Fundamentals of MIMO Wireless Communication
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Lecture - 38
More on MIMO Channel Capacity

We continue our discussion on MIMO Channel Capacity. In this particular lecture, we look at the capacity of SIMO, MISO channels as well as that of random channels, we will also see this situation when, there is low cellular approximation and when there is high cellular approximation you see what happens the MIMO channel and that is going to give us insight into what has to be done when the channels SNR is low or when the channels SNR is high with that.

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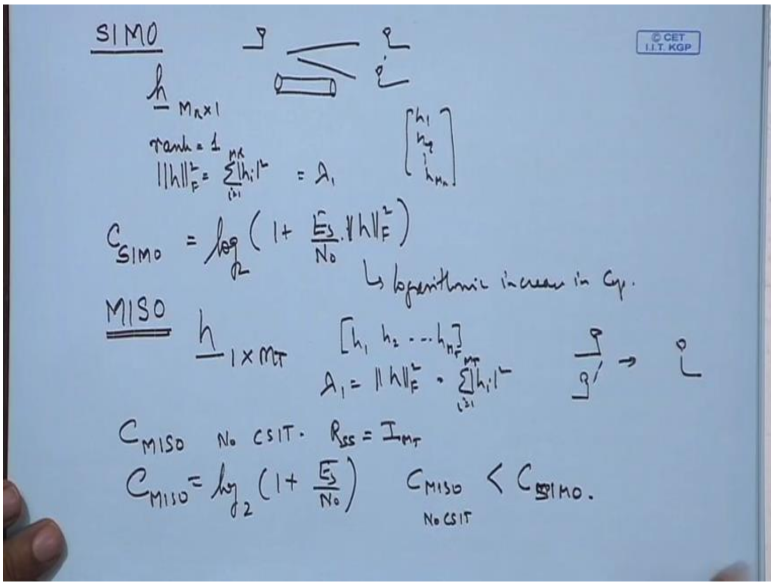

Let us take SIMO channel, let us consider a single input multiple output channel. So, single input multiple output one antenna at the transmitter multiple antennas at the receiver. So, basically it is a receive diversity kind of situation. So, if you have this particular situation where, h vector is basically MR plus 1. So, if it is a single vector. So, definitely the rank of this is matrix is equal to one and we have frobenius norm of HF is equal to some of h i squared i equals to MR and this is basically the lambda of the channel.

So, this one could see as the lambda is basically Eigen value of h h i hermitian, which is basically the this particular term, the does not have any other Eigen value and this is a rank one matrix, one can easily see because its one column other ways also are they have to prove there it is a 1-1 matrix suppose, C SIMO channel capacity of a single input multiple output channel is since there is only one pipe this is we have been saying the diversity more is log plus 2, 1 plus e s and n naught times mode HF square. So, mode HF square is sum over all the channel gains. So, the there is capacity increase, but definitely there is special there is only logarithm capacity increase with MR. So, as MR increases this increases. So, there is increase in a capacity over here and that is it. So, basically there is only logarithmic increase in capacity and even if, we have channels state information at the transmitter you cannot do much because there is only one transmitting antenna.

So, this is single input multiple output case. So, when we go in multiple input single output scenario in this case h vector is basically 1 plus MT 1 cross MT because, you have h 1 h 2 in this format in this case we would have h 1 h 2 up to h MR this was the way we wrote MT this is the particular notation and again lambda 1, there is only 1 lambda is frobenius norm squared of h and that is equal to sum over h i mod square i equals to 1 2 MT. So, here in case of a multiple input single output; that means, we have multiple transmit antennas going in this direction with single antenna.

So, if we do not have CSIT; that means, no channel state information of the transmitter we would put the covariance matrix of the signal as I MT because we would have no specific direction it be sending them with equal power of all of them in that case, what we want to get is log base two one plus e s by n naught this will be capacity of the MISO when, there is no channels to give information.

