Fundamentals of MIMO Wireless Communication

Prof. Suvra Sekhar Das
Department of Electronics and Communication Engineering
Indian Institute of Technology, Kharagpur

Lecture – 39
Capacity of Random Channel

Welcome to the course on Fundamentals of MIMO Wireless Communication. Till now we have seen the expression of capacity for two important cases - in one case, it is no channel state information available at the transmitter where we said the best strategy is to put equal power across all the transmitting antennas. The second case would be when you have the channel state information at the transmitter and then the channel modes are all accessible, in that case the gross result that we got is it is better to do power allocation across the different antennas and the best strategy to power allocation is to put more power on the channel which has better channel strength. So, and that was the result of the water pouring algorithm which we had shown in the previous lecture.

So, we carry on with those concepts and what we are going to discuss initially is about mechanisms or techniques by which you could achieve those results, and then will move on to see the effect of correlation on MIMO channel capacity and also we will move further to look at the random MIMO channel, and what is the outcome on capacity or how do you replace on capacity under those con those conditions will also be looked at.

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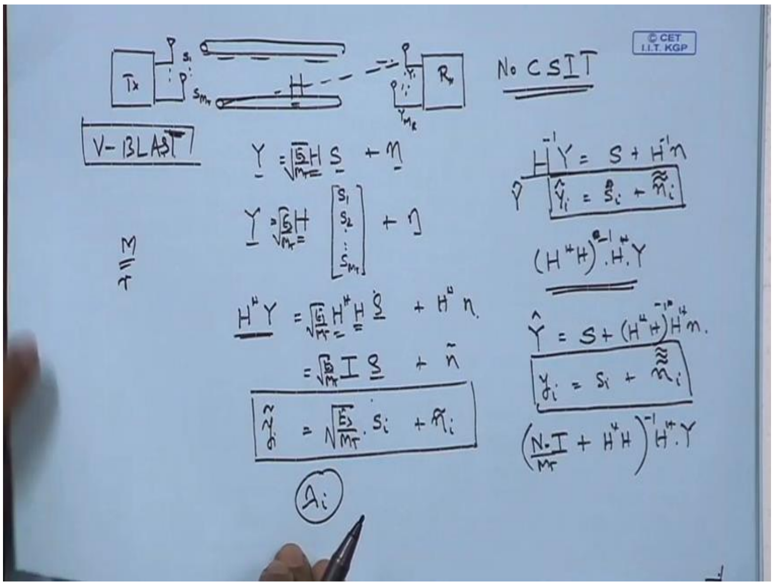

So, when we take the MIMO transceiver system again I draw the transmitter with multiple antennas at the transmitter and at the receiver we have multiple antennas. So, the first thing we said is when there is one directional flow of information, no feedback information. So, in that case there would be signal S 1 up to signal S M T that gets transmitted and the signal that is received Y vector would be H that is the matrix channel which lies between the transmitter and the receiver Y 1 up to Y M r times S plus noise vector this is the expression that is received. So, if we have to transmit these signals that means, you have to put equal power that is what the expression gave us, but the result that we have got is basically putting equal power, but it never told us how you can send symbols across these this lines or these parallel channels, as to achieve the result which we have shown.

So, the mechanism that we are doing over here is will be sending S 1 to S M T that is different symbols. So, will directly sending the difference symbols S 1, S 2 up to S M T let say and we have this channel matrix plus noise across this. So, if H is orthogonal; that means, if the channel coefficients are orthogonal then one of the simplest strategies to receive this would be multiply H hermitian with y, if we multiply H hermitian with Y we will to get H hermitian H times S plus H hermitian noise and that would result in H hermitian H would be identity matrix. So, we can get an identity matrix times S in this case of course, with our notation we are going to have S by M T root over E S over M T right. So, H hermitian H is yeah. So, we have an identity matrix over here and we could also have zeta upon M. So, that the trace ends up as M. So, we could also have the total channel strength constraint that will be the scalar modification of this.

So, when we expand this, this will be H n tilde what we have over here is a diagonal matrix. So, each one that we receive this one would write Y 1 tilde is equal to root over E S over M T times S i because is a diagonal matrix times n tilde i. So, what we can clearly see from this expression is that any symbol that is received in the corresponding branch of the receiver is not having influence of other symbols whereas, in a typical MIMO con communication whatever is been transmitted from all the antennas get added up together in any one of the antennas. So, if the channel is orthogonal you can get that, if the channel is not orthogonal the other simple way of doing getting back the received signal would be to multiply H inverse if H inverse exists. So, we can assume it a full rank M cross M square matrix. So, that is for one particular case H inverse Y would give us S plus H inverse n. So, this is the zero forcing mechanism.

