a finite sequence, finite sum n 1 to 0. It is like a z transform, it is like a case of z transform with sequence that is from 0 to n 1 0 to n 1. So, we have seen z transform that is this summation will be always existing, always finite, but since small n is taking positive values here; 0 or positive 0 1 to also positive will be negative powers of n, negative powers of r; therefore, r equal to 0 will be ruled out. So, this is everywhere, converges everywhere except at r equal to 0, and this one is a purely left side sequence from n equal to minus 1 to minus infinity; that is this part, and for this region of convergence is inside some circle. So, together this means, it will be inside some circle r o c, but this is excluded. Otherwise this whole area or this is excluded. And lastly the more important case of a general sequence; that is both sided.

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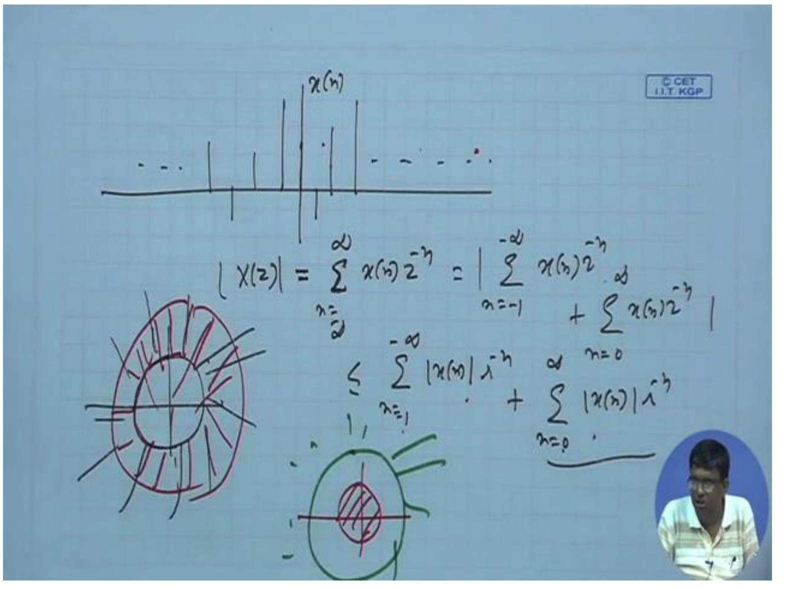

For a both sided sequence, it is from minus infinity to infinity. Here x z and this summation you break into two part n equal to minus 1 to minus infinity. So, mod x z is this, which is less than equal to mod of this plus mod of this, and mod of this is further less than equal to r to the power minus n. again this must be finite, this must be finite, because both are positive and there is a plus sign. If both are positive then only right hand side is less than finite, and therefore, left hand side is guaranteed to be finite.

Now, this is corresponding to a purely right sided sequence from 0 to infinity. So, it is region of convergence is outside a circle up to infinity and this is a purely left sided sequence from minus 1 to minus infinity, it is region of convergence is inside the circle up to origin. Now suppose the two circles are such that there is an overlapping region; that is in one case outside this circle; that is from these up to infinity. And in another case inside a circle; that is from this, then we have to see the overlap this part. This will be the r o c, because here both will exist, but if the two circles are such that there is no overlap; one is, I mean from here it is inside the circle suppose here, and from here it is outside the circle, then there is no overlap between the two shaded region, in that case there is no r o c the sequence is then, so z transform does not exist. See region of convergence is very important, because you see, I can give you the same sequence I mean you can have same z transform expression. Suppose I give you a to the power n u n.