See if we compare these two terms; what we are going to get because we want to get HF square and again there will be ES by MT. So, they will balance out where as at the receiver there is no MT in the denominator. So, what we are going to get is what we could write is C MISO no channel state information of the transmitter is less than or equal to max you can get it some case as SIMO.

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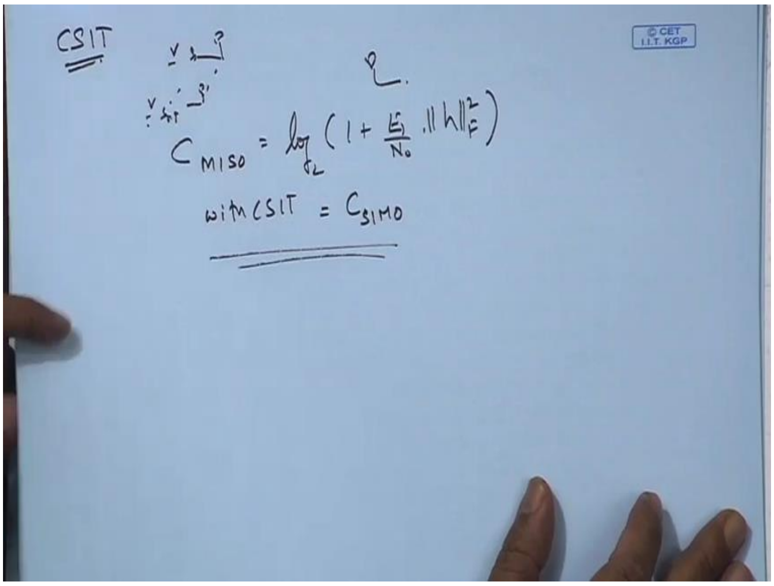

Whereas, if we have channels state information if there is channel state information at the transmitter then of course, you have this case we have s 1 plus MT, but then there is something you are doing multiplied at some point and there is 1 receive antenna. If you remember we have studied the dominant Eigen mode and we had also studied transmit MR c and we said that this dominant Eigen mode is basically one of the forms of transmits MRC along with MRC at the receiver. So, if we will have this particular configuration we can think of MR c or u n m c dominant Eigen mode transmission in this case we would have a capacity of a MISO system given as log base two one plus e s and n naught times HF squared.

So, what we see is which channel state information because we would remember the capacity of or the expression of the SNR for MR c at the transmitter or MR c at the receiver is basically this e s by n naught times h squared which gets reflected here also. So, which c s with channel state information you can write this is equal to c SIMO. So, this is just to remember that when you have single input multiple output, this is the expression of capacity when you have multiple inputs single output and when there is no channel state information the capacity is similar to that of SISO link. So, it is just like a SISO link this is not that you can do, but when there is channel sate information in the transmit the capacity is same as that of a SIMO link. So, this is something which we should remember we move on now from this point to look at the random channel we would remember that we have started with channel condition which is deterministic; that means, given a channel coefficient.

So, now we would like to look at the case where this channel coefficient should be and what would be the meaning of that what the impact of capacity is on when we, look at it averaged over all over several such realizations of the channel.

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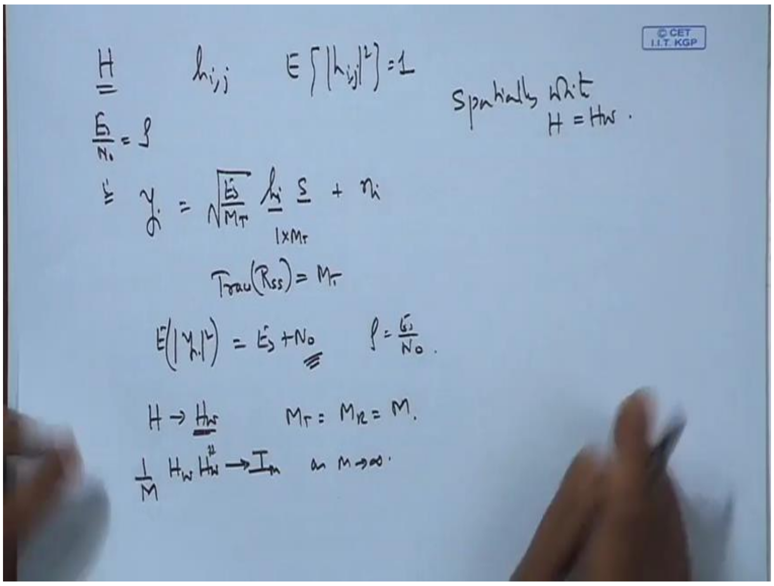