So, even in this the S is a vector. So, basically Y have Y i cap is equal to this is basically the vector which is Y cap you can say is S i plus n tilde I because this will again be a vector of course, this n tilde we can put n double tilde which is different from in this case. Here again we have avoided interference, there is no interference of the symbols. If it is non square then one could use H hermitian H whole inverse times H hermitian multiplied by Y. In this case if we are going to do this kind of equalization to make it square the end result again would be S plus; is the pseudo inverse, H inverse H hermitian H inverse times H hermitian times n. So, this is what is going to be the received vector.

So, here again Y i would be S i plus n put triple in i. So, here again in this format we are not getting any interference from any other symbols. So, in his way the one could be construct this one could also use an NMSE kind of approach where one would use I plus H hermitian H n naught upon M T inverse times H hermitian times y. So, this is the MMSC approach this way we would also get it, but there would some minor amount of interference because of these channel condition. So, what we have effectively is through this mechanism which is also known as vertical bell labs layered space time coding also known as V BLAST; that means, in short is a vertical bell labs layered space time coding is the name for this. Where you could send the symbols directly across the sym, across the antennas and at the receiver you could do different kinds of processing and recover the symbols without interference. There by you are achieving as if parallel transmission and we would remember that this SNR is dominated or is indicated by lambda i where lambda i is the eigenvalue of i-th channel mode.

So, if it is square matrix you have M parallel links or you have r parallel links depending upon the rank of the matrix. So, this is one mechanism where you can achieve the transmission and parallel data transmission when there is no channel state information at the transmitter. We move on further after this to look at the second scheme where channel state information could be made available at the transmitter T x and R x.

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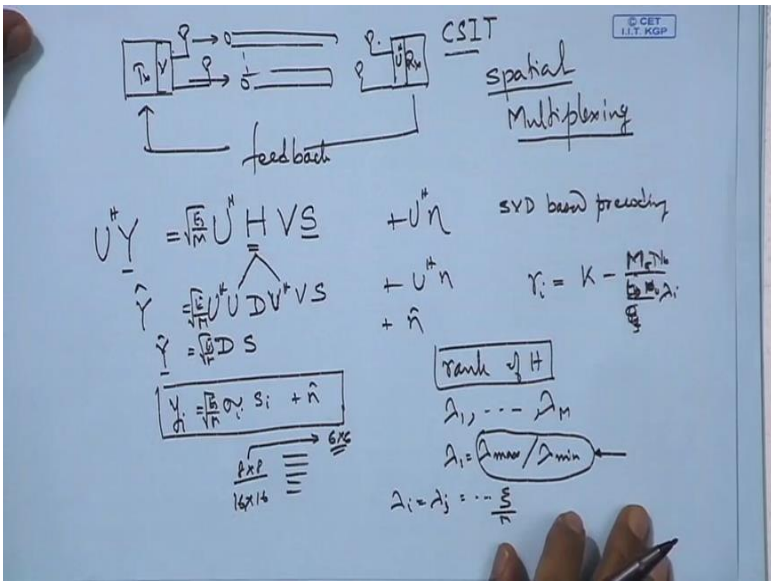

So, basically here there is a feedback and if we would remember the result of capacity in this case told us that we have the feedback about the channel state information. So, we can get the eigenvalues once we get the eigenvalues of H hermitian H we are getting indication of the channel strength of the parallel modes that we can send. So, based on that again we will be distributing the amount of power which is proportional to the eigenvalue of that particular mode of the channel, here again we look at that we have S has a vector of signals to be send there is a this channel H which gets multiplied with S plus there is no noise and this is what is received at the receiver across the antennas. Here again as we can clearly see that these signals get mixed up and you have mixture of signals at the antenna.

Now, what we have said is when your feedback what we could do is we could multiply a V at the transmitter side; that means, there is some processing V matrix and there is some processing at the receiver with U hermitian this is what we had proposed when we had discussed that. So, basically at the receiver we are going to get U hermitian, U and V are (Refer Time: 09:40) matrices. So, then if you would have expand H, H can be stated equal to d V hermitian and U and because of this U we have U hermitian because of this U we have V plus U hermitian n and this we could write it as Y cap. U hermitian U is identity matrix V hermitian V is identity matrix.

So, we have D times S plus n cap we can say Y vector cap. D is the diagonal matrix consisting of the singular values. So, therefore, we could write Y i is equal to of course, there is root over E S by M I tend to miss this term. So, root over E S upon M sigma i S i plus n cap. So, this again we see, through this mode we are not having any interference from any other term. So, in this case also we are able to establish parallel links between the transmitter and receiver and hence in both these mechanisms that we are just now discussed, these provide us we can clearly see the gain of spatial multiplexing. This is an important terminology in the domain of MIMO.

So, clearly what we are achieving here is in both these cases where there is no feedback and when there is feedback we are transmitting S 1 to S M T in parallel. So, there is multiplexing and this multiplexing is in the space domain. So, this is why it is spatial multiplexing and here this particular scheme there is channel state information at the transmitter because of feedback we are doing some pre processing or pre coding it is also known as pre coding. So, this particular method is known as S V D based pre coding. So, if we follow this S V D based pre coding mechanism in this mechanism we can achieve the capacity when we have feedback information at the transmitter and of course, as has been said we have to distribute the power such that gamma i; that means, the power in this channel is such that there is a k minus 1 over E S by M T n naught times lambda i.

