Discrete Time Signal Processing

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Lecture – 13
Introduction to Z–Transform

So, we now move to the next topic. Again a very basic topic and I believe, we must have studied it somewhere, then it is better to again talk a relook into this, this is z transform. We start with DTFT, that is given x n, it has a DTFT capital X, e to the power j Omega which was x n e to the power minus j Omega n, n from minus infinity to infinity. And these are the infinite sum so it is magnitude is a complex sum, of course, because e to the power minus j w n is complex.

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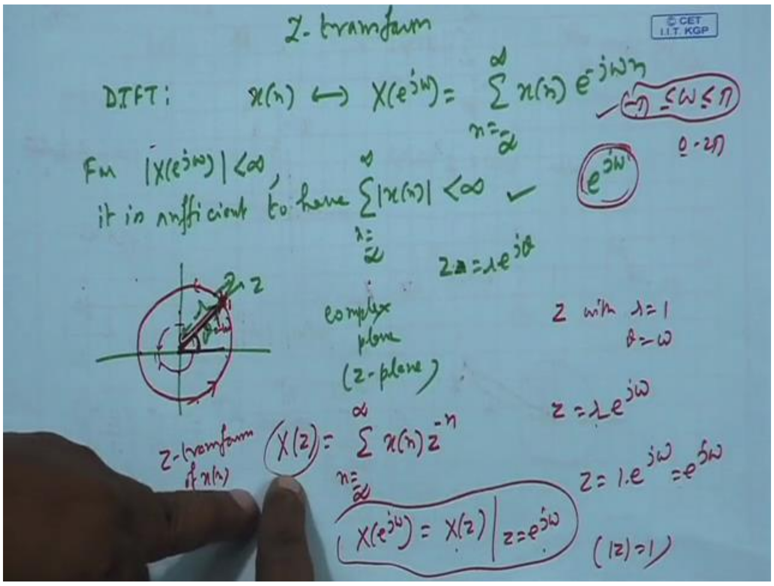

So, magnitude of the overall summation, we convert this may be finite, or may not convert. If it does not convert, then you say this is a DTFT 1, does not increase. So, for that we had earlier derived, 1 necessary and 1 sufficient condition, that is, for this to be finite, or this to hold good, it is sufficient to have, that is x n should be such, that born value, absolute value should be such. I mean if you take a sum, it should be sum of all, that is sum should be finite, if that happens, maybe DTFT will be always exist. This is strong condition sufficient condition, if it happens the DTFT will always will always exist. But it does not mean that, if it does not happen, DTFT will not exist. It may or may not, but this is a safety case, where if it happens, DTFT will exist, that is what we studied there.

Now, here you see what is the term e to the power j Omega, so, this sum, you know, I can generalize, what is say the complex plane, this is a complex plane. We call it z plane. z plane means every point has got, you know, if it is z you can draw line, to and if this is theta, and if this length is r, z basically it is r, it is equal to j theta. Here will be intersecting the polar form, not the rectangular form like a plus j v, but this kind of polar form. That is not r, sorry it is j. Now what is this? This is z, with r equal to 1, and theta equal to Omega. So, e to the power j Omega is a special case of z, where the magnitude part is 1. So, this may be 1, this could be 1, and theta is the given Omega. So that means, depending on the Omega you have, Omega can be 0, and it I mean, it can go up, and then become 90, and then you know 180, like that. But magnitude will be always 1. Omega can be anything from 0 to 2 pi. This DTFT is periodic, over a period 2 pi. So, you can take it from minus pi, 2 pi. So, Omega here, then DTFT, Omega can be from, or equal to Omega can be from 0 to 2 pi also, total period length should be 2 pi.

So, minus pi, 2 pi, normally we take this, equal to 0 to 2 pi, you can view that way also, we normally take this. So, Omega in this equation, we take value from minus pi, because from this minus pi, this much and then it will become less, less negative. It will go in this direction, and will become plus, plus. So, Omega can vary from minus 180, then less, less, less, minus 90, then less, less, less, 0, then again positive to 90, and then 180. So, it covers 360, 2 pi range of 2 pi, in this.

