Fundamentals of MIMO Wireless Communication
Prof. Suvra Sekhar Das
Department of Electronics and Communication Engineering
Indian Institute of Technology, Kharagpur

Lecture - 37
Capacity of Channel known at Transmitter

Welcome to the lectures on fundamental of MIMO Wireless Communication. In the previous lecture we have seen the capacity of MIMO system, when channel is not known at the transmitter. So, now, we will take look at the capacity when, channel is known at the transmitter. So, basically we will with this figure that we used in the previous lecture.

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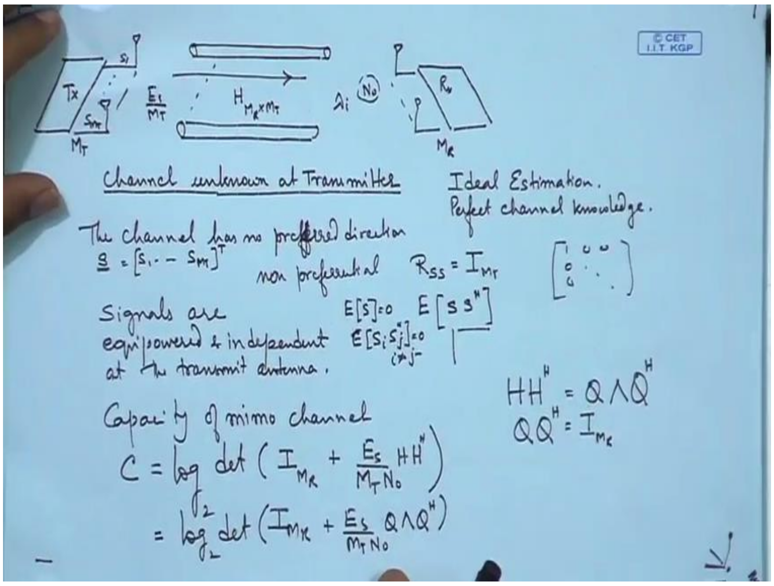

We now have a feedback in the system.

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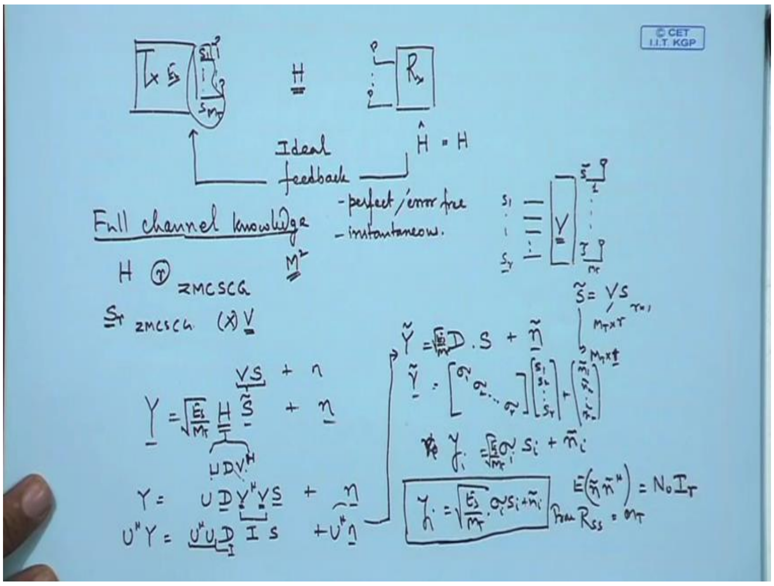

So, in brief we could draw an image transmitter block in general and a receiver block in general, and that is this channel in between the transmitter and the receiver. This channel is estimated at the receiver usually write this H cap, but if it is ideal that we would say that it is h. This H is fed back where is feedback system which is again an ideal feedback system an ideal feedback system would mean that there is perfect; that means error free and it is instantaneous, which is never a reality. So, these assumptions are never true in practice. In practice you will always get some estimation errors as well as you will be getting delays because that is natural. When signal is transmitted from a source by you required certain time for the frame duration if required certain amount of estimation signal processing time further also you would require time to send back information from the receiver back to the transmitter. Which will again we used during communication.

So, in that case one might question that why such an analysis is is being done, why is it? So, important the importance of this is to find out what if at all you have the best channel information at the transmitter, what the best can you do. So, although this is not achievable in practice at least this going to give us hints towards what is the gain that can be achieved. If such a thing is realized which would further tell us that whether we need do it or not or what is the loss. If you do not have the exact channel information could be further investigated using these knowledge again practical schemes could be devised which would take care of errors as well as which would work in certain amount of feedback.