So, this is the kind of sorry M T n naught this E S upon M T n naught. So, this is 1 upon SNR. So, this is basically the amount of power which again leads to higher lambda values higher values of SNR. So, if we add this kind of power distribution along with this then that would naturally lead us to a parallel connection parallel sending of signals so that we achieve capacity of channel state information at the transmitter.

So, summarily when we do either of these two streams we are sending parallel information, we are sending parallel information from the transmitter to the receiver. The number of such parallel links is clearly is as the many number of independent links available and this is indicated by the rank of H. The rank of H would indicate as many numbers of independent links. There are many details to it sometimes it is not very easy to identify what is the rank of a matrix and one would deal with the eigenvalues of H hermitian. So, one would be interested to know what is the ratio of lambda one which is basically lambda max to lambda mean. So, this would give us condition number which tells us what is the distribution of eigenvalues and we have seen this capacity is maximized when all the eigenvalues are equal.

So, sum zeta upon M let say. So, this going it is maximized. So, we would like to see this distribution is very large then capacity is not maximum. So, we would like to choose the number of modes; that means, it may happen that I have 16 cross 16 system or an 8 cross 8 system, but if instead of taking an 8 cross 8 if I would choose a subset 6 cross 6, it could give me a better capacity better some total capacity, so that one has to try and see out of these combinations which is giving a better capacity and one chooses and this is one way of quickly circumventing problem instead of looking around for all possible combinations.

So, we have seen at least two ways of which MIMO channels could be exploited - first we have seen what is expression of capacity and then we have looked at how the signals should be send. First mode where you send the signals as it is, in second mode you would be doing some pre-coding the pre-coding is based on S V D of the channel or the single value decomposition of the channel and we have to use the V matrix at the transmitter and U matrix at the receiver. So, that you get r parallel independent data pipes and since you are getting parallel data pipes with the help of multiple antennas. Now these mechanisms are known as spatial multiplexing. Now what we had seen before a when before we are looking at the capacity we are talked about diversity as, that was spatial diversity. So, seen two very important things - one through spatial diversity we are able to improve their probability and through spatial multiplexing we are able to improve the capacity. So, with this we move forward and take a look at some more things on capacity.

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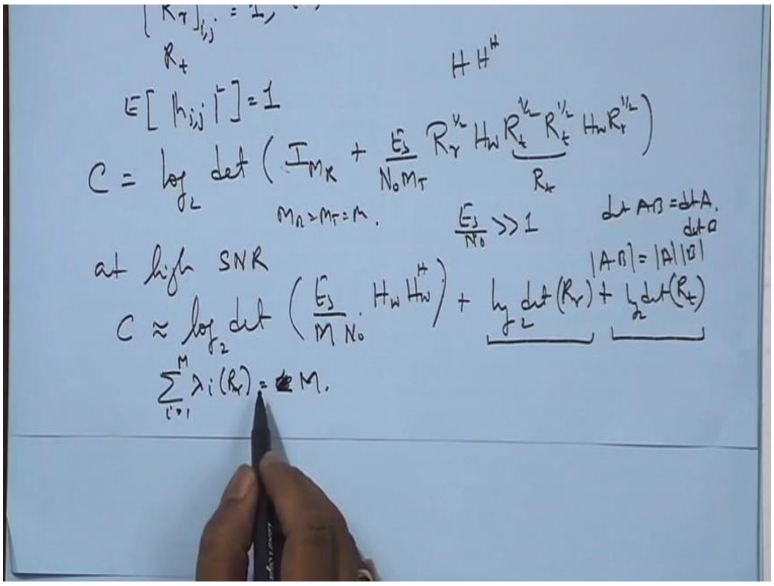

So, the first thing we are going to look at is the influence of spatial correlation right. So, we have seen that H which is a channel could be written in terms of the correlation matrix at the receiver raise to the power of half times H w which is spatially wide channel times R t raise to the power of half, is a spatially white channel. So, this captures the correlation at the receiver, this captures correlation at the transmitter. So, of course, we have R r the elements i j set equal to 1 or i equals to 1. So, basically we are saying that it is normalized all these R r coefficients are normalized and. So, R r t coefficients R t coefficients are also normalized and this would result in expected value of H i j mod square to be equal to 1.