But magnitude remains always 1, so; that means, as Omega changes, in the z plane. I am moving on a circle, we call it unit circle. Unit circle because, radius is 1 and then depending on where you have, for the particular Omega, suppose this, this much this theta is now Omega, this much Omega. So, this is your z now, r is 1 this much is 1. So, you are here. For that z, which is equal to e to the power j Omega, you are running the summation.

So, for a general z, in this plane, that is if I come out of the unit circle, and go here, here anywhere for general z, a more general sum would be like this. Where z is, sum, or and sum, into the power j Omega. And this is general sum, I call it xz. Provided the sum even though it is infinite sum, it may exist, it may not exist, this mod, may be finite, may not be finite. When I write this, I am assuming this is finite, exist. Then the summation is more general summation, and I call it z transform of xn, xz is called j transform of xn.

Why this more general? Because in this summation, z can take any value over this z plane, not specially on the unit circle, number 1, number two, if I restrict z to be on the unit circle, then magnitude of z is 1. So, z is just 1, into e to the power some j Omega, that is, you will require j Omega, if I put that back here, I get DTFT. So, DTFT becomes a special case of z transform, where either you write as z equal e to the power j Omega or equivalent, you write, mod of j with this magnitude this is r, 1, ok.

So, this is a more general summation, and it allows you the flexibility, of choosing z from anywhere in the z plane. This has a tremendous advantage, you will see later because it will give rise to the concept of poles and 0s, which can be anywhere in the z plane, and not necessary on the unit circle. That is why you have to allow, this variable to go out of unit circle, because we have more general variable z, whose special case will be, this to the power j Omega, for mod z there is r, equal to 1.

So, I am allowing now, the complex variable to go out of unit circle, if it were in unit circle it would have been e to the power j Omega. Now I am making this general. So, it is not restricted to be on the unit circle, it can be anywhere on this complex plane, typically on the form j, this r e to the power j Omega, r is the magnitude, Omega is the angle. And then this summation becomes a more general, general summation. I call it z transform, x capital xz, it is special case is DTFT, when z is e to the power j Omega, magnitude 1, that is why I have 1 on the unit circle, angle Omega. Then the existing z transform, transform to be nothing, but DTFT at Omega, because we have replaced z by this, you get nothing, but the DTFT formula. I am assuming this exists.

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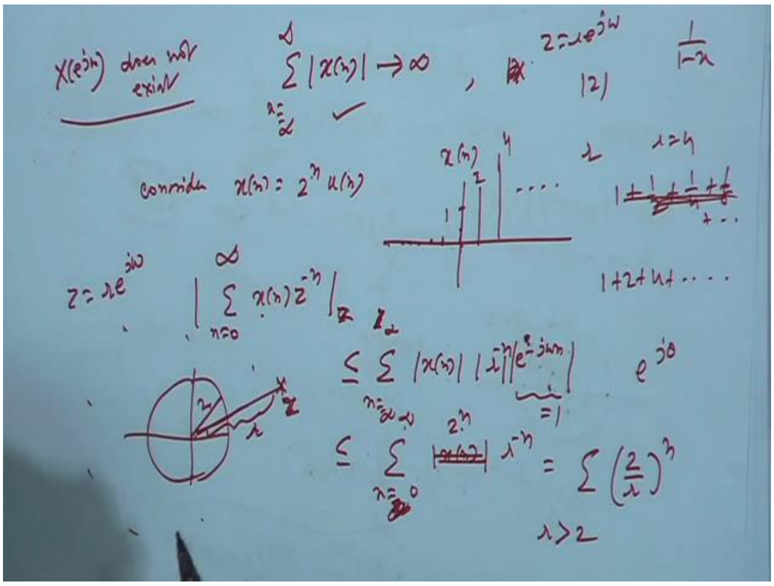

This summation, this has some also additional advantage, that suppose, x n is such, suppose x n is such that, this is, this should have been less than infinity, which is a sufficient condition for DTFT to exist. But suppose say this is not, not less than infinity, is going to infinity, again, if it goes to infinity, DTFT may exist, may not exist, we are considering the case where it does not exist. If it is less than infinity, it will always exist. When it is going to infinity, it may or may not. I am considering the case where it does not exist, but if instead DTFT, we go for z transform, then for (Refer Time: 08:54) of r, magnitude of r is what? Mod z, because z was r e to the power j Omega; so for (Refer Time: 09:01) of r, magnitude of z, we can still have the z transform converging, existing.

