Discrete Time Signal Processing
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Lecture – 10
Nyquist Interpolation Formula

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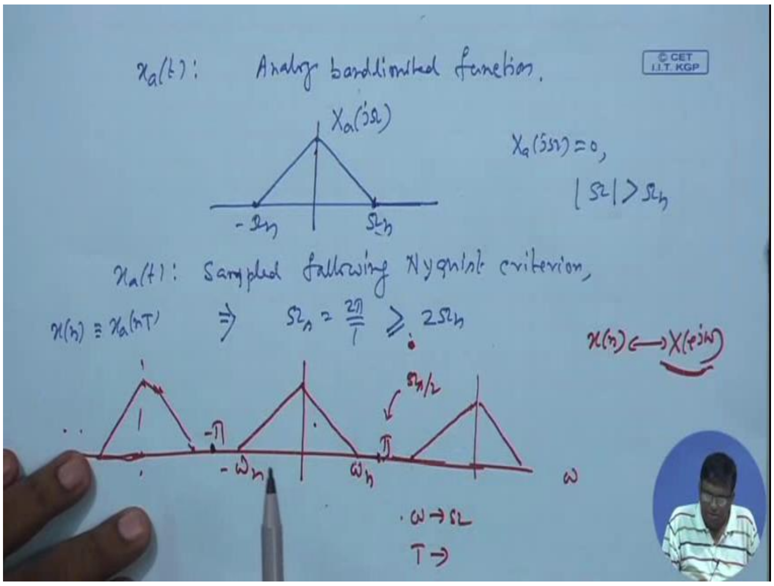

So I repeat, an analog signal band limited, which is a band limited function, or band limited signal. That is, eh, if you take the analog Fourier transform, it is limited from minus Omega h, to Omega h, h for highest frequency. So, h, actually the magnitude is limited, from minus Omega h, to Omega h, that is a 0 outside these (Refer Time 00:48) I did a sample now, to generate a sequence, sequence x n, where x n is nothing but, x a, the n into t, n into t is a time point, at that point, whatever is the value of the analog function, that I calling n is sample of the sequence. That sequence, and I am sampling following nyquist condition, that is, I am sampling at an analog sampling rate of 2 pi by t, t is the period, sampling period, that is greater than equal to, twice the band limiting frequency, then you have seen, that the sequence, which is obtained, by sampling analog, this analog function, at that rate, there is a period capital T, is GTFT will take this same, where, from minus pi to pi, whatever you see, that is, your replica of original analog Fourier transform, only thing is it was a function of capital Omega.

It is a function of small Omega, and of multiplication by 1 by capital T of these 2 place, because that was the formula, 1 by t times this, otherwise same point, it is same.

Therefore (Refer Time: 01:56) interesting question, that, what is this DTFT? And again, its capital Omega is greater than equal to 2 Omega h, half something frequency Omega based by 2, is greater than equal to capital of Omega h, half something frequency maps to pi, capital Omega h is maps to small Omega h. So, pi is greater than this. So, whenever there is no (Refer Time: 02:26) half sampling frequency, is greater than equal to the band limiting frequency, or in digital domain, pi is greater than equal to small Omega h, small Omega h comes from capital Omega h, these are the situation.

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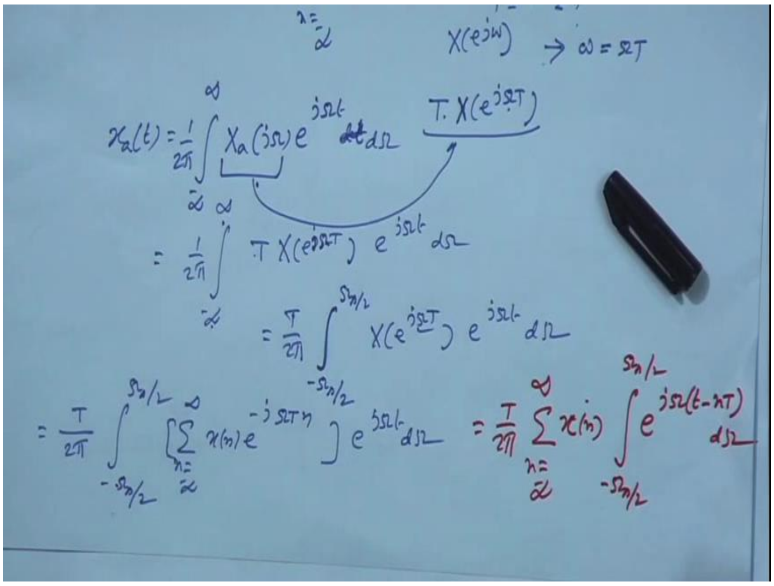

So, we are assuming no (Refer Time: 02:38) because nyquist conditions was satisfied, then, what is this DTFT? We know DTFT, for any small Omega, on this axis of your choice, is, these are DTFT, but DTFT, if I restrict small Omega from minus pi to pi that is, I plot only this function, then, this looks very much like this, there is no right half. There is no left half, only this much, otherwise 0.

