Discrete Time Signal Processing

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Lecture – 11
Rational systems

Today we will begin a new topic.

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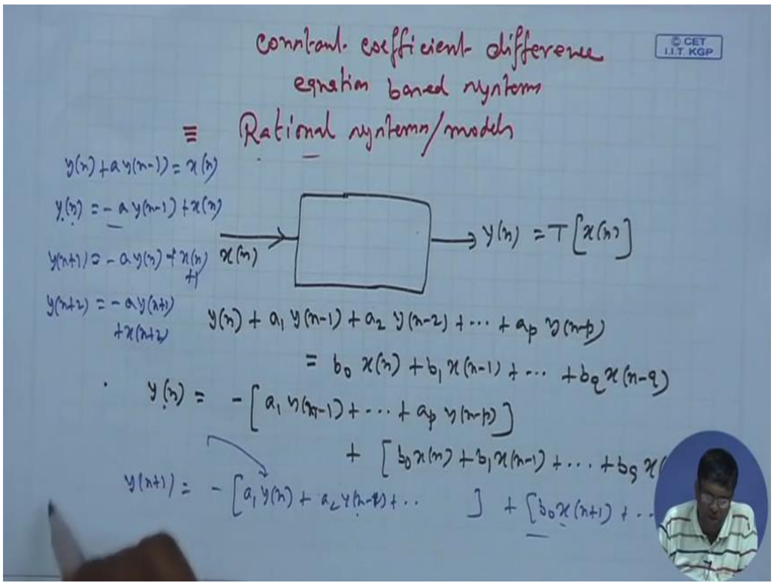

It is one special kind of systems called constant coefficient difference equation based systems, in short equivalently we also call it rational systems or even models; I prefer these descriptions. Suppose there is a system, I am not sure whether is linear at shift variant that has to establish but there is system. That takes an input x n produces an output y n, y n is system operator T working on x n.

Now, it is giving this system works in a particular way; that is output and it is past samples are related to input that is first samples in these manner y n there a constant coefficient a 1 times y n minus 1 then may be y a 2 y n minus 2, dot, dot, dot a p y n minus p is equal to b 0 x n plus b 1 again this is a constant coefficient b 0 this is a constant coefficient b 1 then b q. Giving this you see you can work out in this way keep y n on the left hand side then you have bought these thing the distort from y n minus 1 to y n minus p you can keep them together right hand side, there is a 1 y n minus 1 plus, dot, dot, dot a p y n minus p.

And then again the x n terms b 0 x n. Here, if suppose the past output y n minus 1 to y n minus p they are known and input of course current input and past input they are known then you can carry out the right hand side calculation and you get y n. Then when you go for next index y n plus 1 then this becomes minus a 1 y n then a 2 y n minus 2 sorry, n minus 1 and, dot, dot, dot. And again b 0 x n plus 1 because n plus 1 is the current index, dot, dot, dot. Essentially, the y n which you compute now that put back here, at this past output terms are already available there is a current input will x n plus 1 and past input terms are available so this is not a problem and you get y n plus 1.

Then again at the next index n plus 2 put it back here so at this way it continues. There is in an example, suppose here equation is like this y n plus a y n minus 1 is equal to that suppose x n. So, you keep y n on this side left hand side is right hand side is a y n minus 1 minus x n then. So, if first output is known a y n minus 1 is known x n is known right hand side is known, so y n is known. Next when you move to y n plus 1 in this equation it will be plus x n, y n already calculated so put it back here and it will be x n plus 1 x n plus 1 is current index is n plus 1 this is a known. So, a y n plus 1 is known.

Then go to y n plus 2 it will be minus a 1 index lays so n plus 1 plus x n plus 2 these are current input known y n plus 1 already obtained so y n plus 2 known, and this way. From y n you get to know y n plus 1, y n plus 2 uses y n plus 1 which is output it already. So, recursively you forward from y n to y n plus 1 y n plus 2 like that. But alternatively you can also write it like this, so this equation let me go through this I mean use this as an example.