So, let us define that, we assume that elements of h basically the h i j elements are normalized. So, that expected value of h i j mode square is equal to 1. So, that is the first normalization we will look at and we will use e s by n naught as the variable row. So, the signal at the ith receive antenna that is y i is equal to root over e s by MT h i is the vector which is one cross MT and see we have talked about the ith receive antenna. So, it is getting transmitted from all the antennas at the transmitter plus n i and we have to ensure that RSS trace of RSS is equal to MT. So, that there is total power of e s that is used, if we take e of y i squared we will get e s plus n naught. So, this is the average SNR at any one particular antenna e s plus n naught.

So, the average SNR is row which is e s upon n naught. So, next new move on further with this particular channel and we say that h is basically h w. That means, specially y and we also have MT is equal to MR equals to m. So, when we have h as a h w MT equals to MR equals to m we would have one upon m h w h w this terms to i n as n tends to infinity. So, let us look at this if this is true or why it is like this. So, if it this is h w dimension.

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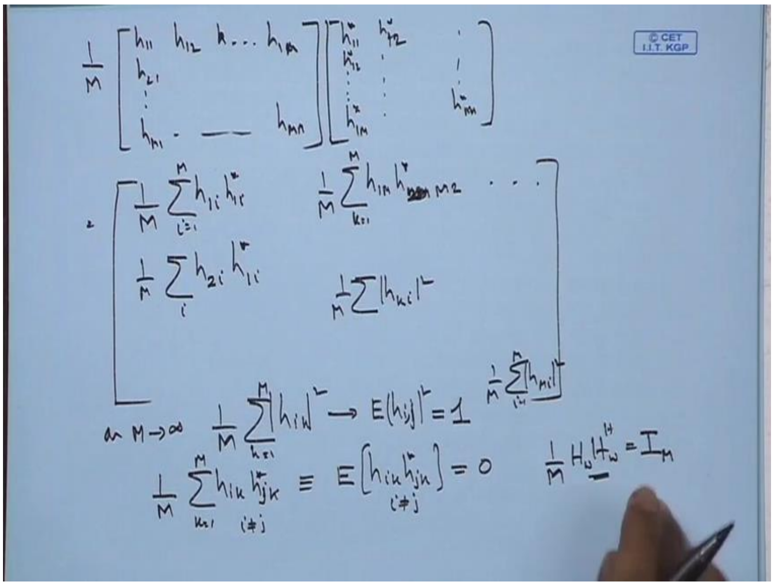

So, if we have one upon M the particular look at this particular expression is h 1 1 h 1 2 dot, dot, dot, h 1 c I would write it all over h 1 n h 2 1 h m 1 h n M times h 1 1 star h 1 2 star up to h 1 m star h 2 1 star like that h n n star. So, when you do this product the big matrix that gets formed as elements 1 upon m i equals to 1 to n h 1 i times h 1 i conjugate right in the first one the next, one would be one upon n i or k equals to 1 to n h 1 n times h n 2 by h this gets multiplied over here. So, we have h 2 m h 1 m gets multiplied by h 1 two sorry h M 2 conjugate and. So, on this particular element would be one upon m sum over h 2 i, if I put the index i times multiplied by h 1 i conjugate.