So, you see, capital X e, j Omega, if I take form Omega, from this point to this point, sometimes I can go up to these fully, there is if 2 pi by t is equal to 2 Omega h, that half something frequency will be equal to 2 Omega h, pi will be equal to small Omega h. So, we will start directly here, in some cases there will be gap, that is why, to be more general I take from minus pi to pi. I take the DTFT, there is I do not look at frequency points Omega to the right, or to the left, I see only here, these function, I look at it, and then here, then, then here, small Omega, wherever this small Omega I (Refer Time 04:26) write it by, capital Omega t. So, it will become a function now, capital Omega, and then I multiplied by capital T, because this was obtained by dividing by t. So, this can be obtained, in 1 band, by multiply by t.

So; that means, t times capital X, e to the power j, DTFT is not analog Fourier transform, DTFT, but small Omega replace by capital Omega t, radian per second into second, that is how it is radian, these are DTFT. This function will be identical to this, because I am multiplying by capital T, and replacing small Omega by capital Omega. So, I should get back the original 1, from here to here. Now do not include this side ones, because they are not present there, that is why, first thing I am doing is, restricting Omega, in the range, minus pi to pi, and then doing the reverse frequency mapping, for small Omega to capital Omega, in the DTFT itself, multiplying by capital T. So, resulting, to give the function of capital Omega, that is same as this, which follows from here.

Therefore, now look at this original function, x a t, what is x a t? Is the inverse, analog Fourier transform, of this, but this guy is band limited, this form minus Omega h to Omega h, all right? Here, first this part can replace by this, you have already seen, that they are same. So, t, sorry this is a Omega, I am very sorry, this is Omega, I am not only looking to this Omega, this inverse Fourier transform, all right? But there is no point in taking the integer from minus infinity to infinity, because this function is 0, outside this range. So, I take it from, just 2 minutes, I can take it from minus Omega h to Omega h, but to generalize, suppose I take it from here to here, because sometimes this can question with this, outside this is 0. So, no point in carrying on the integration, this capital T can come out, this is this, but this is DTFT, at this t. So, I can replace it now. by the DTFT expression, j, this much analog digital frequency Omega t, into n, this will be e to the power j, Omega t, d Omega.

Now, there is a double summation. So, you know very well, about the next step is, interchange, these are 2 summations, all right? So, next step, t by 2 pi. This inner summation goes out, x of n, it does not depend on Omega, inter summation with is respect to capital Omega, this does not depend on so, this Omega. So, this can come out as common, and I am left to this inside, e to the power j, capital Omega, t minus, n t, d Omega, alright, these we integrate, which is to 2 Omega, then what we get?

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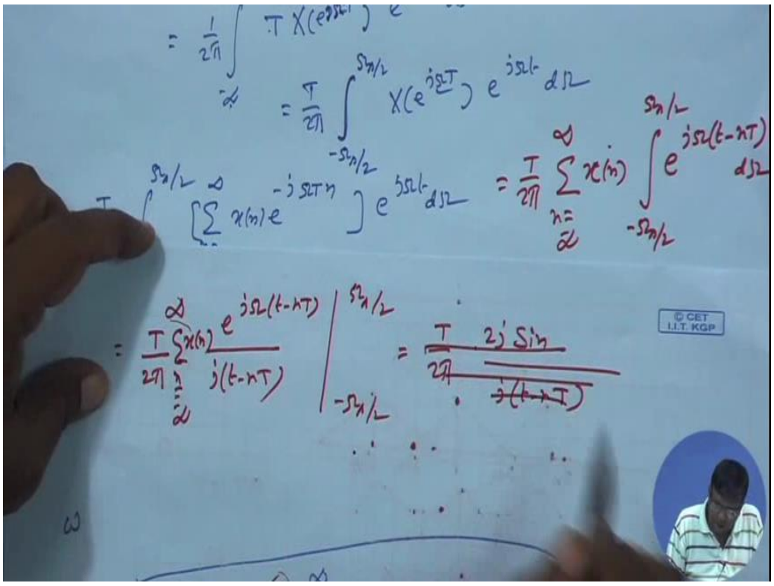

This integral, which is put to Omega, so j, t minus n t, will come below, it will be same as j Omega, t minus n t, and Omega is by 2, then what happens? e to the power j theta, minus, e to the power j theta. So, it is twice j sign theta. So, t by 2 pi, twice j, sin, whatever I am writing that, here j, t minus n t, opposite summation, x n, that I am forgetting there, is a summation, this part I am, this I write again, may be I write on a first page, this page does not have much space.