But, as an example, consider, x n to be say, 2 to the power n, un, it is like, this x n is, 2 to the power 0, that is 1, 2 to the power 1, 2, 2 to the power 2, 4, dot, dot, dot, dot, dot, dot, All positive, and this side is 0, because we are multiplying it by u n, u n is 1 on this side, and 0 on this side. So, 2 the power 0 into 1, 2 to the power 1 into 1, 2 to the power 2 into and you get those values, and 0 on this side. This sequence, for this sequence, this holds, because you see, all values are positive. So, mod of x 0, 1, then mod of x 2, 2 then mod of x sorry, mod of x 1 is 2, mod of x 2 is 4, and dot, dot, dot, dot. This summation you can understand, that it goes to infinity, which is increasing 1, 2, 4 then, 8 then 16, is going on. It is (Refer Time: 10:10) series, but as it goes to infinity, it will go to infinity ok?

So, this is not absolutely summable. So, for this DTFT, actually does not exist. But if I take a z transform, z transform you see, x n, z to the power minus n, here n can start from 0, because values to the left hand side for x n are 0. So, 0 times z to the power minus n it makes no sense you know, because 0 times this is 0, so I do not consider them, I start from n equal to 0, and go on to infinity Because for a less than 0, the sequence x n has values 0. So, product is 0, so no point in considering them.

So, this means 1 into, you see this summation, at z you write as, r, e to the power j Omega. This summation is mod value, as it will be less than equal to, as I told you last time, summation of mod, by the triangle inequality is less than equal to, sorry mod of a summation mod of a summation by using the triangle inequality again, again, again mod of a summation, is less than equal to summation of the mods, and mod will come here, mod of a product, mod will come on this x and it will get to the power minus n, but that is a product of 2 terms, mod of a product is product of mods. So, this, into mod of z to the power minus n, z is r e to the power j Omega, so r to the power minus n, e to the power minus j Omega n.

Again mod of a product is product of mod, or mod of e to the power minus j Omega n, this is equal to 1, and e to the power j theta, has magnitude 1, so 1. So, you are left, with r, r is a positive number, r is the magnitude. So, r to the power minus n, still a positive number, mod has, there is no point in taking mod because mod of a positive number is a number itself, r to the power minus n.

Now, what is mod x n? x n itself is 2 to the power n, u n I will not consider because un, it is present, I will start from 0, 0 to infinity. And then 2 to the power n, all positive numbers, that is 2 to the power 0 1, 2 to the power 2, 4, 2 to the power 3 positive numbers. So, no point in taking mod, you can just take 2 to the power n only, and I am considering from 0 to infinity, I am not considering negative terms. So, u n presence of u n is taken care of.

So, this summation is actually 2 by r, whole to the power n, you see if I take r, greater than 2, that is if I, in the complex plane, if you draw a circle of radius 2, and if I am here, here, here, anywhere where mod, suppose this is z, this much is mod z, is greater than 2, if I put z here, 2 by r, because r will now be this much, r is greater than 2. So, 2 by l is less than 1. So, less than 1, at this, this power half of it becomes lesser, and lesser, less than 1 suppose it is half, so half to the power 0, 1, half to the power 1, half, half to the power 2, 1 by 4, half to the power 3, 1 by 8 dot, dot, dot. So, it will be decaying thing, and the summation will exist, you understand, for instance there is what I was saying, suppose r is 4, so you will have half, half to the power 0, that is 1, then half to the power 1, half, then half to the power 2, 1 by 4, 1 by 8 dot, dot, dot, dot this summation will exist.