So, will be filled with such components where this diagonal elements between sum over h k i or h i j mode square if you look at this one upon M. So, this is what we are going get and finally, it will be h m i mode square i equals to one to m one upon M. So, that is that is the term we are going to get. So, as M tends to infinity if we look at this first term it is one upon m sum over h i k any one of the terms h i k squared k equals to one to n the ith row you can see of the ith column. So, as m is very, very large as m grows this is basically the mean value and we have already stated that we would like e of h i j squared to the equals to one this is one of the constraints that we put; that means, on an average the channel each channel has a average channel 1. So, with due to law of large numbers this one becomes this can be approximated at the expectation of h i j mode square and which could be said equal to one. So, all the diagonal elements would be equal to 1. If we take the non diagonal elements we are going to get one upon m sum over k equals to one to m h i k and h j k conjugate for i not equals to j.

So, this would be equivalent to expectation of h i k, h j k conjugate. Where i is not equal to j, we this particular term because these are independent; that means, we are taking specially white channel. So, in that case seems these are all independent where this could turn out to be a 0. So, because by law of large numbers we could write this as expectation, what we have is all the off diagonal elements will turn out to be to be 0. So, we had over here specially white. So, basically h is equal to h w so; that means, its specially white all are independent components. So, if all are independent components this term is equal to 0. So, all the non diagonal elements should be 0 and all the diagonal elements would be one including this particular term. So, what this whole thing? So, this is particular product what we started with you can write one upon m h w h w hermitian is equal to i of size n.

So; that means, this again proves that this is orthogonal. So, this is what we have. So, using what people write.

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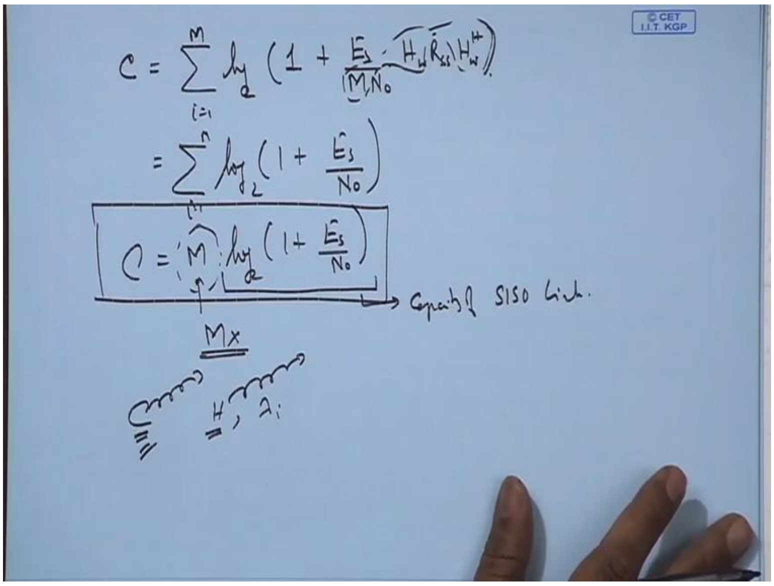

The capacity c is equal to we had sum of i equals one to m log base to one plus e s upon m by n naught times h RSS h. So, we had stated that we will put RSS to be identity matrix and RSS trace of RSS is equal to MT. So, that is what we have said because there is no preferential direction of the channel, in that case this become identity matrix. So, we have h hermitian h and this is h w. So, one upon M times h hermitian h w is also equal to identity that is what we have shown over here in this particular expression. So, what we have is basically sum of i equals to one to n log base two one plus e s upon n naught and if we see inside there is no variable of i. So, we could write this as M times log base 2 1 by 1 plus e s n naught and you could recognize this as the capacity of SISO link and this is M times right. So, what we see is when we have specially white random MIMO channel and m grows very large we have a linear increase in capacity over that of a SISO link.

So, if again we receive the same results. So, we are repeating the result several times in a very general sense we are saying that the capacity of SISO is enhanced by linear increase of capacity. When it is on orthogonal channel we achieve the same when, it is a specially white channel and m grows very, very large this summations for one upon n h hermitian h w h w hermitian the component becomes such that the diagonal elements are identity whereas, non diagonal elements are 0.