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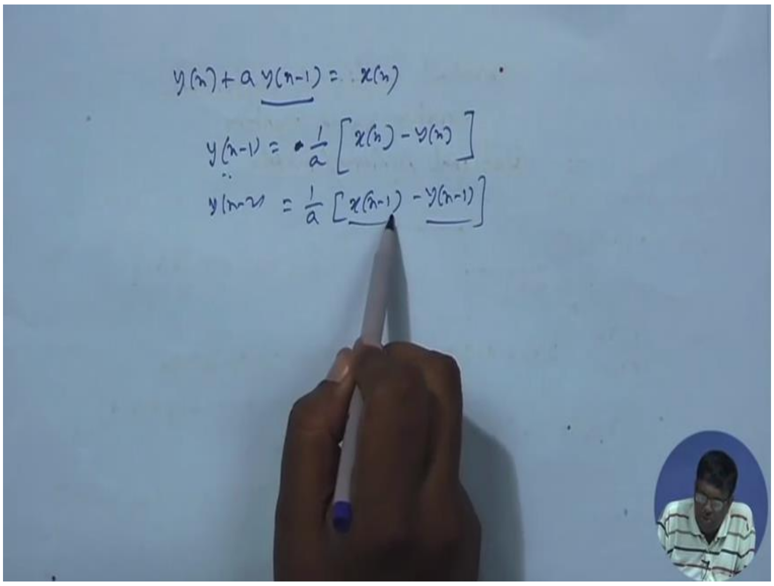

That is systematic the general equation I can as well explain the same thing so just these example let me do this. These are the equation, then you can keep this of the left hand side take the other one of the right hand side and divide by a. In this case if y n is known x n is known you can get back y n minus 1, then from y minus 1 you can go further to the left that will be one by a x n minus 1 minus y n minus 1, but y n minus 1 is already obtained and this is the input so again you get y n minus 2 then while you can get back y n minus 3 this way. So, it is a left ward movement from y n to y n minus 1 y n minus 2 etcetera there is a non causal movement. We will not consider that movement.

There is given a difference equation we will assume first output values are known you calculate y n. Then using y n and other first values calculate y n plus 1, then using y n plus 1 y n and still first values calculate y n plus 2. So, going the forward direction y n y n plus 1 y n plus 2 etcetera this is the model we will take.

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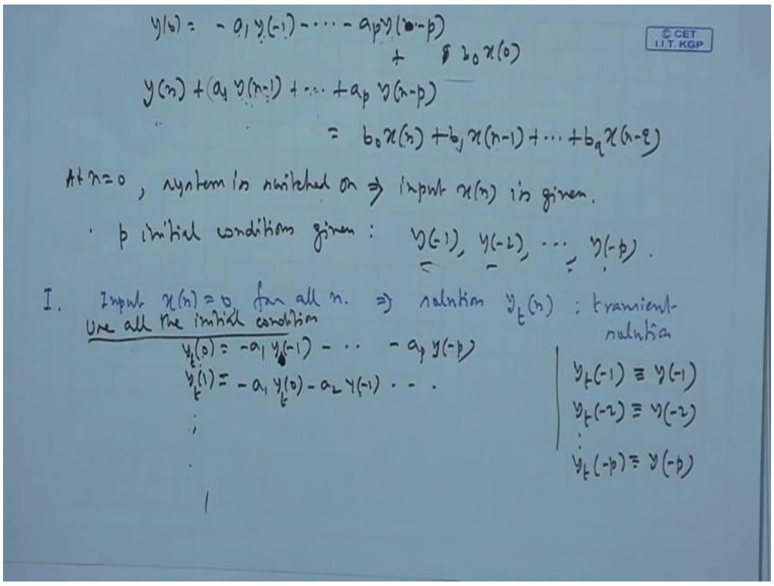

So, I rewrite again this is the equation I just re writing. And with assume let me assume that the system is switched on at a particular time so n equal to 0, whereas at n equal to system is switched on at n equal to 0 system is switched on within you give input x n x n is given. And at that time we have bought p initial conditions given, that is at n equal to 0 if use that y minus 1 y minus 2, dot, dot, dot y minus p, this p values are given to you. This p first values are given to they could be 0 or they could be anything.

Now, you are using the system from n equal to 0 onwards that is you give me your first input at x n at n x point of time at n equal to 0 x 0 and using that first input they will be 0 you can calculate they are 0, because you are switching on at n equal to 0, so you calculate y 0. How, y 0 will be you take this things to the left right hand side minus 1, dot, dot, dot minus a p y n sorry, minus p and then you have only b 0 x 0 because first input there are the switching not at n equal to 0 so no input was given input was 0 before that so just b 0 x 0 using that you find out y 0.