So, with this background that although what we are doing is really of theoretical interest there lies lot of practical message behind it and lot of practical designs could be driven or could be motivated based on the results that could arrive out of this particular analysis. So, here the again it is assumed that full channel knowledge is available full channel knowledge means complete H H is the matrix. So, if it is the m plus n matrix so; that means, there are m squared elements m squared elements and further that each of these elements are continuous which will be of full value. So, which is not possible which is technically not possible, but still it is idealist assumption as i have already described and we would also consider that it is rank r; that means, it is not a full rank matrix which is, in general we want say that H is of rank r it is 0, mean circular symmetric complex Gaussian, we have described what is the meaning of 0 mean circular symmetric complex Gaussian and this vector s r is basically s 1. We usually have s m t, but now we are saying that no this is not what we are doing instead if i need draw it again channel number one up to channel number m t right. There is some processing unit before this and here there is some up to s r number of symbols which are to send from the transmitter.

So, s r is the vector which will be used again this is the 0 mean circular symmetric complex Gaussian. That is what we will assume this will be multiplied this will be multiplied vector v with the matrix v this x multiplied with the matrix v. So, there is some processing with this matrix v at the transmitter prior to transmission. So, what goes into this is basically s tilde which is equal to v times s. So, if s is r plus 1 this has to be m t plus r those we cannot do this. So, end of it we want get m t plus m t plus 1 as the size of s tilde. So, will have finally, s tilde one to s tilde m t this is being transmitted.

So, this signal at the receiver would be our normal form y equals to H s tilde plus noise y indicating matrix and vectors. So, s tilde is basically v times s plus noise. H we could factorize into singular value decomposition we have described that before u d and v hermitian yes it is u d and v hermitian that is how you could break into d is the diagonal matrix composing of the singular values. So, we have r singular values v and u are unitary matrices containing of vectors which are of length 1. So, with this in mind; that means, H could be broken down into this factors of diagonal singular values unitary matrix over a unitary matrices u and it is s is what is being transmitted is pre multiplied by v. So, what we have d times v hermitian times v times s this is what we have u. So, we hermitian time v is basically i d is the diagonal matrix s plus eta. So, just by pre coding or preprocessing with v we have identity here that it is a diagonal n there is a vector so; that means, each of the diagonal entries get multiplied by each corresponding s.

Now, at the receiver if you would multiplied by u hermitian what we are going to get u hermitian u, and u hermitian times n. So, this again u hermitian u is identity matrix. So, this would lead us to y tilde that is what is being received we go to this point is equal to d times s plus eta or noise tilde. So, u is a unitary matrix, even after processing the noise variances is not going to change. So, we simply change notation to n tilde. So, if we look at this what is happening over here is t is a basically the singular value 1 singular value 2 up to singular value r and s is s 1 s 2 up to s r and this again would be n tilde one n tilde two up to n tilde r. So, what we have as y tilde is basically each of these lines y i y i is equal to sigma i times s i plus n tilde i right this is what we are going to receive and of course, e of n tilde n tilde hermitian we take this is going to be n naught times identity matrix of size r this is what we have indicated over here.

So, this is what we are writing. So, we have this particular a received expression we would also have to satisfy that R ss is going to be the trace of R ss or trace of R ss should be m t; that means, the total transmit power should be m t that is what we should have. So, we get this expression what this expression means is that of course, we have to just modify this expression little bit. We have root over E s on m t at all stages we should have root as over as E s over m t indicating the total transmit power is constant to E s and. So, root over E s up on m t is there. So, here also total E s on m t we write this expression clearer y i is equal to root over E s upon m t times sigma s i plus n tilde i; y i is ith received signal and s i is ith transmitted signal.

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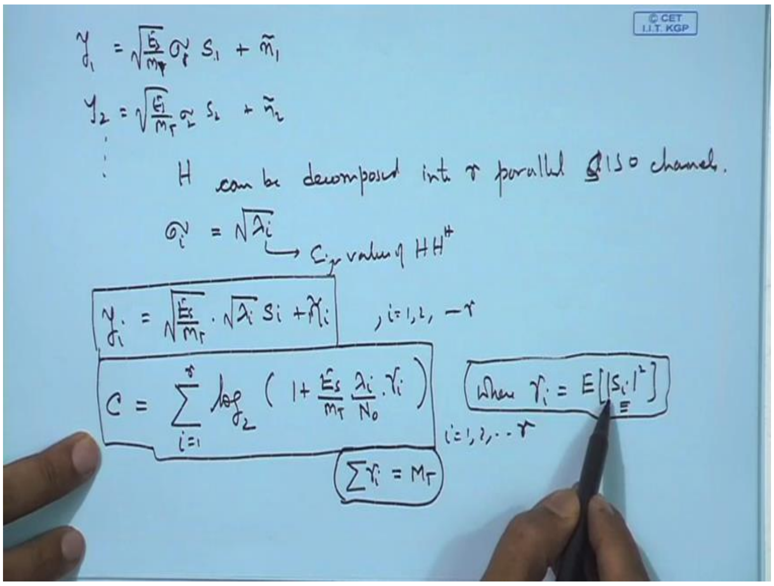