So, again that leads us to the situation where the capacity that we achieve is M times the capacity of SISO link, which is one of the most important things that MIMO gives us and because of this this particularly topic has becomes. So, important for us, with this particular message we like to remember this particular message and we would like to move on further. So, now, what we would like to see is we have taken MIMO channel and we have taken random channel,, but what we have seen is when n is very, very large is the particular expression that we have a received. So, where as if you look at this c given a value of h its depended upon h coefficients when m is not very large. So, basically c is depended upon h and c is depended upon lambda is. So, for any given realization we have 1 value of c. Since h is random basically h keeps on changing values c would also keep on changing values.

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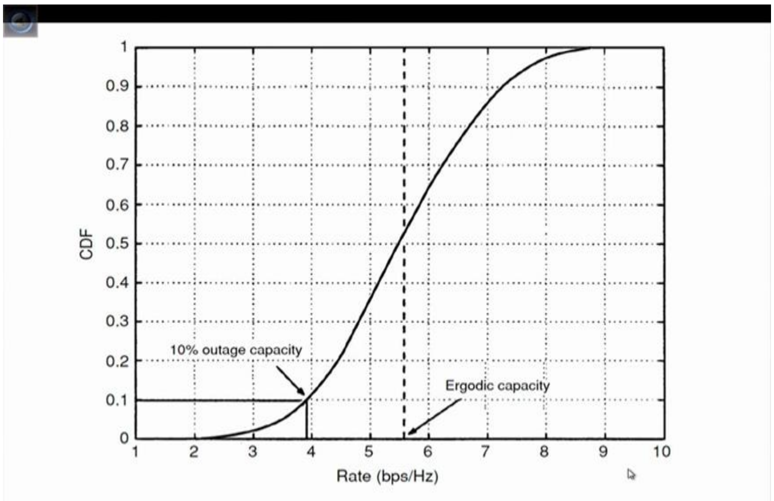

Fig. 1: CDF of information rate for the $\mathbf{H}_w$ MIMO channel with M_T=M_R=2 and a SNR of 10 dB

So, in that case how would be measure c is very, very critical. So, what we have is i-i will show you particular graph. So, this particular figure shows us the distribution of the capacity laws since we have seen that h is random c is a function of h definitely what we have is, what we have over are here is distribution because c itself is random c is random we have plotted the cumulative distribution function of c and whenever we plot the cumulative distribution function the curve would look one of the typical difficult curve would look like this this is for the case.

Where MT is equal to MR equals two and SNR of ten d b again the reference is we have pointed out here. So, in this we have to note 2 important things; one is there is a ten percent high point which is two less the outage capacity that is very, very important what it means is from this if i read particular value as let us say 3.9 bits per second per hertz what it means is that ninety percent of the time this is 90 percent. 90 percent of the time this system provides the capacity value which is more than 3.9 bits per second per hertz along with this there is also another important parameter which is the ergodic capacity or the expected value of capacity for basically c bar. So, this point is c bar the average value of capacity this particular point is expected value of capacity because we are getting equal each realization of channel we are getting a particular value of c so.

When change the channel because of time variability we have studied this in the early part. So, whenever channel changes because of time variability we get a different value of c. So, if we collect the large number of cs what will see that c is also random variable. So, to characterize the random variable we can do it by the c d f or the cumulative distribution function and two important numbers. If you can take instead of d f c d f we can take the expected value which is also known as ergodic capacity and we could also take outage capacity which we have just explained.