We all know 1 plus x, plus x square plus x cube, dot, dot, dot, dot we take these and is given by 1 by 1, minus x, if mod x is less, than 1 mod x is half, then less than 1, this also this exists; that means, in this case z transform will exist, the mod value will exist, because this is finite.

So, because I am allowing the complex variable to go anywhere, I can choose z anywhere in this z plane. In this case I take z to be such that it is mod value that it is magnitude is greater than 2, and therefore, the summation exists. So, this is an advantage, that when your DTFT does not exist, by clever choice of the magnitude of z, there is r, you can still have the z transform exist, this is another motivation, but I tell you the main motivation is that, you are now free, you are from the unit circle, you can be anywhere in the z plane. So, you get a more general expression, and this will give rise to something called pole and 0, which we will consider later, and that will be related to stability (Refer Time :14:52) there is the main region of this generalization.

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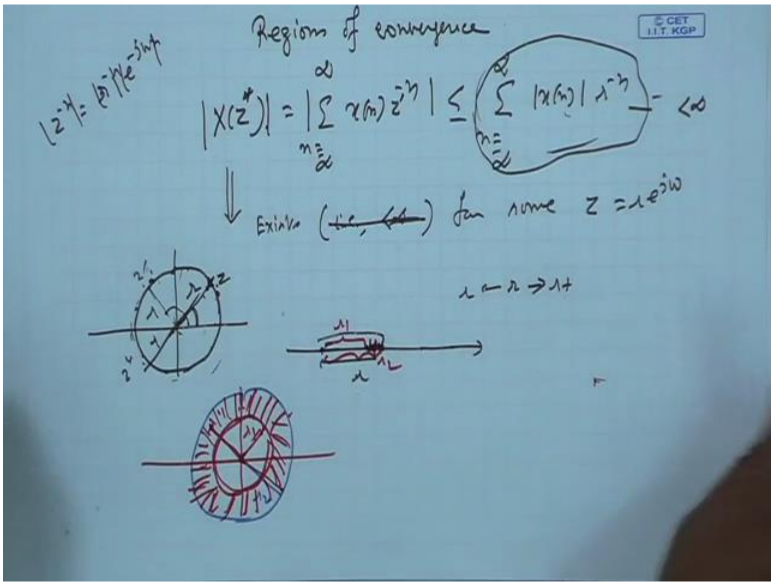

Now, we will talk about the convergence. Convergence is, we call regions of convergence. Before that consider x z, mod x z is repetition, mod of this mod of a summation, is less than equal to summation of mod, is just repetition of what I just told, and mod of a product is product of mods, into mod of z to the power minus n, from that I will get only r to the power minus n, like before, because z, z is, z to the power minus n, is r to the power minus n, e to the power minus j Omega n, mod of this is, mod of this, into mod of this. So, mod of this is 1, at r, to the power minus n, is always positive, r is positive number. So, any power of r, positive, negative, whatever, that will be positive. So, mod of a positive number, is the number itself, that is why I put down put a mod here, this is, in general true.

Now, suppose this x z, mod x z exists, suppose this exists. That is this, that is, that is, this finite, for some z, some z with r into e to the power j Omega. Then; that means, if you draw a circle in the z plane, of radius r, Omega has no effect on this summation, if it is exists, that is if it is finite, suppose this is finite, this is finite, suppose this is finite, and therefore, this exists, this it is finite, for some z, whose magnitude is r, r, e to the power j Omega. Now you see this summation has got nothing to do with Omega, it is free of Omega.

So, therefore, if I draw a circle of radius r, any point on the circle, will have the same magnitude r. Omega will vary, but since this summation does not depend on Omega, therefore, whether I am here, whether my z is here, or here, whatever z I put, for all of them the same r will come here, and if it is less than infinity, that is it will be finite for any r, any z, it will be finite for any other z, because there r ,value is same so; that means, this summation if it exists for any z, then you draw a circle through that z taking the magnitude of that z, as radius there is r s radius, then on the whole circle full circle, any point if you move to, from z, to z prime, to z double prime, anywhere, at every point this summation will be finite, and therefore, z transform will exist.