So, what we see is in this form y one contains only s1 E s on m t or E s upon m m t, we have m t antennas over here sigma one is the singular value corresponding to the first singular value times s1 plus n tilde one this is the variance of n naught. So, y one does not have anything else other than s1. So, there is no interference between the symbols in this case although we have seen initially that when, you send MIMO transmission whenever send something is here everything is received in all antennas. So, first antenna is going to receive s1 to s m t; however, because of the preprocessing which has been introduced here we are not using we are not going get any such interference. So, what we have done over here pre multiplied by over here.

So, that is there is some pre multiplication because of which this interference is gone and we are receiving this. So, y 2 is equal to root over E s by m t sigma 2 s 2 plus n tilde 2 and so on. So, you have clearly got them separate out there is no interference and we can say that H can be decomposed into r parallel channels SISO channels right and if you look at what is sigma i sigma is equal to root over lambda i where lambda is are Eigen values of H H hermitian right. So, we could write the expression is y i is equal to root over E s upon m t times root over lambda i s i plus n tilde i - i equals to 1, 2 to r.

So, we have also achieved parallel transmission and in this case we could write the capacity of the MIMO channel as sum of i equals to one to r log base two one plus E s upon m t times we have lambda i because this is square root of lambda i. So, the square this would be available lambda i upon n naught times gamma i where gamma i is equal to e of s i mod squared for i equals to 1 2 up to r. So, in the previous1we had R ss now in even after R ss we had i over there because each of them had the value of one, but now we are saying that no instead of having each of them having in the same value, which we have done the case for more channels information at the transmitter now we have gamma i indicating that there is a certain amount of power to be given to each of the received each of the transmitted signal; however, subject to the constrain that sum of gamma i is equal to m t. So, this constrain has to be maintained.

So, now what remains for us to do is this good as a capacity is to find what is the maximum value of this expression c which could be defined as the so called capacity of the system when channel knowledge is available at the transmitter. So, till now we have said that we could give different powers to it because channel knowledge is available at the transmitter. So, we have to work out what kind of powers can be given because each of the channel modes can be accessible by the means of a pre coding we could rather write the expression of capacity or mutual information maximization problem.

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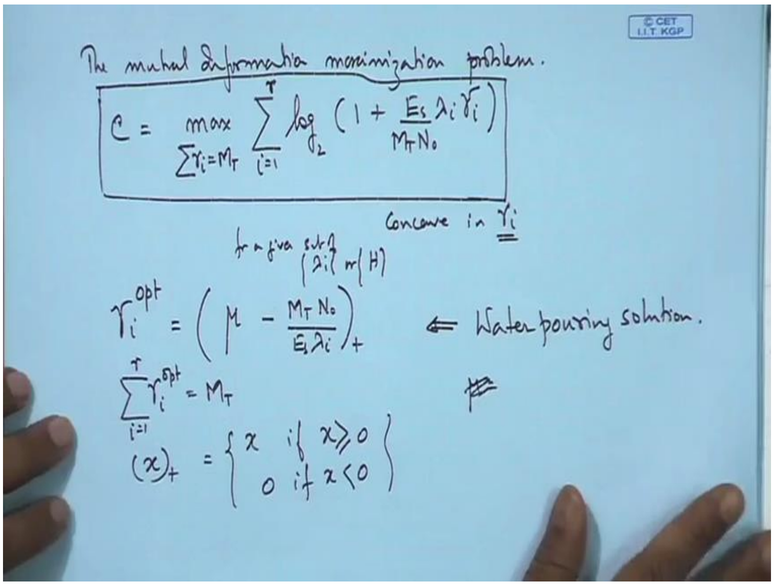

We could write the mutual information maximization problem can now be stated as c is equal to the find the maximum value of i equals1to r log base of 2 1 plus E s upon m t n naught lambda i plus gamma i subject to the constraint that sum of the gamma i is equal to m t.