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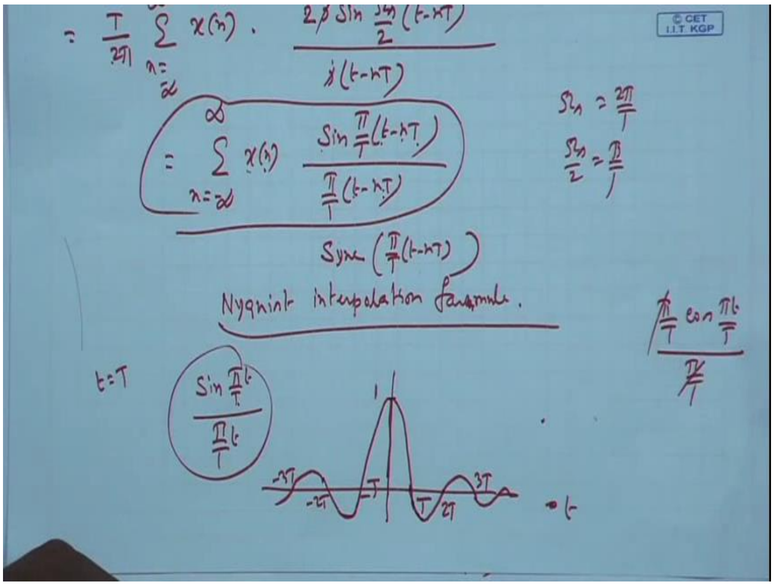

So, this integral, t by 2 pi, summation, x of n, and then this, twice j sin, this Omega was, Omega is by 2, and then minus Omega s by 2. So, it is Omega s by 2 into this quantity, if you want, you can see, look at that thing, is equal j Omega t minus n t remains, j t minus n t here, replace Omega by Omega s by 2, and there again minus Omega s by 2, that minus this. So, it would be twice j, sin, Omega s by 2 into t minus n t, divide by this. So, twice j sin, Omega s by 2 into t minus n t, divides by that j, t minus n t. So, j and j cancels, 2, 2 cancels, t by pi, you can bring the t by pi here, below this as pi by t, and t minus n t, this x n, sin Omega s by 2, Omega s by 2, what is Omega s by 2? pi by t. So, what is Omega s by 2? It is pi by t. So, if we bring pi by t here, pi by t into t, minus n t.

So, that is the sin function, delayed sync, it is sync of this, if you call it x, sin x by x. So, sync. So, you see this, a very, very interesting, very important result, that, we started with original x a t. So, if the function is band limited. And it is sampled, following nyquist theorem. Then from this sequence, the samples, because you are sampling so, you are getting a sample train, and that forms sequence, for the sample values, you can get back your full original signal, analog signal, just by this summation. These are fixed functions, and when n equal to 0, x n, x 0 into the sin, pi t by capital T, and pi t by capital T, then x 1, delayed version of that, x 2 delayed further, further delayed version of that, and they give superimposed and by superimposition, you get a function of small t, which turns pot to be x a t. These the fundamental theory mean, DSP that is the function is, if I start with an analog function, and is giving that is band limited, then I can always sample it at, more than twice the band width, and all information of the envelop x a t. All information over envelop x a t, that is call, that will be contained, in the sequence in the samples, because from the samples, I can recover that my x a t, ok?

So, if the function is band limited, I should sample it at, more than at least equal to twice Omega h, or more than, that twice the band limiting frequency, then the samples I get, which form a sequence x n, that carries all the information about the full analog signal, because using the samples, I can get back by a formula like this, the full analog function correctly. This formula is called Nyquist Interpolation Formula. We can try to explain the, try to understand how it looks like. Consider this function sin, pi by t, into t, and divide by pi by t, into t. This function, how it look like? There is small n equal to 0, after all it is small equal to 1, this will be shifted to the right by capital T, and small n minus 1, this entire thing will be shifted to the left by capital T. So on and so forth. So, I am considering n equal to 0 cases.

