Discrete Time Signal Processing

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Lecture - 12
Properties of Rational Systems

First we will show that this system which has a zero initial condition at takes x n produces y s n this is linear and shift invariant.

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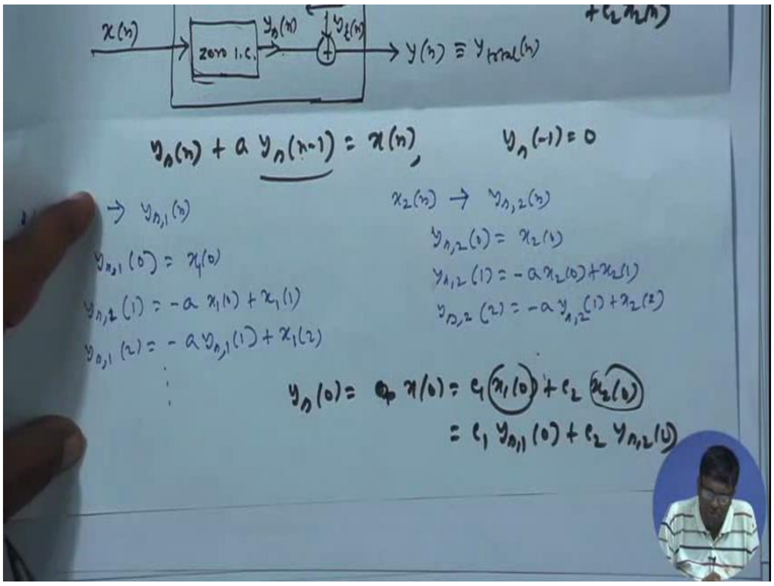

And then we will show this is overall system is neither linear nor shift invariant because of the presence of this y t n this is my interesting. So, consider this system to explain clearly, I will just take a very simple example. Now, you can generalize it to the general difference equation after that, that is y s n plus one difference equation I am considering a y s n minus 1 is suppose x n and initial condition is 0, you need only initial condition here y s minus 1 equal to 0. Suppose I give an input x 1 n and it gives me y s 1 n. So, it will be satisfying this way y s 1 0 will be this initial condition as 0 if you put 0 here y s 1 minus 1, whether it is y s 1 is to what about initial condition is not given that is of 0, so that means, this part is always 0. So, you get x 0, x 1 0, then y x comma y x comma 1 index 1 then will be minus a y s comma 1 y 0 y s comma 1 0 which is minus a x 1 0 plus x 1 1, then y s 1 2 will be minus a y s comma 1 1 plus x 1 2, dot, dot, dot, dot.

Similarly, if we give x 2 n, y s 2 n, you get the same manner another input. So, y s comma 2 0 will be x 2 0 y s comma 2 1 will be minus a x 2 0 plus x 2 1, dot, dot, dot, dot, alright. Now question is instead of giving either x 1 or x 2, suppose I give you linear combination of this c 1 x 1 plus c 2 x 2 we will output c 1 y s 1 n and plus c 2 y s 2 n answer is yes. That time your if you call y s 0 that will be new input, you are giving input now c 1 x 1 n plus c 2 x 2 n plus c 2 x 2 m and we are calling it x n. So, y s that time corresponding output y s 0 will be new input x 0 and x 0 is c 1, but this much was your y s 1 0 this much is your y s 2 0 that is c 1. So, at least at index 0 it is linear, that is total output is c 1 times output for system 1 case 1 and c 2 times output for case 2 that is case 1 means input x 1 n, case 2 means input x 2 y 2 n sorry input x 2 n right, very simple.

That is I am repeating again what I am doing, suppose I keep x 1 n as input transferring output I denote it as y s 1 n and I work out from this equation the values y s 1 0 is x 1 0 because this is guy is 0 into we will have value minus 1 index is (Refer Time: 05:27) if any 0 this total is minus 1. So, past samples are the past values are 0, because 0 initial conditions are assumed because simply y s 1 0 is x 1 0, y s 2 are will be minus a, y s 1 0 plus x 1 0 and that I have replaced back and dot, dot, dot. Similarly I gave another input x 2 n the same way I find out the out then my question is - if neither I gave neither x 1 nor x 2, but anyway a linear combination then call it as x n, x n is these, how do the new output. So, new output again I have to solve with 0 initial conditions. So, I know solution if I call it y s n, y s 0 will be nothing but x 0 because past I mean y s minus 1 is 0 because of 0 initial conditions. So, y s 0 will be a x 0 which is c 1 x 1 c 2 x 2 0, but x 1 0 was y s 1 0 x 2 0 is y x 2 0, if their linear combination of the outputs for the 2 cases same the way.