Then you go to y 1, y 1 will be what you take this side on this terms on the right hand side so minus a 1 y 0 because n is 1 now so where minus a 1 y 0 y 0 is already calculated put it back then minus a 2 y minus 1 y minus 1 initial condition is given, dot, dot, dot and then b 0 x 1 and b 1 x 0. These two input values are known use them get back y 1, then from that you get back y 2, dot, dot, dot, these how we proceed.

But this initial conditions on y are required because at n equal to 0 first input are 0 current input is present b 0 x 0, to find out y 0 you did these values because n is 0 y minus 1 y minus 2 up to y minus p. Then again at n equal to 1 here y 1 plus a 1 y 0 y 0 will be obtained, but then after that you have got a 2 y minus 1 still you need first samples then is t y minus 2, dot, dot, dot put those first the initial conditions. Right hand side b 0 x 1 plus b 1 x 0, so calculate that input is known so right hand side is not problem using them find out y 1. Then y 2, y 2 means y 2 is take this from the right hand side minus a 1 y 1, y 1 already calculated then minus a 2 y 0 already calculated then minus a 3 y minus 1 use the initial condition, dot, dot, dot plus b 0 x 2 plus b 1 x 1 plus b 0 x 0 they are known input is always known that is not a problem.

This is how recursively calculate, this is the situation. Question is; is it a linear system, is it a shift invariant system that the question that we do not know. But before that before ensiling that we will see one particular property of this system that is first I will consider two scenarios. One input x n is 0 for all n there is I will not give any input is permanently held at 0, and you solve this equation solution. The solution of these equation we will call y t n is basically Trans we call it transient solution; t for transient. That is in this equation how you proceed y 0 will be minus a 1 that is why using all the initial condition use all this is very important use all the initial conditions. There is input is 0 right hand side is 0 left hand side to find out y 0 y 1 y 2 you use all the given initial conditions, the given initial conditions I use here.

So, minus a 1 so y 0 what will be y 0, take this part of the right hand side minus a 1 y minus 1, dot, dot, dot minus a p y minus p y 0 obtained. Then again y 1, if it is y 1 then right hand side will have minus a 1 y 0 minus a 2 y minus 1, dot, dot, dot y 0 already obtained y minus 1 is used like that. So, this is how you build up these solution which you get y 0 y 1 y 2 I call it transient solution instead of calling it y I put a subscript t. These are initial conditions so these are the no need to put a t here whether given to you in that form; y t 0, y t 1, y t 2, y minus 1 y minus p are these initial conditions that I am (Refer Time: 13:38) put back.

When I am y using y t 0, I need y minus 1 y minus p those values I put back I get y t 0 then y t 1. I need minus I have to find out minus a 1 y t 0 because n is a 1 y t 0 already obtained that part then minus a 2 y minus 1 put the initial condition that is I am equating y t minus 1 with given y minus 1 y t minus 2 with give him minus 2, dot, dot, dot, y t minus p with given once. That is y t 0 will be minus a 1 y t minus 1 but y t minus 1 I am taking to the given initial condition y minus 1 minus a p y t minus p y t minus p I am taking to y minus p. Then full initial condition I am using here for y t.

Then in this solution satisfies the initial conditions for t n equal to minus 1 minus 2 to minus p that is the assumption. So, whenever I have y t minus 1 I write I replace it by y minus 1 y t minus 2 I replace by y minus 2 that is what I have been doing in this equations you get a solution. These I call it transient solution. That means, if I put back the transient solution here on the left hand side right hand this will among to 0 because right hand side was equal to 0 and that is why I form the equation right, this is one solution.

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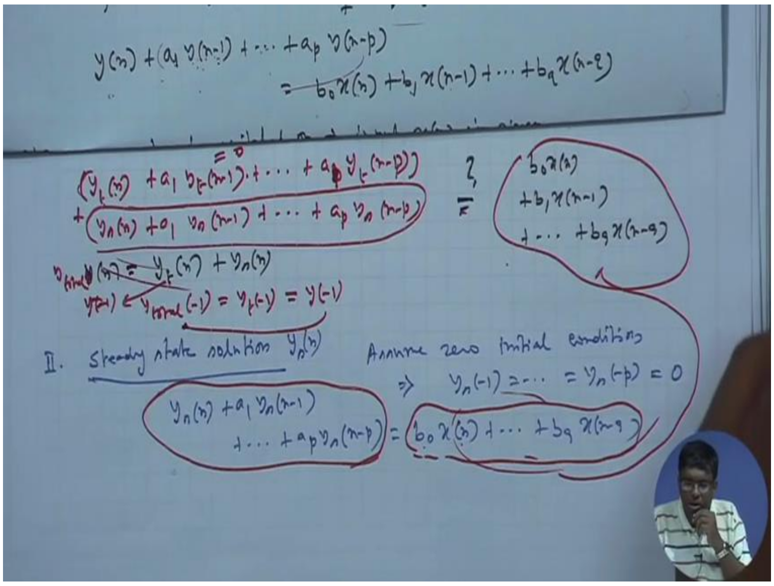