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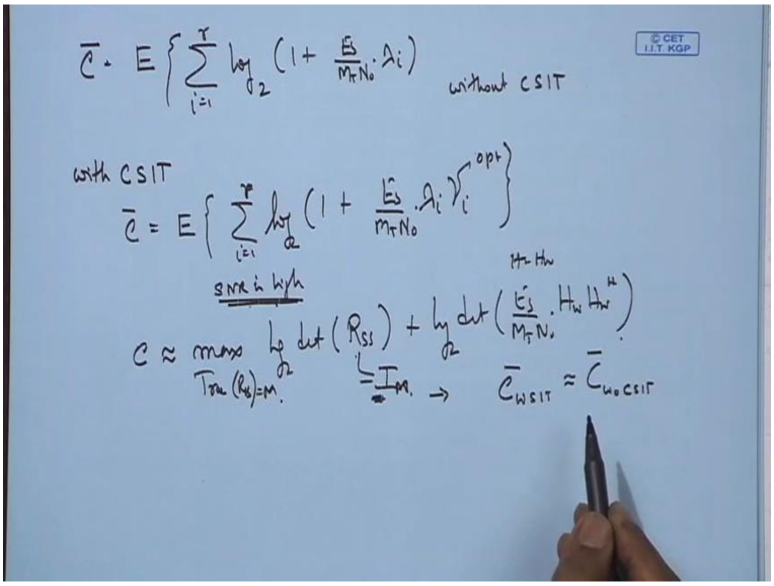

So, moving further this when we look at the expression of a c bar; that means, it is the average value of capacity it is basically expectation of sum over i equals 1 to r. So, we are taking a general channel log base 2 1 plus e s upon MT times n naught times lambda i which is the ergodic value of h h dimension. So, this is without channel state information at the transmitter. Now if there is channel state information with CIST which channel state information at the transmitter we would write c bar as the expected value of i equals 1 to r sum over log base 2 1 plus e s upon MT n naught times lambda i; however, there is a comma i optimum which we have seen how to arrive at in the previous lectures what we will see is again when suppose we take the condition when SNR is high when SNR is high.

So, when we have very high SNR, we could say that c is approximately equal to maximum. So, basically we are neglecting this we look at the other terms. So, sum of this is basically log of the product of this. So, log of product of this basically log determinant which expression we have used to could write it is log two determinant of

RSS times log base 2 determinant of e s upon MT times n naught times HF or specially white we have h w h w hermitian and of course, it is maximized over trace of RSS being equal to m.

So, when we look at this particular expression; that means, we taken for a high SNR this expression is maximized for a given h w is when RSS, when this is equal to identity matrix right. So, when this is equal to identity matrix and what do you get to be get all the lambda is sorry. So, this is this is maximized for IIN; that means, when there is equal power on. If we have to maximize with respect to the power distribution across all the; all this signal sources, when this comes arithmetic quality again we could go to that and show this.

So, when we are saying that this equal to i n. So, this would in turn mean that we are putting equal power across all the branches; that means, lambda i gamma i are also almost equal to each other and that is also the case, when there is no CIST. So, what we are saying effectively is under high SNR and when h is equal to h w and it is high SNR and h is equal to h w. We could say almost that c without channel state information at the transmitter is almost equal to channel without channel state information at the transmitted average the average channel capacity of the ergodic channel capacity.

So, as n increases we can say that the capacity gap between these two increases, but with increase in SNR. When SNR is high the capacity gap between these two decreases that that we will are shortly see in in one of the results yeah.

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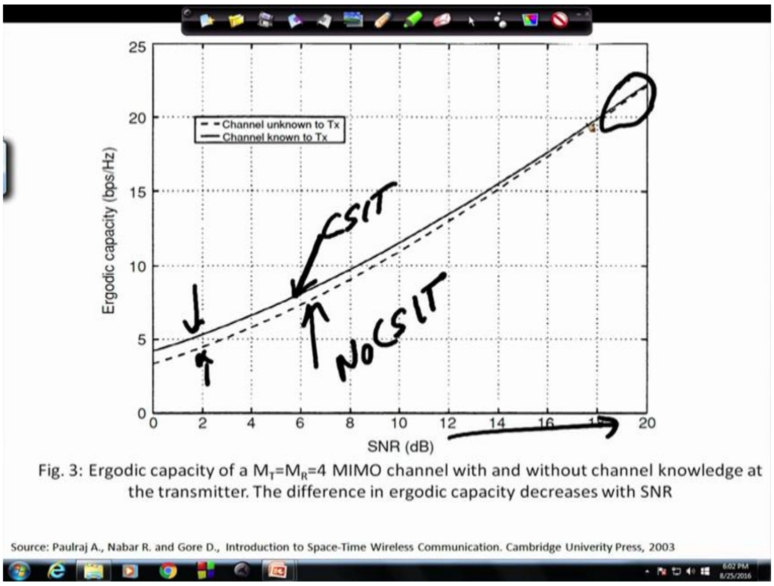