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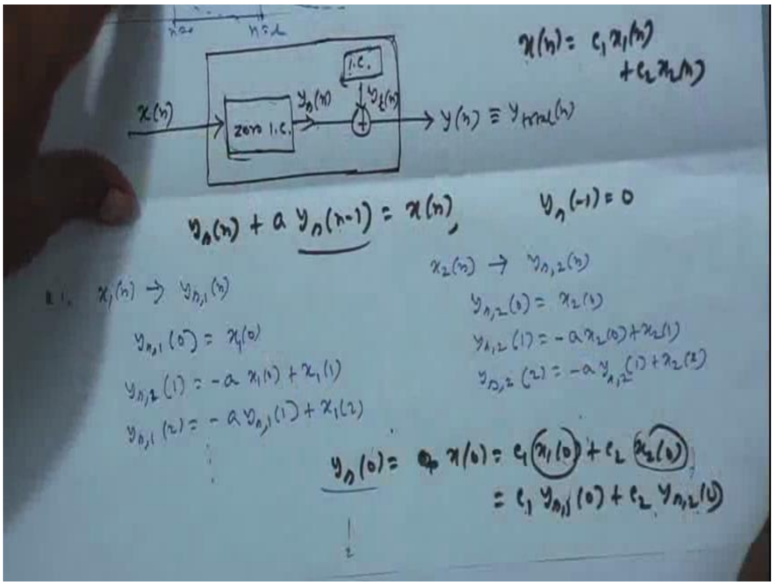

So, linearity satisfies that n equal to 0 and then you can see linearity will be satisfied at any other index, we can provide by induction. That suppose, y s n is equal to c 1 y s 1 n y s 2 n for n up to say m minus 1. Then I will show it is true, we will show that it is true at n equal to m also.

Look at this is how I put by index (Refer Time: 07:31) you know, remember you proved binomial theorem in a cool - a plus b whole to the power n equal to some formula how did you prove. We assumed that these to be true, for again up to some index p did you prove that is true for n equal to p plus 1 and then you validated these assumption by taking any special case. We have already taken the special case here that n equal to zero linearity satisfied by direct investigation.

Now, I am assuming the linearity to be valid for n up to some index m minus 1 then, we should show it is true at n equal to m also. So, which means it is true for any, if it is true up to n equal to say 0 same is 1, it is true I mean n up n up to 0 because which is already the case which we have seen to be valid. So, that is valid assumption. But then this will be this is true at n equal to m that is it is true at n equal to 1 because m is 1 then if it is true for n equal to 1 it is true for n equal to 2 and dot, dot, dot, dot. So, what is y s m from this equation? You can take it to the left hand side minus a y s m minus 1 plus x m, but I have assumed that up to index m minus 1 system is linear. So, if it is y s m minus 1 this will be c 1 times y s 1 m minus 1 by the assumption plus c 2 times y s 2 m minus 2 and x m I know - it is always c 1 x 1 m because I am getting this linear combination input and did you combine you take this together 1 part here, multiplied by c 1 take the c 1 out a arrange from here also c 1 out. So, x 1 m and another part c 2 from here c 2 out, minus a y s 2 and from here c 2 out x 2 m alright.

And what was this part? From this equation, if I put m here, this is within bracket sorry from this equation, what is y s 1 m? Y s 1 m will be a y s 1 m minus a y s 1 m minus 1 plus x 1 m because that thing input is x 1. So, this is nothing but your – you input y s 1 n here that is equal to minus a this minus a y s 1, y s 1 n minus 1 m minus 1 and then plus 1 m because I am giving input extra. So, this is y s 1 m because the same way this is also y s 2. So that means, at the index m also I should put it in bracket sorry at index m also output is linear combination of the 2 outputs. So, it is linear at index n equal to m also.