Another solution is I am constructing some solutions transient solution another is called steady state solution. Steady state solution means assume 0 initial conditions this solution I call y s s for steady state n. So, here I assume 0 initial conditions mean I assume y s minus 1, dot, dot, dot equal to y s minus p that is this sequence this solution at minus 1 minus 2 minus p they are all equal to 0. And putting them I solve this equation y that is I solve y s n plus a 1. Right hand side does not change, these equation I solve like the way I told you how to solve y s 0 will be minus a 1 y s minus 1, but minus 1 is 0 then minus a p y s minus p because n is 0 but that is 0, only b 0 x 0 will come out so and so forth.

This solution is y s n. Then my claim is total solution if you solve this equation for the given initial condition at given input you have to solve this equation. Then my claim is the solution if I call it y n that y n will be summation of y t n plus y s n. Then is what I have done first I consider two special cases of this equation; in one case input was 0 at this equation was solved there is right hand side was fully 0 and the equation was solved by using the initial conditions given the initial conditions here, I assumed I call the solution to be y t n transient and if I put that y t n here I get y t n from this first samples. So, y t 0 requires y t minus 1 y t minus 2 but those y t minus 1 I took to be minus 1 y t minus 2 will be minus 2 dot, dot where y minus 1 y minus 2 are given to me using them I records this build up the solution I call it y t n. That is one two special case, so right hand side was 0 and initial conditions were used out.

Now another special case of this equation is assume 0 initial condition and solve solution is call y s n steady state solution I am assuming y s minus 1 equal to, dot, dot, dot y s minus p equal to 0. And then if you solve under that special case I call it y s n. Then when I am having both initial conditions if I get no there is initial conditions are not 0 and also input is present input is not treated to be 0 then here I took input to be 0, but I am not assuming input to 0 now. Similarly, here I assumed initial conditions to be 0, but now I am not assuming initial conditions to be 0 that is I am now considering the general case, input present initial conditions present. And then I solve this equation what will be the solution my claim is that solution will be summation of these two special cases solutions.

How we will see easily; that if I put back this y n on the left hand side does it satisfy the right hand side that is the question. See if I put back y n that is I have y n minutes y t n plus y s n, y s n I am writing here then a 1 y t n minus 1 and here a 1 y s n minus 1, dot, dot, dot a p y t n minus p plus, dot, dot, dot a p y s n minus p. And right hand side is as it is b 0 x n plus b 1 x n minus 1 plus, dot, dot, dot b q x n minus q. Question is, is this true I am putting a question mark. Is this left hand side in reading there is equal to right hand side that is the question I am putting this question mark answer is yes, because what is the property of this transient solution; transient solution must obtain by equating by putting this y t n in the left hand side y t n plus a 1 y t n minus 1 plus a p y p n minus p that is what I have right hand side was 0. That means, this part was obtained this part is this part is equal to 0 because that is how the solution y t n was obtained by equating right hand side 0 no input and using this initial conditions this solution was obtained.

That means, I put back y t n on the left hand side a y t n plus a 1 y t n minus 1 plus, dot, dot, dot a p y n minus p there is equal to 0 because right hand side was equal to 0 so this will be equal to 0. So, this does not contribute anything on the other hand this part, this part corresponds to the left hand side here that was equated to this point. So, which is right hand side here, this two are same which means left hand side is satisfying right hand side. Also how would initial conditions why this y minus 1 is it equal to that me give the one answer is yes because y minus 1 means y t minus 1 and y s minus 1, but y s minus 1 taken to be 0 y t minus 1 was taken to be the given initial condition y minus 1. So that comes back here.