So, this particular results shows the ergodic capacity when channel is known at the transmitter. So, that is the solid line CIST and this is ergodic capacity for no CIST. So, there also we see as SNR increases this gap between these two decreases becomes better and better for higher values of SNR we move further and we will see some more important results we will derive some important results further and we will give you some important analysis that is again very important.

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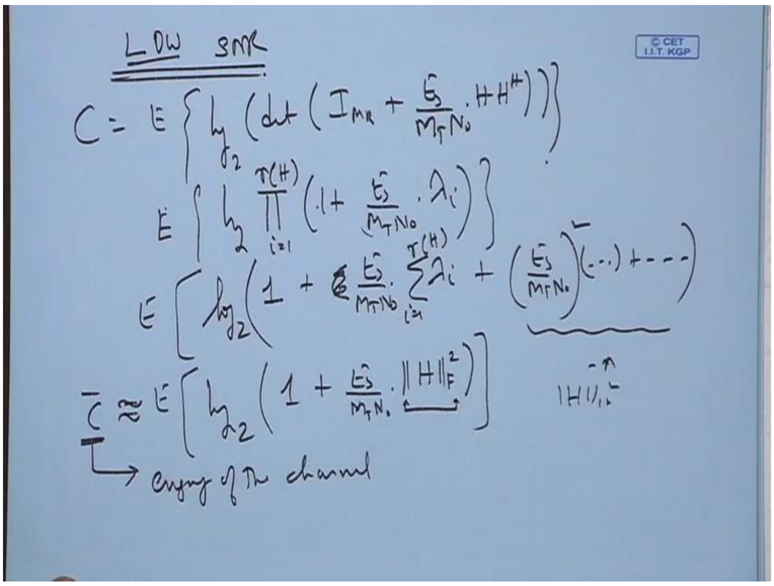

So, one approximation which we make is one that we need to study is low SNR regime. So, when SNR is low. So, if you look at the expression of capacity c is given by or the ergodic the capacity is expectation of log base 2 determinant of i MR plus e s upon MT times n naught times h h hermitian this is the expression.

So, this in other words you could write it as log base two product of one plus e s upon MT n naught times lambda i i equals to one to rank of h. So, we taking the expectation, right? So, if we have to expand this particular product we could write this you could do it yourself you are not doing it exactly we could set it as one as one of the terms we would get sum over you get basically e s upon MT n naught. So, basically this when one gets multiplied with this, when one is multiplied by second and so on so forth, we will get some over lambda i i plus one two rank of h plus we will to get e s upon MT by n naught squared times something plus, plus, plus, plus, plus and this is inside the log we have taken the expectation of that.

So, again we could say it is expectation of log base 2. So, when e s by m n naught is very small these terms can be neglected and we are left these terms. So, these terms would mean one plus if we take a look at this it is basically e s by MT n naught times the frobenius norm squared of h approximately. So, in the low SNR regime; that means, when SNR is low these terms are neglected what we see is that the ergodic capacity is determined by the energy of the channel which is indicated by this term; that means, the total power this gives us a hint that under low power or the low SNR conditions, What we need look at base the for frobenius norm of h and this frobenius norm of h would give us what? This frobenius norm of h. if you would see was which appearing in the error probability expression.

So, when we had this error probability expression there this frobenius norm of h was reduced in in this exponent. So, we had HF squared to the part of m minus n and so on. so forth. So, basically what this description hint is when SNR is low, we have to strengthen this the more we can strengthen this more energy we can bring into the channel the better it is. So, we would rather go for a diversity mode is the hint that this particular expression is has given we would like to stop this lecture at this point and move on with further discussion the next lecture in the next lecture.