So, if it is linear up to index n equal to m minus 1, it is linear at index a n equal to m also and this is goes on and how to show that it is true for n up to m minus 1 at least 1 plus directly you should show that we have seen here, by direct substitution that is we have assumed 0 initial conditions. What is y s n y s 0? That will be, give me the x of 0 what was the x of 0 x of n was the combine input, so c 1 x 1 0 c 2 x 2 0, but what was x 1 0 - x 1 0 was y s 1 0 because y s 1 minus 1 is 0 0 initial condition and x 1 0. Similarly, that is y s 1 0 is x 1 0 this I call at y s 1 0 by the same way x 2 0 is y x 2 0. So, c 1 y s 1 0 plus c 2 y s 2 0 which means it was linear and index 0.

We have from that proof it is linear at index 1 because if it is m minus 1; that means, m is 1. So, it is true at m equal to 1 at 1 and then if it is m minus 1; that means, m is 2. So, this true at 2 and dot, dot, dot, dot. So, at y s y s 0 y s 1 y s 2 all satisfy linearity all right, it is a linear shifting. It is also shift invariant because on the time x axis at n equal to 0 I am giving the input and before that system had no stored value or all output all initial conditions, all past samples past output samples, all past input samples they are all 0 past input was 0, past output was 0. System had nothing, no value all past output samples were 0, all past input samples were 0. Then I gave input I got a solution y n. Therefore, instead of giving an n equal to 0 if I hold that 0 initial conditions situation for further and give the input at n equal to r, the new output will be same as the near the output because still n equal to r the same 0 initial conditions for output and same 0 value for input where we can leave.

So, whatever happens started happening from n equal to 0 onwards previously same thing will start happening from n equal to r onwards because I get the same input or a time scale might new input is the delayed version of the previous input. So, my new output will be a delayed version of the previous output the same delay. With which it is shift invariant, if I delay the input by some amount or current output also will be delayed by the same amount or like if it was like this, now the same thing will start happening here like this because before these n equal to 0 system was at rest all initial conditions were 0, all past output were 0 and all input were 0. Instead of giving the input at n equal to 0 if I continue that 0 initial condition and 0 input condition t n equal to r and meet the same input system will work in the same way because system properties time invariant it is not changing with time. So, I will get the same output or the same time screen if I plot. Input is delayed by r, so output also is delayed by r. So, it is shift invariant.

Probably this define consider the total output y n then we will see some difference, because y t n is obtained from the initial conditions. Suppose I do this exercise, instead of giving the input at n equal to 0 I here, in this system I extend the 0 initial condition, condition that is 0 initial condition up to n equal to r, that is t n equal to r this system it is past output samples will be 0 past input will be 0 and then I give the input to input I delay it by r.

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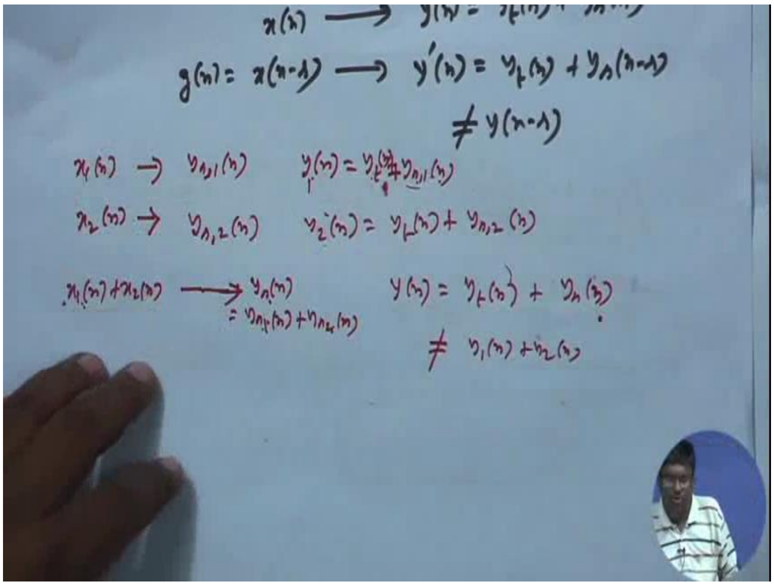