So, all we are saying is that the channel coefficients are normalized unit power and the expression of capacity that we have (Refer Time: 16:47) at earlier is log determinate of I

M R plus E S by n naught M T you had H H hermitian. So, instead of H and H hermitian we had H H hermitian. So, we could write R r of H w R t half for h. So, when we have H hermitian you would have R t half H w R r half right we are expanded this. So, this R t r T half we could write it as R t right this is what we could do and we will make the assumption that M r equals to M T equals to M that is will make the full rank app approximation. So, when we write it in this form since it is the determinate we could say that at high SNR this effect of one are from this could be neglected. So, we could say that it is dominated by this because we are saying that E S by n naught is much much greater than 1 because the determinate would be having the eigenvalues and in the eigenvalues you are going to have a (Refer Time: 18:08) matrix and that of the other matrix. So, basically this will lead to 1.

So, when the right hand side is much more than one we could say that one can be neglected and this capacity expression we could approximate it to log two determinate of E S by M we are said M T equals to M r equal to m. So, E S by M n naught times H w hermitian see now we do not have this. So, we are having log determinate of this things. So, log determinate of determinate of AB is equal to determinate of times determinate of B right or determinate of AB is equal to determinate of times determinate of B.

So, what we have is determinate of these terms. So, determinate of these terms we have H H w. So, we have basically H H w as one of the terms plus log base two determinate of R r this really what we have and then again we have R r half so basically plus log base two determinate of R t right. So, if you put them together log determinate of this times this times this. So, basically we have this term and that is the result of this expression when right hand side or towards the plus of this is much greater than 1.

So, if what we see from this is the effect of correlation at the transmitter and at the receiver are basically similar. So, if we study one of them we can satisfy. So, we can do either at the transmitter side or at the receiver side. So, we will put the constraint that sum of lambda i R r is equal to 1. So, this is one particular constraint that we will maintain 1 to n. So, if sorry this is equal to M this equal to m. So, total power constraint is that we have to maintain.

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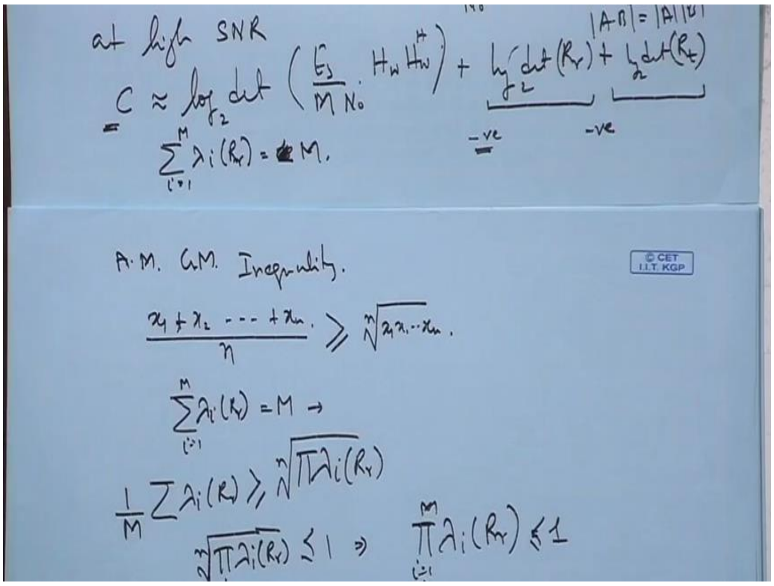

So, now, we will use one inequality; that means, arithmetic AM, GM inequality known as AM, GM arithmetic mean geometric mean inequality to our (Refer Time: 20:27). So, when we use arithmetic mean and geometric mean inequality. So, what we get is that if there is x 1 to x n set of random numbers a set of numbers if you say. So, we take the average we will find it to be greater than or equal to the root of x 1, x 2 up to x n and what we have is in our case is sum of I equal to 1 to M lambda i of r is set equal to m. So, this is what we have and according to the inequality that we want is one upon M sum of lambda i R right this is of course, equal to 1, this should be going by this is greater than or equal to nth root of pi of lambda i of R r, right, that is the received matrix this is what we have said.

So, now if you look at this, this is equal to one according to this gives a small upon M this equal to 1. So, what we see is that square nth root of pi I lambda i R r is less than or equal to 1 or in other words product of the lambda i of this co variance matrix is less than or equal to 1, let i equals to 1 to m. So, if you apply this back here. So, determinate of matrix, determinate of R r is equal to pi i equals to 1 to M and lambda i R r that is the product of the eigenvalues and this is less than or equal to 1, that means, inside the log the (Refer Time: 22:17) is less than or equal to 1, so this whole term is negative.

So, what we have is negative term and a negative term, otherwise the best value that you can have is when they are equal. So, that is the best value that you can have. So, basically you can have 1 as the best value in that case this will be 0. So, all these eigenvalues being equal means that this is an identity matrix, and that being identity matrix means it is an orthogonal channel, orthogonal channel means that R is not correlated.