The sin x by x, at small t equal to 0, at small t equal to 0, it will be sin, 0 by 0, 0 by 0 form, and then you apply l'hospital theorem. If say, the numerator denominator, you have pi by t, into co sin, and here you have just pi by t. So, they cancel, and a t equal to 0, what you have 1. Sin pi by t into t, if you differentiate, pi by capital T comes out, and you get co sin of that, and here pi by t, the 2 cancels, now if t becomes 0, is 1. So, it goes to 1, and then small t, suppose at capital T. So, you look at the numerator, numerator is a pure sin sinusoidal function, numerator, if I have this numerator only that suppose sin pi t by t.

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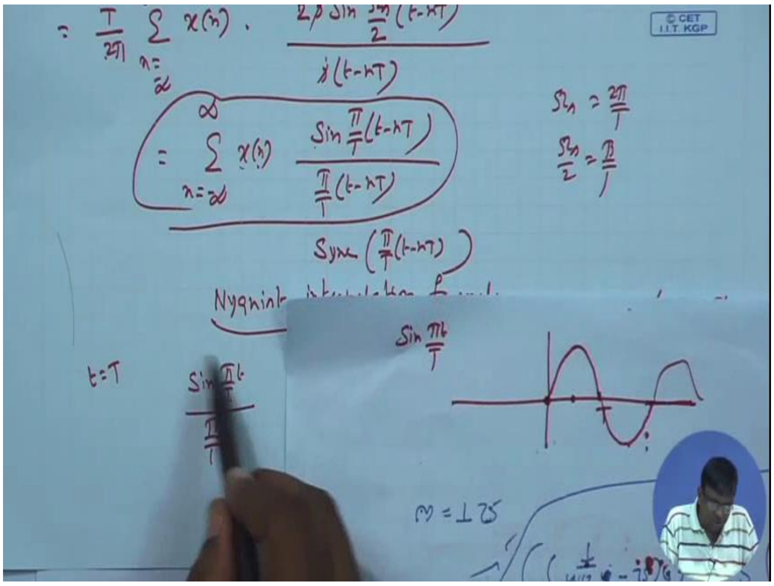

This appears in just numerator, this is a pure sinusoidal function. At t equal to 0, it will be 0. Then, at t equal to capital T, at sin pi, it will be 0, and at t by 2, it will be maximum, t by 2, it we will sin pi by 2, and like this, it will be periodic, and it will go on like this. So, it will be crossing 0, at capital T, 2 t, 3 t, like that, but the denominator function, we should leave out the point t equal to 0, at t equal to 0 you have to apply l'hospital, because it is 0 by 0 forms. So, that is why it is not 0, it is 1, but after that, this function is periodic, but this is if t is positive, it is growing in magnitude. So, denominator is increasing as t increases. So, the sinusoidal oscillators will get damped. So, damping, same for the negative side, magnitude of the denominator is going up. So, it will damp.

So, at t equal to capital T, the denominator you have t, t cancelling pi, but the numerator 0, and its varying sinusoidal. So, it will come like this, and will its getting damped. So, it will not go fully up to minus 1, it will come back here, at 2 t, it will be like this, then 3 t like this, then like this, like this on this. On side also you can verify, minus t, minus 2 t, minus 3 t, like this. This will be this function, where n equal to 1, it is t minus capital T, t minus capital T.

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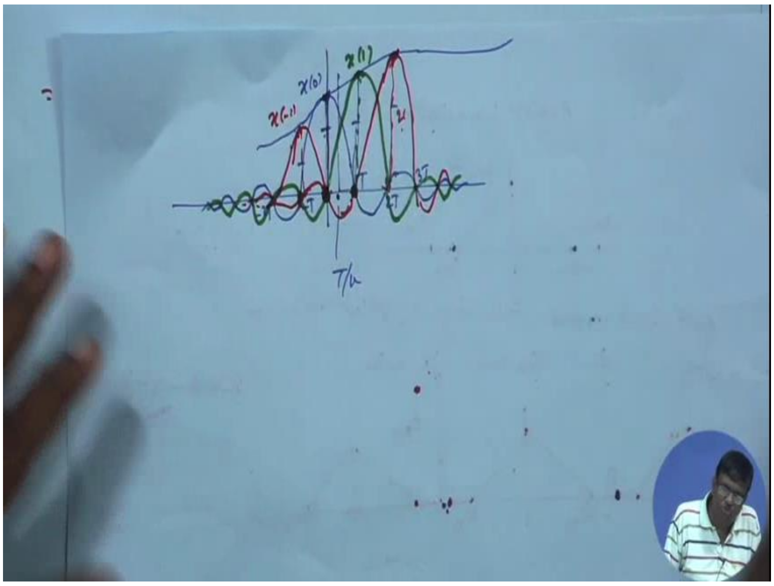

So, the entire thing will be shifted to the right, by capital T. t equal to 2, twice t, like that. Now let us try to understand, how to, how, what this formula means. Nyquist formula implies, take the case of x n equal to 0, you have the term x 0, into sin pi by this, this sin pi small t by capital T, divided by pi small t by capital T. So, this function is to be multiplied by x small 0. So, it was 1 here. So, height will become x 0, and then it will be like this, this is t, 2 t, minus t, minus 2 t, 3 t like that.