So, I give a new input g n which is a x n minus r this we have seen this shift invariant this will be n minus r, but y t n has nothing to do with input right. So, y t n will not be delayed. So, new output will be previously I had total output was y t n plus y s n when I get x n as input that was my output now I am giving x n minus r as input which I call a new input g n. Corresponding output o if I call it by y prime n, that will be what? This part will be output of this system this sub system, this subsystem is shift invariant. So, therefore, we have seen it is shift invariant here in this example because initial conditions were 0 I am extended the same, if I delay the input start giving the input instead of at n equal to 0 at n equal to r. Till that point of time system no output will be generated, so output initial condition will be still at 0, input also 0.

So, whatever I observed by giving the input at n equal to 0 from n equal to 0 onwards same thing I will observe from n equal to r (Refer Time: 17:14) if I give the same input. So, that is why this was shift invariant which means this x n if I delay by r new y s n will be delayed by the same amount. But y t m because of this will not undergo change because I am giving input, it will delay at the input this y t m will be not delayed by that because y t m is not controlled by the input it is controlled by another subsystem and governed by it is the initial conditions. So, this will remain y t n as it is, but this is not equal to delayed version of this, this is not equal to which is y prime is it is not, but delayed version of previous output y n with the same amount of delay r, but you see if y t n was not present then this is shift variant, alright. That means, if initial condition was 0 y t n, obviously has to be 0 because in the past initial conditions were 0 so, y t 0 will be 0 y t 1 will be 0 y t 2 will be 0. So, no y t will be formed. So, if this is absent y n is y s n y prime n will be y s n minus r this which means y prime will be y n minus r this will be equal to y n minus r, that will be shift variant.

But as long as there are initial conditions which are non zero, the overall system will not be shift invariant. Similarly linearity can be viewed that suppose I gave 1 x 1 n here, I found out 1 y s 1 n and output is y n as y t n you can call it output y 1 n as y t n plus y s 1 n. Next time I give x 2 n corresponding output, so overall output will be now same y t n, y 1 involve at y t n y t n is common that has nothing to do with the input side. Input is hitting this system and y is in this form, y t m is built in solution, built in from the given initial conditions built in solution it has got nothing to the input that is y t n plus y s 1 n same y t n plus y s 2 n, but if I now give a linear combination as a (Refer Time: 20:04) very simply linear combination just x 1 n plus x 2 n. If the overall system is linear my new output will be y 1 plus y 2, but it will not be because what is the new output if I give x 1 plus x 2 here, this system is linear. So, it will produce y s n as this summation of y s n 1 plus y s 2, right because this is linear.

So, now y s n will be response due to x 1 n that is y s 1 n, and response due to x 2 n that is y s 2 n. They were added x 1 and x 2 added at the input side. So, output also added. So, up to these no problem linear y s n will be summation of previous 2 outputs - one due to x 1, one due to x 2. But in this case the total y n if I call it total output y n will be still summation of y t n only and y s n.

Now, this y n you can easily see this not equal to summation of y 1 n plus y 2 n this is not equal to y 1 n and y 2 n because if you add the 2 you have put y s 1 plus y s 2 they are present - y s 1 plus y s 2 y s 1 plus y s 2 which is y s n y s n present. But y t n, y t n they are added, it is twice y t n, but here you have got only 1 y t n. So, it is twice y t n if you add there here you get only one y t n that is the begin this equality is not satisfied. So, it is not linear. But once again if y t n verse 0 if it was not present, that is initial conditions were 0 so built in solution was 0. So, if this is 0, this is 0 then there is no problem this part is 0 means whether you have got y t n plus y t n that is twice y t n or only y t n - y t n is 0. So, 2 into 0 or 0, they mean same. So, y n will be just y s n.

And y s n means y s 1 plus y s 2 that is y n will be y 1 plus y 2, linearity will be satisfied. So, this detail things shows that such a constant coefficient difference model it has got 2 solutions one is called steady state solution, one is called transient solution. Transient solution because it comes from the initial conditions built in solutions it basically they try to decay down if it is a stable system, and y s n is the solution driven purely by the input, but assuming system 0 initial condition system in overall thing is neither linear nor shifting variant, but this sub system 0 initial condition is linear in shifting variant.