So, when r is not correlated this determinate will turn out to be 0 this will also turn out to be 0 sorry this is log of 1 would turn out to be a 0. So, there will not be any effect if there is any non zero effect that will only be negative effect so; that means, the effect of capacity when we are studying at (Refer Time: 23:09) especially that becomes very very clear, whenever there is correlation be added in the term then the capacity decreases. We had also seen similarly just to remind you when we studied the effect of correlation on error probability what we had seen whenever there is correlation the error probability increases. So, what we seen both the cases when there is a correlation in case of diversity again the error probability increases relative to the non correlated case, but still it is better than SISO link. In case of MIMO what we are seeing again is correlation reduces capacity gain, but overall (Refer Time: 23:47) will be better than a SISO link.

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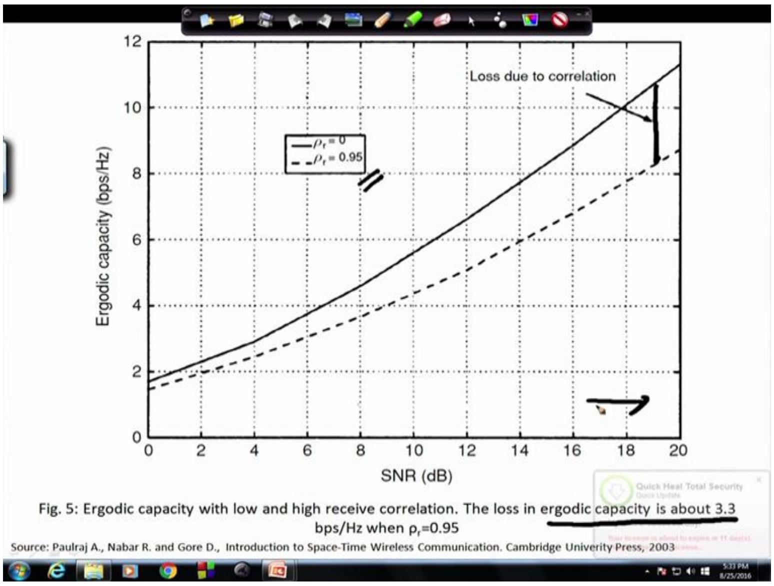

So, these are some of the important results that we can gain over here and we take a look at the result of correlation. So, if we see this particular picture what we would see is, when we look at this particular picture this is the case where correlation is said as 0.95 and this solid line is a due to no correlation case and if there is a 95 percent correlation between the antenna branches this dashed line what is there and this is a from the book introduction to space time values communications by (Refer Time: 24:30) the Paulraj Rohit Nabar Dhananjay Gore from Cambridge university have got taken this picture from that particular book and could generate this result also.

So, there is a loss which is indicated by this particular line, there is a capacity. So, we will discuss ergodic capacity about 3.3 bits per second per hertz which is significant at higher SNR regime. So, what we can conclude from this is that when there is correlation there is loss in capacity and this is well understood in terms of expressions a when we look at high SNR regime, for lower SNR regime also there is a loss, but again we have taken higher SNR all these approximations that we are making is only to ensure that we get an inside in to it and from which we can clearly see as we have seen over here that is expressions lead us to certain clear understanding from directly by looking at it and also which is supported by results.

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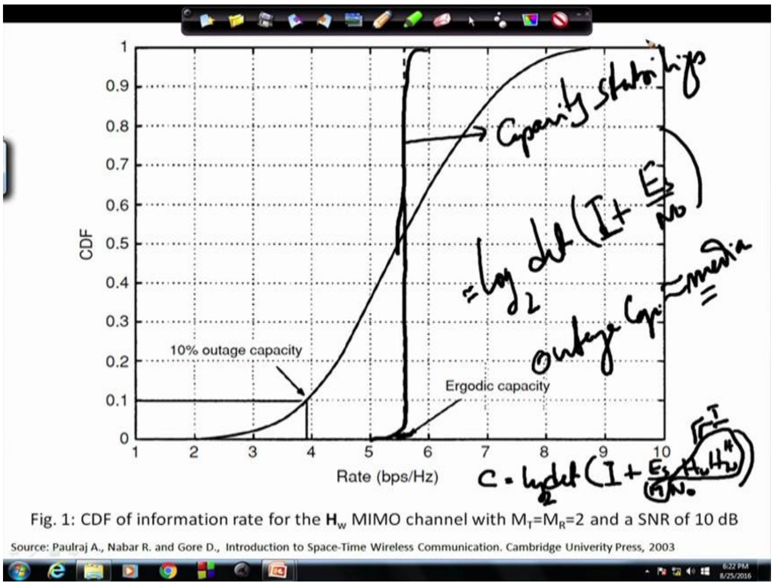

So, now we take a look at the outage capacity. So, we have seen this particular figure before because c is random as we have said, the capacity would be described more by its distribution and at some point which is 10 percentile or p percentile we would say this corresponding value of capacity is the outage capacity c out; that means, in this case we have taken the 10 percent outage capacity. So, what we are seen is that above this is 90 percent, basically this side is 90 percent. That means 90 percent of the time the capacity is in this range. So, this is very very important value and it determines the lowest point in performance of course, ergotic capacity is what we have already described and seen how it is influenced under lower SNR and higher SNR regime. So, we will take a look at the outage capacity performance.