Because at every point mod z, or mod z prime, or mod z double prime, they will be r, r, and if it is, we have already seen that, since z transform I am saying, it is existing, it is finite for 1 z, for which this is r so; that means, this summation is finite. So, if move to z prime, z double prime, anywhere on the circle radius is still r, magnitude is still r, and that is if I have x z prime, for that also, if I have z prime, suppose x z prime, if I have z prime to the power minus n, it will eventually be less than equal to the same thing, because with difference between z, and z prime, is interference of the Omega, this much Omega here, this much Omega here r, is same, and this summation is independent of Omega.

So, whether we have z here, z prime here, z double prime here, whatever, there is z prime here, z double prime here, z here, this is this will always be less than equal to this, and this is free of Omega, this is free of Omega. So, if it is finite, because finite for any z, z prime, z double prime, because it is r everywhere, same r. Therefore, it means that if z transform exists, for a particular z, then if you draw a circle, through that z of the same of the radius, of what radius? mod of that z, which is r .Then for any z, any on this circle, this, this summation still will be, this mod of x z, x z for any z on this circle, will be finite therefore, it will exist.

But there is a little more than, if this is finite, for some r, r is the radius of this circle, then it cannot be, it cannot go to infinity immediately, if I increase, for if I go from r to, higher side, r plus, or r to r minus, that is because of the continuity, if I it, is finite summation is finite, for some r, for some r, this much r, the real axis, this much r then it, it is not possible, that immediately to the right of r, immediately to the left of r, this summation will suit up to infinity. It is not possible, there will be a band, because of the continuity; however, small that band.

Within that, this summation will still be finite; it cannot be arbitrary just for a point. If that be; that means, I mean if I draw a circle, of this much radius, or of this much radius, or of this much radius, for all of them I will get things like this, that is suppose this, this, this part is r 1, this r plus r 2. If I draw 1 circle of radius r 1, draw another circle of radius r 2. So, since is then, this is this, this will be finite, for r taking not only 1 value here, from r 1 to r 2, this entire band, r can take, r 1 greater than r 1, still greater than r 1, up to r 2, for this entire band, this summation will be finite, it can be finite, just for a specific r, and then all r to the right, all values to the right all values to the immediate left, immediate right, shooting up of the infinity, does not happen because of the continuity (Refer Time: 21:38).

Therefore at least there is; however, small; however, narrow, there will be band around r, that r. So, this band may be from r 1 to r 2. So, from r 1 to r 2, any value you can give to r, this summation will be finite, and therefore, if I draw 1 circle in radius r 1, and there is r 2, then the entire region, will be my part of region of convergence. Because if I pick z from here, this much radius, this is between r 1 and r 2. So, r, there is a magnitude of z, it will be between r 1 and r 2, this will falling in the band, and if I falling in the band, for that r this summation is finite. So, DTFT will exist which means, we will have this annular ring, this ring, which is shaded here, and I will call it region of convergence. Sometimes r 1 can swing to 0. So, it will be whole inside of a circle, sometimes r 2 can go to infinity; that means, the outside circle goes up to infinity. So, it is outside, a smaller circle of radius r 1, sometimes there can be multiple rings.

But this is a typical shape of region of convergence, that within this region, if we pick up z from anywhere, this z transform will exist. This is called region of convergence. With that in mind, then let us let us consider various sequences, finite lane, right sided, left sided etcetera, etcetera.