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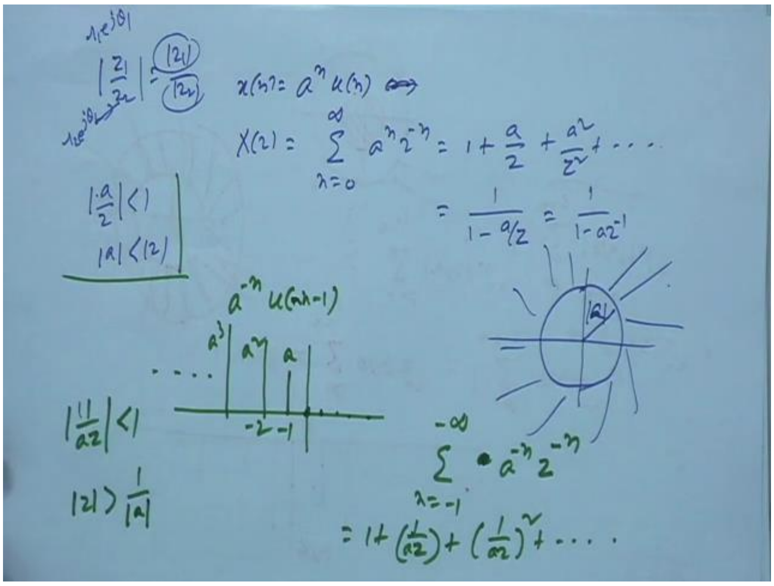

What will be the z transform and r o c, that mean this is x n will give raise to what x z summation this u n is present which is from 0 to infinity, and therefore, a to the power n z to the power minus n; that means, a to the power 0 that is 0 1, if a by z a square by z square dot, dot, dot, dot this infinite sum will converge if this is less than 1 we all know, and it will then converge to 1 by 1 minus a by z, this is some basically they have done. We write just 1 minus a z inverse.

These are right hand side sequence, it is region of convergence should be outside a circle, and what does this give. Mod a by z is less than 1. You know that mod of two complex numbers; z 1 by z 2 is same as mod z 1 by mod z 2. If you do not now remember, you do not understand remember z 1 if you write as r 1 e to the power j theta 1, this you write as r t 2 e the power j theta 2. So, division based r 1 by r 2 into e to the power j theta 1 minus theta 2, mod of that will be r 1 by r 2 into mod of e to the power j theta 1 minus theta 2, but e to the power j theta 1 minus theta 2 has mod 1. So, it will be r 1 by r 2, and r 1 is this r 2 is this. Now mod of a by z less than 1 means mod of a less than mod of z, which means r o c is what. There is a circle of radius a, and mod z greater than mod a, because radius is mod a, the whole region outside this.

Because mod z greater than mod a. So, any z here means mod will be greater than mod a, that is the r o c. I consider, and in this the r o c the z transform is this I consider, a defined sequence. Now, a to the power minus n u minus n minus 1. What is this sequence, u minus n minus 1? So, at n is equal to minus 1 into the minus minus plus, so plus 1 minus 1 0. So, if the sequence is like this, at n equal to 0 it is u minus 1, value is 0, so 0. At n equal to 2 n equal to 1 minus 1 minus 1 u minus 2. sorry at n equal to 1 u minus 2 u minus 2 is 0, n equal to 2 u minus 3 0 this at 1 equal to minus 1 minus minus plus 1 minus 1 0 u 0 is 1 a to the power minus 1 n is minus 1. We have got a, then at there is minus 1, then at minus 2 a to the power 2 a square, because minus 2 means plus 2 minus 1 is plus 1 u plus 1 is 1, this way this side will be a the power 3, dot, dot, dot, dot.

If I take the z transfer of that this sequence, summation will be from minus infinity, or minus 1 to minus infinity a to the power minus n that is n minus 1 means a to the power plus 1, n minus 2 means a to the power plus 2, z to the power minus n alright, and what will be the summation. It is like you know 1 plus 1 by a z plus 1 by a z square and dot, dot, dot, dot, dot. If mod of this is less than one, then you have to converge this, but mod of this less than one means, mod of 1 divided by mod of z a z, and mod of a z is mod of a into mod of z. So, if you simplify it will be this. I changed the problem a little to make it more convenient, I just make a little change here. this part is fine, this is also fine, but and then it will give raise to, if this happens 1 by 1 minus 1 by a z, this is fine, but I will just change the problem a little bit.