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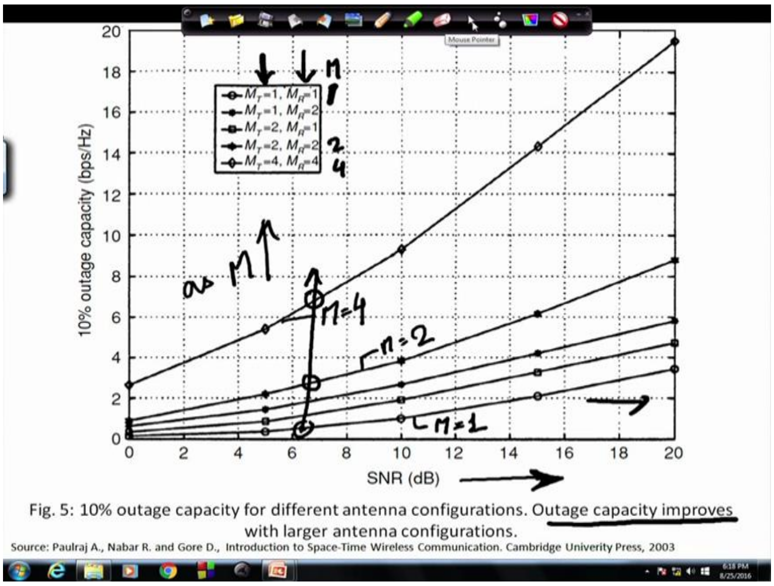

So, this is the 10, percent outage capacity performance for different MIMO antenna configurations, this side is the transmit antenna configurations, this side is the receive antenna configuration. So, as we increase the number of antenna. So, basically M is equal to 1, M is equal to 2, in this case and M is equal to 4 in this case, this is M. So, we have M that means, this diamond this particular curve is M is equal to 4 and this particular star is n is equal to 2 and this particular line M is equal to 1. So, what we have seen is that as we increase M, as M increases what we can observe is as M increases what happens to 10 percent outage capacity that also increases. So, basically the 10 percent outage capacity increases. Clearly this is for M equals to 1, this is for M equals to 2, this is for M equals to 4, and we clearly see that outage capacity is increasing. So, if you are here this is 2, this is more than 4. So, this is one important thing as we increase SNR the outage capacity also increases. So, this is one important observation.

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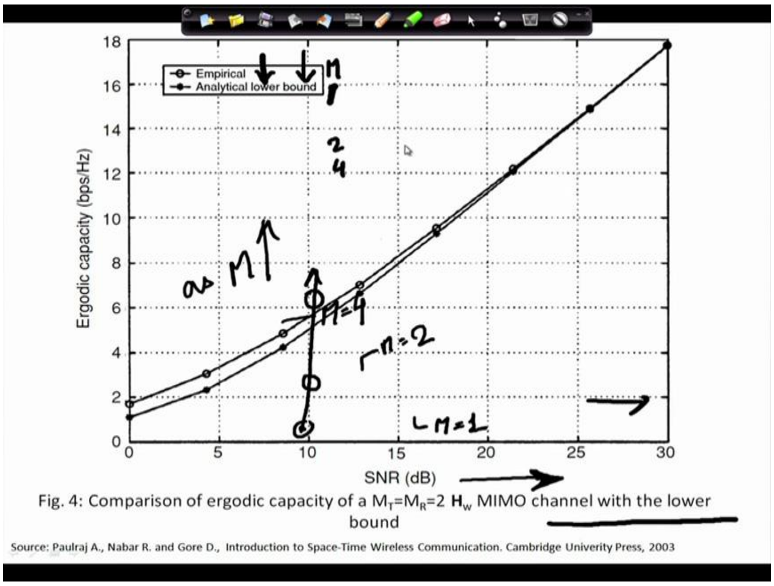

Fig. 4: Comparison of ergodic capacity of a $M_T=M_R=2$ $\mathbf{H}_w$ MIMO channel with the lower bound

Source: Paulraj A., Nabar R. and Gore D., Introduction to Space-Time Wireless Communication. Cambridge Univerity Press, 2003

And if we look at the ergotic capacity what we have is the same over here is as we increase M, here also is the same - this is for M equals to 4, this is for M equals to 2. So, we have almost the same - this is for M is equal to 2, I think this is M is equal to 2 in this particular figure this is M is sorry. In this particular figure we have this one as M is equal to 2 and this one for M is equal to 1. So, what we are seeing here also its capacity is increasing linearly and as we increase m. So, in both the cases it increases with m.