But overall system will be linear and shift invariant if initial conditions were 0 then the built in solution is 0. We will assume for our purpose that either initial condition are given to be 0 and therefore, there is no built in solution no transient solution, we are only giving this sub system done. So, that is linear as shift invariant else we will assume that let their initial conditions are the solution built in solution y t n for our purpose we will be bother about only this sub system, we will consider the property of this. Once the y s n is obtained to that we will add y t m separately, where as far as analysis is concerned as far as understanding is concerned we will consider only this subsystem. So, in either case we will take this constant coefficient difference equation to have 0 initial condition, like this and it will give me input find the output the system in this case will be guaranteed to be linear and shift invariant.

In one last thing we have taken DTFT studied; one thing we did not see, but it is a bit trivial thing suppose x n is the sequence it has got a discrete type furrier transform x e to the power j omega.

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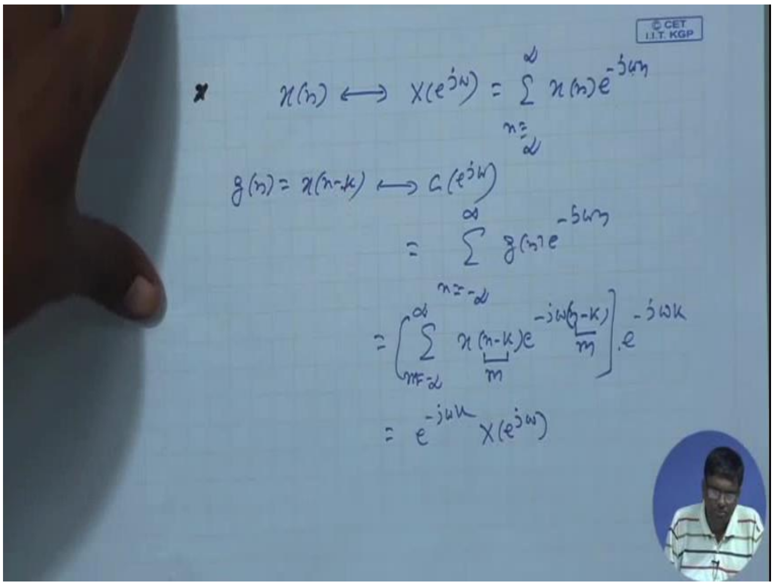

That is summation, DTFT at any digital frequency omega from minus pi to pi put that omega here carry out this sum, omega is continuous from minus pi to pi omega is not discrete, this is DTFT. Question is if I get a new sequence g n which is a shift inversion; n minus say k what will be DTFT of this g n very simple you run the same summation and then g n is nothing but x n minus k. Now here, suppose I bring n minus k here; that means e to the power minus j omega minus k. So, it is your plus j omega k has come in, so to cancel it, to bring e to the power minus j omega k and this can this is depending upon k summation is about n.

So, this can go outside the summation as common and what is this part - n minus k if you call m. If n is minus infinity m also goes to minus infinity, if n is plus infinity n is n goes to plus infinity m goes to plus infinity. So, summation x m e to the power minus j omega m which is nothing but original DTFT, so original DTFT multiplied by e to the power minus j omega k let me say x n has DTFT capital X e to the power j omega if I shift it by k x n minus k is d d f t will be original DTFT multiplied by e to the power minus j omega k, k is the amount of shift.