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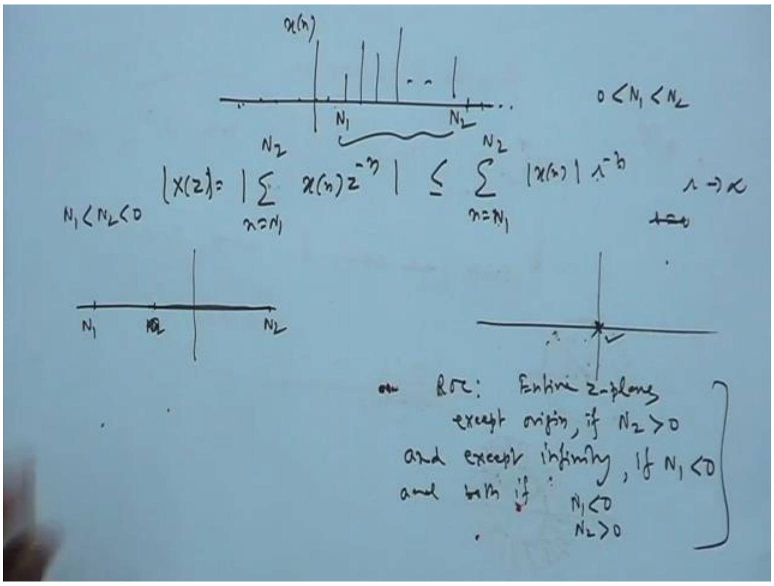

So, consider one sequence, say in general, say from finite duration, and I am starting from right hand, side going up to right hand, z 1 to n 2 to start with, that is n 1 less than n 2, greater than 0. If the sequence is over this range, outside this range, this is all 0 values, this is my x n. So, in this case x transform, no need to consider the summation from my minus infinity to infinity, we have to start from n 1 to n 2, x n, z to the power minus n. So, mod of z x, will be this, as before summation of mod,, less than equal to, mod of summation, mod of mod of summation less than equal to, summation of mods, mod of x mod of a product mod of x n, and from here up to here minus n, will come there, we have done.

Now, you see this is not an infinite sum; this is a finite sum, alright? This is a finite sum, and this finite sum will always changes, only if, you take r equal to 0, then there is a problem, because 0 to the power you see, n, small n is taking value from capital n 1, which is positive, then further positive, further positive, further positive, up to n 2. So, there are negative powers of r. So, if you take r equal to 0, then this summation will not qualify, but otherwise it will converge, always.

For any r, that is, in this case, the entire z plane, that is you can pick any z of any r, and move a circle around that of that r, all points on that circle, with the same r, and this summation we say, finite summation, it will exist, for any r you choose. You can move the circle, you can expand r, contract r, only thing is, origin cannot be included, if you have got negative powers of r. On the other hand, if you have got the summation localized here, n 1 n 2, and n 1 less than n 2, less than 0; in this case n 1 negative n 2 negative. So, there will be positive powers of r, but even then, this is a finite sum, this will be finite, only thing, thing you cannot take r, going to infinity, if r is taken as infinity, this summation will be infinity.

So, in this case, origin is allowed, is a z plane, origin is allowed, but infinity is not allowed. And if you have a situation like this, n1 of this side, n 2 on this side, then in the summation you will have both negative powers of n, positive powers of n. So, both origin and infinity will not be allowed, alright so; that means, here r o c means, entire z plane, except origin, if n 2 is positive, because if n 2 is, n 1 can be negative, but if n 2 is positive, there are some negative powers, because n small n will be positive. So, minus n will be negative.

So, there will be some negative powers of r, which means r equal to 0 will be ruled out so; that means, except origin, if n 2 is greater than 0, and in that case, that is if, n 1 n 2 both are this side, r is r is equal to infinity, no problem, because negative for infinity will be 0, that does not cause a problem. On the other hand if I have got n 1, n 2 both on this side, you get negative, that the small integer negative value means, minus of integer positive value, r equal to some positive number here, or r equal to 0, or it is no problem, but r equal to infinity is a problem. So, in 1 case, r o c is entire z plane, except origin if n

2 greater than 0, and except infinity, if n 1 less than 0, and both, that is except origin, except infinity, both, if n 1 less than 0, n 2 greater than 0.

This is r o c of the finite, otherwise this is a finite sum, you can use this for any finite art, or any non 0, or any non infinity or, therefore, z transform varies. That is for the finite duration sequence.

Thank you very much. In the next class, we will go for the other sequences.