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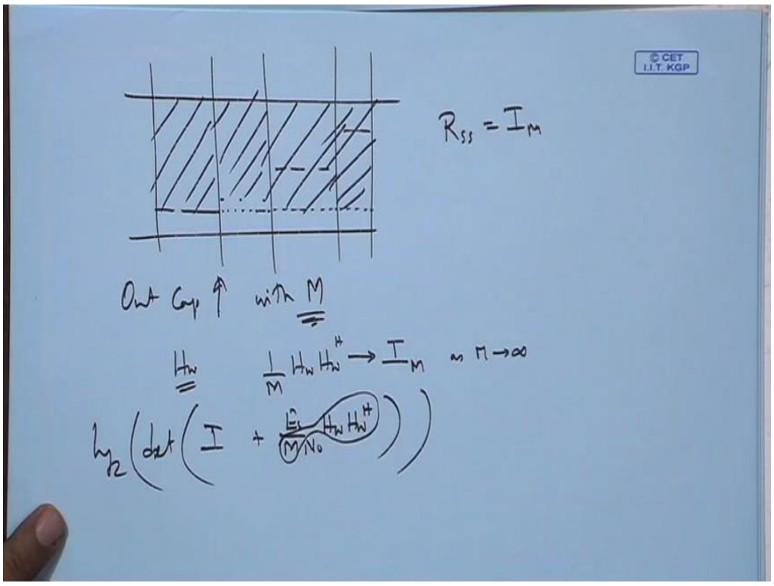

Now, the important thing that we are going to look at is here. So, outage capacity increases now with M clearly. So, now, suppose we have an H w channel. So, for an H w channel we have seen 1 by M H w H w hermitian tends to I of M as M tends to infinity. So, what does it mean? That a capacity expression inside the capacity expression you have E S by M n naught and H w H w hermitian right plus of course, i M r this is expression of capacity for given channel what we have seen is that this tends to i. So, if this tends to i, what is the meaning? The meaning is that the capacity expression is becoming independent of channel coefficients; that means, every time I see measure the value of capacity for a very very large H w channel and I keep changing the channel coefficients the capacity values is not going to be different from each other.

So, under such situation what I will see is, under such a situation what is going to happen if we see this capacity axis most of the time my capacity will be very very close to one value because what we are seeing c is equal to log base 2 determinant of i plus E S upon M times n naught times H w H w hermitian. So, 1 upon M H w H w hermitian if that is equal to i for M being very large that is what we have seen what do we get capacity is log 2 determinant i plus E S upon n naught. So, that is the capacity of a SISO channel. So, this is in the limiting case. So, if it is not a limiting case will be very, very, very close to this value that means, will be very very close to one particular value. So, this distribution which is source spread out will almost be like this that means, under such condition the outage capacity is almost equal to the medium capacity or in other words what we can say is that under this case the capacity stabilizes.

So, one we are saying is that if we are able to provide very large number of antennas and if the link that we are providing is, what we have said is if we have very large number of antenna elements at the transmitter and the receiver side and the links are specially wide; that means, they are orthogonal what we can achieve is almost a stable link which does not vary, the capacity does not vary. So, we are almost able to provide the capacity of additive white Gaussian noise channel. So, that is very very important.

In fact, it is M times the capacity of spatially of additive white Gaussian noise channel so; that means, it stabilizes the link is not fluctuating any more similar situations would happen when there is line of sight you are getting a very high signal strength the channel is very very stable there is not much variability in it, but of course, there are different (Refer Time: 33:00) on the channel capacity we have just seen the situation when there is lower SNR. What we need to see now is the situation when we have very high SNR.

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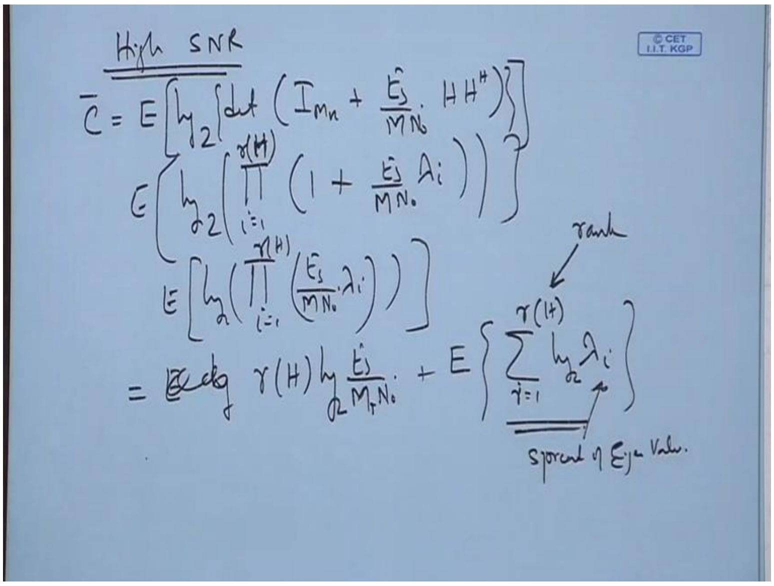

So, we take at the high SNR condition we have seen it once we will see it again in a more exact manner and we will see certain interesting outcomes. So, the ergodic capacity is basically expectation of log base two determinants of i M r plus E S by M n naught times H H hermitian. This is what we are quite used now, this particular expression. So, again determinant being pi of i equals 1 to rank of H times rank of H of 1 plus over E S by M n naught times lambda i, where lambda i is the eigen value of H H hermitian log 2 determinate is replaced by this and the expectation sign. So, when SNR is high when E S by n naught is very high this could be approximated as E S by M n naught times lambda i log 2 determinants.