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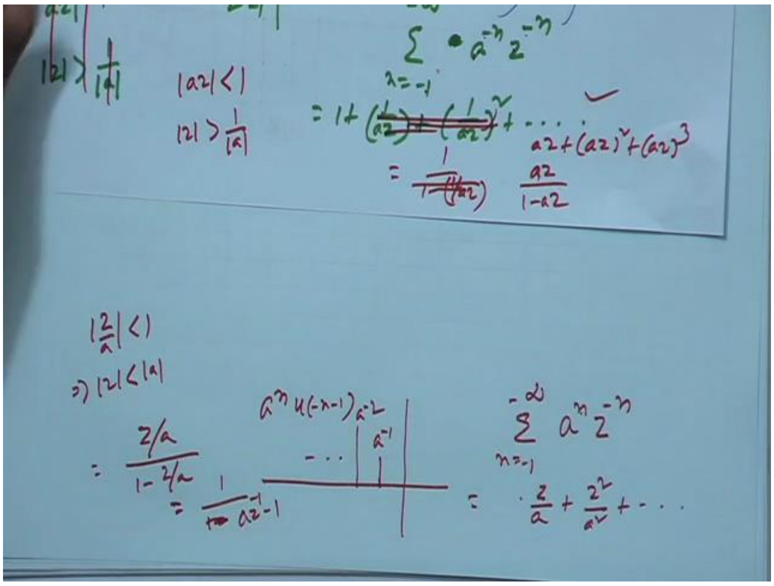

Instead of a to the power minus n u minus n minus 1. Suppose I consider a to the power n, in that case the sequence will be, form here only, but n is minus 1, so it will be a to the power minus 1. Instead of a then a to the power minus 2 and dot, dot, dot, dot. In this case z transform will be the same way a to the power n; that means, a to the power minus 1, so z by a, then z square by.

Here there is a mistake I was making sorry. Here n was negative right, so it cannot be 1 by a z, a to the power plus it was a to the power plus 1, because n is minus 1. So, a is equal to plus 1 z to the plus 1. So, it will be 1 plus a z, then, a z square a z cube dot, dot, dot, dot. So, this is nothing mod a z will be less than one, which means mod z should be greater than one by mod a, mod a z should be less than one then only it converges, and this will be 1 by 1 minus this one. I am sorry this, because n is taking negative value. So, a to the power minus minus 1. So, a to the power 1, it is minus 1 z to the power minus minus 1. So, that a z plus a z square a z cube dot, dot, dot, dot. So, you will take a z common. So, there will be a z here also alright, because a small mistake.

Here it is z by a plus z square by a square dot, dot, dot, dot, see if mod z by a less than 1, which means mod z less than mod a then this converge this, and if you take z by a common it will be z by a divided by 1 minus z by a it is common. So, it is 1 plus z by a plus z square by a square dot, dot, dot, and under this condition will be this. If you divide numerator denominator by this quantity, it will be 1 by 1 minus, sorry 1 by a by z that is a z inverse minus 1. If I further make a change, if I put a minus here minus there is (Refer Time: 30:07) sequence is minus a to the power n u minus a minus 1 so the minus sign there is a minus sign here, equal to there will be a minus sign here, and if we put the minus sign inside, it will be 1 by, same expression as this.

So, I mean two different sequences see that was our right sided sequence, that was the right sided sequence a to the power n u n has got these expression. And now this has got a left sided sequence, this 1 that was these, these, these both of them have this form same z transform, but what does it differ. In one case it was r o c was mod z greater than mod a; that is you draw a circle of radius mod a is greater than that, and in this case it is less than mod a; that is inside this. So, that is the difference you see you cannot for two different sequences you can still have the same z transfer expression, but what they will differ is (Refer Time: 31:07) from r o c's.

So, that is all for today we will continue further in the next class.

Thank you very much.