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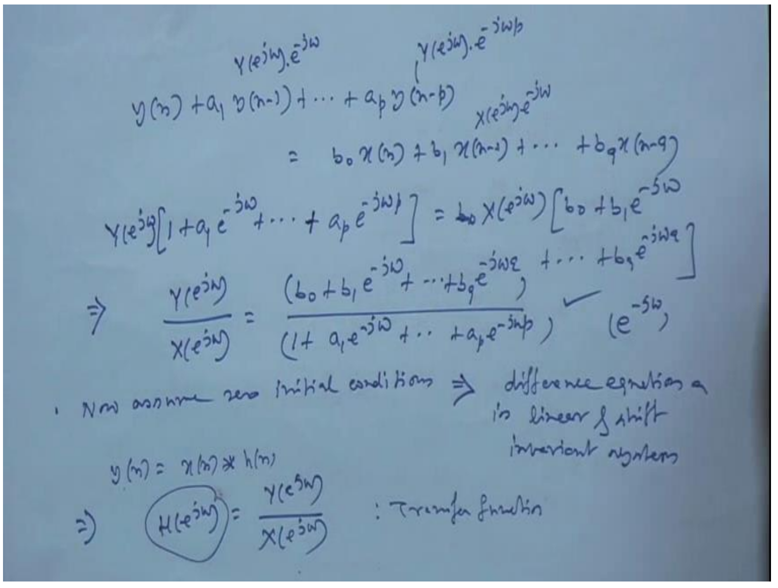

Now, you consider the difference equation, suppose I take DTFT on the left hand side and right hand side. So, again there will be they will be equal this DTFT is y e to the power j omega take common, so 1. This DTFT is y e to the power j omega into e to the power minus j omega k, k is the 1 here - j omega, this will be original times e to the power minus j omega instead of k it is p, so p; as same on this side. So, from all of them I take y e to the power j omega common. So, 1 plus a 1 and here capital is will be equal to omega, then this is capital X e to the power j omega into e to the power minus j omega because k is 1, so on and so forth.

So, you take b is common and b 0 then b 1 e to the power minus j omega plus, dot, dot, dot, dot b q e to the power minus j omega q. And if I now take a ratio output DTFT it will be. So, this is general difference equation, I have not bit have been assumed this is linear shift invariant that is whether the initial conditions are 0 or not, irrespective of all that this is valid, this is I am telling you I took a general difference equation in behalf non zero initial conditions. So, over all solution of this may not satisfy linearity shift invariants even that this is true.

Now, assume 0 initial conditions, during difference equation, difference equation is a linear and shift invariant system, but for linear and shift invariant system we have also seen output y n is convolution between input x n and the unit sample response that is true this difference equation with 0 initial condition that is making a linear and shift invariant. If you give x n is equal to delta n whatever output you get that is h n, and if you converge between these two then you can same as y n you get. And we have seen that is if you take DTFT here is a product from the 2 DTFT's which means h e to the power j omega into x e to the power j omega is y e to the power j omega or y, if you take this ratio like this then this is the DTFT of the unit sample response which we say transfer function. Transfer function in frequency domain for the time being we equally transfer function only.

So, if the system is linear shift invariant then there is a definition of transfer function because output is a linear convolution between input and unit sample response and in that case DTFT of this unit sample response even output DTFT by input d t ft which you call transfer function. But if it is linear and shift invariant and if the system is given by difference equation and I am assuming system to be linear and shift invariant that is difference equation as 0 initial conditions.

In that case this will be equated to this because y e to the power j omega by x e to the power j omega is for a difference equation is always this, whether you have got 0 initial condition or not. But if we have 0 initial condition then it is linear and shift invariant, in that case this ratio is the transfer function because if it is linear and shift invariant then output y n will also be written like this x n (Refer Time: 31:32) with h n and DTFT of a h n is called the transfer function which is this ratio. So in that case this ratio itself will be the transfer function. So, in this case what is this transfer function you see? It is a ratio of 2 quantity a numerator quantity and denominator quantity it is a polynomial (Refer Time: 31:51) e to the power say minus j omega. So, e to the power minus j omega 2 to the power 1, e to the power minus j omega 2 to the power 2, dot, dot, dot, dot, and here also, so numerator polynomial divided by denominator polynomial, polynomial in terms of e to the power minus j omega. Since it is a ratio of 2 polynomials we call it rational systems.

That is all for today, by the way if I give a problem, if I give a small difference equation and with some initial condition you should be able to find the solution. How you will consider the total solution as a summation of two solutions? For one, you will assume the a consider the transient solution where right hand side will be 0 and build up the solution from the initial condition; at the another case take the initial condition to be 0 change that solution, put the input on the right hand side, calculate the output from the input only and from the 0 initial condition. Then if you add the 2 solutions you get the total solution, that is all for today.

Thank you.