So, this you could also say is we could write it as expectation of, basically this term is constant if that term is constant you will have r H because this raise to the power of r H times log base two E S by M T n naught this is the constant term plus expectation of sum over I equals to 1 to rank of H times log base 2 lambda i, this is constant, so not of much importance, this is what we should look at. So, what we looking at is the ergotic. So, when SNR is high the capacity is determined by this spread of eigenvalues.

So, it is determined by two things - the rank of the channel and spread of eigenvalues. Again clearly this thing will be maximized when all the lambda i are equal and when it is the full rank. So, if rank is less then capacity is less. So, we would like to have full rank as well as lesser spread of eigenvalues. So, lesser spread of eigenvalues, has multiple meanings - one of the meaning is if lambdas are almost the same, they are equal to then we are having an orthogonal channel which you have seen before and all eigenvalues being same has another influence. So, if you have all same eigenvalues you can easily guess what would be the result when channel state information is available at the transmitter.

So, we have discussed that under such situation when the channel state information is available at the transmitter, we would like to follow the water pouring algorithm. The water pouring algorithm says the allocate power to that particular eigen mode proportional to the eigen value of H H hermitian.

So, if all eigenvalues are same the power to be allocated in all the modes are again going to be the same and; that means, that R ss would lead to identity. So, again we are getting the situation when under high SNR condition, we see that the power allocation would mean would be equal amongst all the antenna branches and that is the same whether you have channel state information at the transmitter or you do not have channel state information at the transmitter. So, again what we say is that when channel state when SNR is very very high extremely high under that condition the gap between no CSI transmitter and with CSI transmitter again reduces.

So, in summary we have two points that rank of the channel determines the capacity and spread of eigenvalues, if eigen values are very similar you have very similar power involve of the channels under that condition the capacity of CSI at transmitter and no CSI at the transmitter would again be the same.

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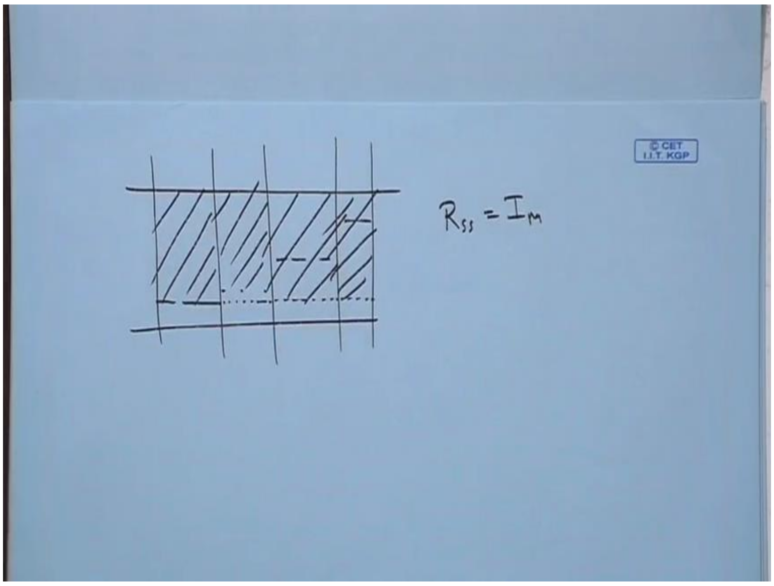

So, I would briefly give you an explanation. So, we said that this is water pouring algorithm when these channel state information available at the transmitter and we said if this is inverse of SNR levels this is the amount of power that is allocated in each of the links. Now if all the eigenvalues are the same, then SNR is also the same, in that case the power allocated would also be the same - power allocated, in all them being the same means R ss is equal to I of M. So, there again we see that if the equal power is allocated under such conditions; that means, when you have very high SNR all you need to know is the rank of the channel and you can give equal power to them. Whereas when SNR is low and when eigenvalues are not equal in that case it is better that you identify the rank of the channel and allocate power to those modes where the eigenvalues are non zero or where eigenvalues are significant and you give the power depending upon the strength of the eigenvalue.

We will also see another important aspect in the next lecture where we will talk about the impact of large M or as H w channel for M being very large what happens on outage capacity.

Thank you for attending this particular lecture.