So value y minus 2, this y if you call it y total n then y total minus 1 is same as y t minus n plus y s minus n, but y s n was 0 so it is becoming y t minus 1, but y t minus 1 as taken to be given initial condition y minus 1. So, this solution total solution satisfies the initial condition at minus 1, similarly at minus 2 minus 3. That means, it satisfy on the initial conditions and also it satisfies this difference equation because left hand side if I put back write it as y t plus y s n y t n plus y s n and then left hand side is call to right hand side, because this upper half which is in terms of y t n plus a 1 y t n minus 1 plus a p y t n minus p; for a y t n is the transient solution this part is equal to 0 because that is how it was obtained.

Or linear difference equation was solved by right hand side to be 0 there is no input is given all initial conditions are absorbed on the in building up the solution there y t 0 y t 1 y 2, so if put them back here left hand side was equated to 0 so this base equal to 0. So, this does not mean anything at y s n was obtained by solving this difference equation that y s n plus a 1 y s n minus 1 plus a p y s n minus p that is equal to this right hand side. That means, left hand side all together is this right hand side which is this here, so equality is satisfied.

And initial conditions are satisfied as I told you some repeating actually. If I call that solution as y total n this summation of y t n plus y s n then y total minus 1 is y t minus plus y s minus 1, but y s minus 1 taken to be 0. So, y total minus 1 is y t minus 1, but y t minus 1 was obtained by equating y t minus 1 as the given initial condition y minus 1, so this is satisfied same for y total minus 2 is y minus 2 y total minus 3 is y minus 3, dot, dot, dot. Obviously, we can write the solution y total n as summation of y t n minus y s n.

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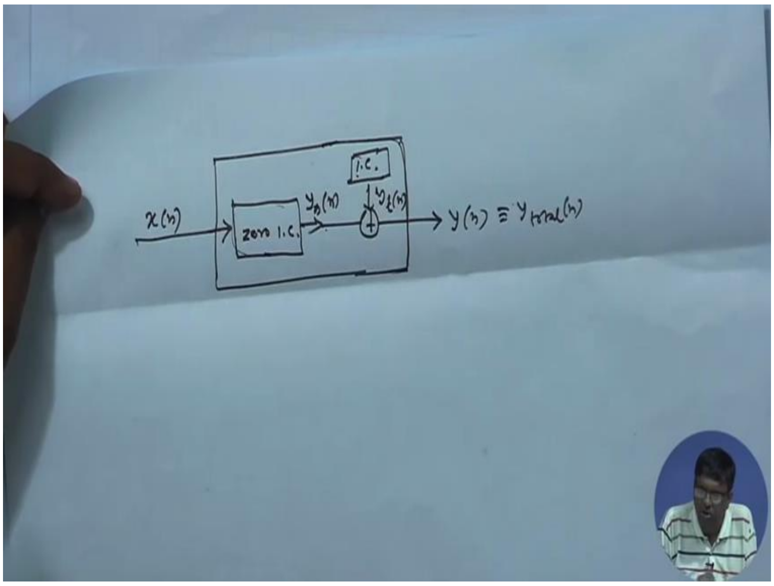

Now, these are your input. See this steady state solution is obtained by solving this difference equation and that has x n as the input and y s n as the output and same difference equation, but 0 initial condition. This is one input one system and that has 0 I C, I C means initial condition. It takes input x n we have show that solution y s n. To this we add y t n transient solution then only y total n we can call it y n equivalent to y total n, because what was the total n the summation of the two. So, we add y t n.

How y t is n generated? y t n is obtained you see by using by taking the input to be 0 and you solve this equation so no input and solve this equation by using the given initial conditions that y t 0 y t 1 y t 2 all obtained first that the initial condition given initial condition that y t 0 is minus a 1 y minus 1, dot, dot, dot minus a p y minus p they are given. So, y t 0 known then y t 1 was minus a 1 y t 0 y t 0 already obtained then minus a 2 y minus a 1 use the condition and, dot, dot, dot.

So that means, it is also output of a system, but that system takes no input. It uses only full initial conditions I C, based on the initial condition the difference equation with input 0 give rise to y t n. Both these boxes correspond to the same difference equation, but here there is no input means input 0. And it uses the all the initial conditions that is I C and build up a solution y t n. This one the same difference equation it uses input x n output y s n and no g initial condition. So, solution is y s n and these two are then added then you get y n which is actually y total n as used here. Now, question is from this we will be able to understand whether the system is linear and shift in variant. This I do in the next half.

Thank you.