Then consider n equal to 1, it will be x 1, sin pi by capital T then (Refer Time: 21:27) bracket t minus capital T, divided by pi by capital T, within bracket again, t by minus capital T. So, instead of t, it is t minus capital T, which means the entire function of the shifted to the right, move to the right, by amount capital T. So, which means this, and then multiplied by x 1; that means, the entire set sync function, origin will move here, that is multiplied by x 1. So, height will be x 1 say, x 1, it will have 0 crossing, t to the right, further t to the right, here also, like this. Then if n equal to 2, this is further shifted to the right, multiplied by x 2. So, it will be here, origin will move here, 0 crossing like this so and so forth. For this side also, here again you can have, another 1, multiplied by x minus 1, going like this, like this, ok?

Now, when you super impose all of them, you see, consider n equal to 0, at n equal to 0, Everybody else, going through origin, only this fellow remains as it is, summation will be x 0, because others are 0. Similarly at capital T, this is going to 0, this is going to 0, and this is going, only this fellow remains, its value is x 1. So, at capital T, value remains x 1, at 2 t value remains x 2, and minus t value remains x minus 1, at intermediate points, they get added, and that is how this function is formed, function I know, at x, at n equal to 0, resulting function analog function has sample, how much? Just a minute.

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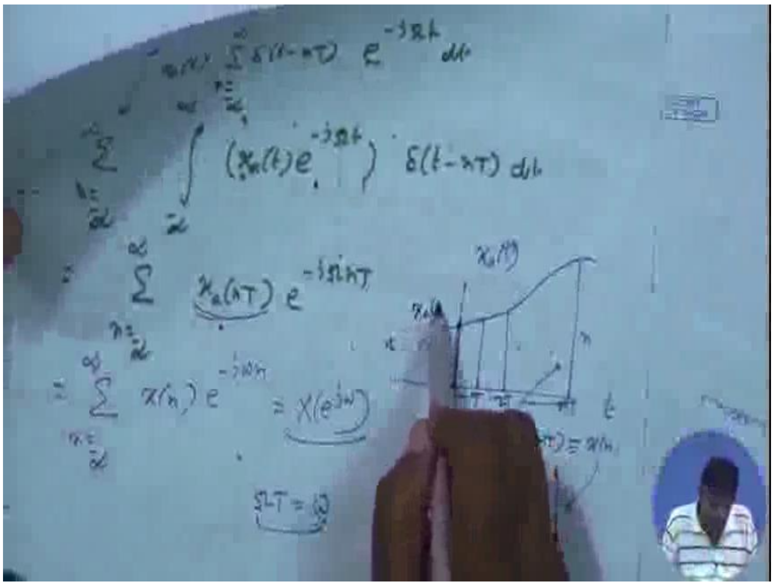

At n equal to 0, analog function has value x a 0, which is same as x 0 (Refer Time: 23:55). So, that will come up, then at n equal to t, you have got x a and x a t, which is same as x 1. So, this will come up, this x 1 is nothing, but in terms of analog x a t. So, x a t coming up, x a 0 coming up, x a 2 t coming up, and intermediate points they all get added, because they are overlapping, they are added, and the envelop gets formed.

There is a building of that summation, but these again a fundamental thing, it means that. If you have got an analog function, x a t, and that is band limited, then you do not need, you can sample it at twice or more than twice the band limiting frequency, you get as sequence of samples, train of samples, which forms a sequence. Using those samples you can get back the original envelop, by way of interpolation formula, which is call nyquist interpolation formula, this interpolation formula. In fact, given this samples, we are giving only this samples, you can now constant envelop, and you can get the value at any arbitrary point, any arbitrary, may be t by e, some k, so that kind of thing. So, this interpolation that way very useful, you know. You can have fractional, fractional sampling point, the value there; can be obtained in terms of these samples. This is very interesting topic in GSP. So, that is all for todays class, in the next class I will consider a special kind of linear time (Refer Time: 26:04) systems, called rational systems.

Thank you very much.