So, this ones we solve this we would find the values of gamma i which would lead to max value of this looking at this expression this is a log function. So, log is concave in gamma is now this is the characteristics of the channel. So, we are not saying that we want to find what is the property of lambda i which we did orthogonal channels. So, we are not assuming orthogonal channels because we have already taken the rank r. So, we are saying that given set of lambda values that is given a set of Eigen values. That means, for a given set of lambda is or for given set of H can i find gamma is which would maximize it. So, look at the maximization problem you can derive this, but what the result is is what is more important as of now the optimum values of gamma i can be written as sum mu sub constant minus m t n naught by E s lambda i. We indicated the plus and we also have the constraint i equals to one to r gamma i optimum, opt is indicating optimum is equal to m t and mu is the constant and this plus; that means, x plus is equal to x if x is greater than or equal to 0 and 0 if x is less than 0.

Now, this solution is known as the famous water pouring solution what we will do is the exact steps for going from this to this would be written down and it will be attached along with the videos. So, that you can follow through the steps and arrive it this expression, the expression mu that is usually done is we want give it as. So, there is algorithm by which we will show i-i will show it later, but before we go to the algorithm, let us try to see the meaning of it and try to understand.

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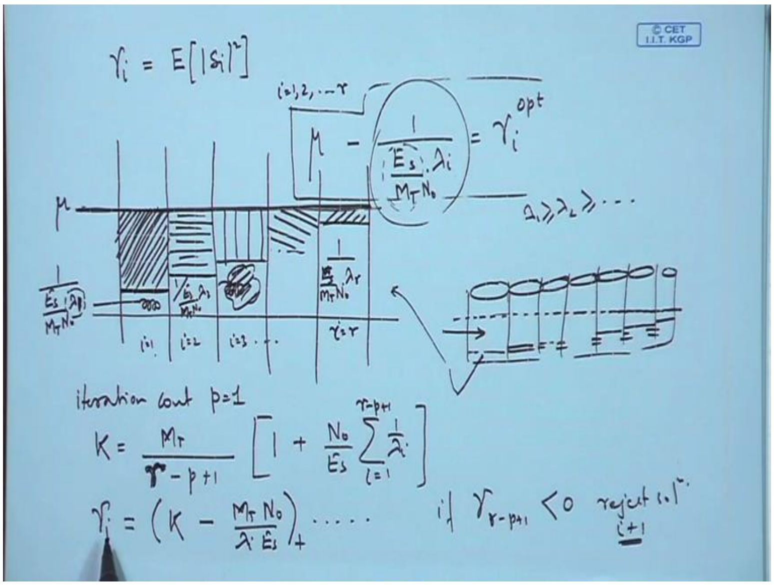

So, let us assume as of now that mu is some constant if we look at the expression of gamma i opt. So, what is gamma i gamma i is e of mod s i squared. So, that is the ratio of the power on the ith symbol compare to all other symbols. So, if i have this basically i am talking about power distribution amongst the different transmitted symbols i equals1two up to r indicating the different modes of the channels.

So, when we have this we could write this as mu minus1upon E s by m t n naught times lambda i. So, what is this? So, this some constant take away one upon what is this E s by m t is the equal power multiplied by lambda i lambda i is basically the channel strength you can say because lambda i is the Eigen value of the ith Eigen value of H H hermitian or the ith mode of the channel.

So, now if we take this and try to draw a figure and say that this is i equals to 1. This is i equals two i equals to 3 up to i equals r and we have these bins this power bins these are power bins and say we have some levels which is known as the mu. So, this is the total power that is a mu, one upon this right is basically if I would write over here lambda i is the first Eigen value lambda, one is the first Eigen value now we have earlier said that lambda one is greater than or equal to lambda 2 is greater than or equals to and so on, because lambda 1 is the is the highest value rest of it is constant so; that means, this is the smallest value right.

So, the second Eigen value lambda 2 is smaller lambda than lambda 1; that means, this value is bigger than case of 1. So, we have it here this is still remaining constant. So, this indicative of the amount of the power that we are going to give so this is gamma i optimum to be ith symbol. To the first symbol you want to give mu minus1upon this value for the second it is mu minus 1 upon this with lambda 2 and what we are saying this lambda one is bigger than lambda 2. So, this number this whole this big number is smaller for lambda one that lambda 2. So, basically this is the part which is E s upon m t n naught times lambda i one upon that lambda 1. So, this is1upon E s by m t n naught times lambda two and so on.

So, basically like this up to lambda r. So, this is 1 upon E s upon m t n naught times lambda r. So, what we have effectively is the remaining portion; that means, mu taken away this value is the amount of power that could be associated with the ith mode of transmission. So, it is the inverse of SNR right. So, or in other words you can say whichever channel has a large value of lambda i, you are giving more power to that to that particular string and if we look at what is happening as if these are buckets with connection at the bottom and one side pour watered the water gets filled up in each of the basket according to the amount of volume that is left unused and the volume that is used up that is this space and this space is basically proportional to the amount of the noise that particular bucket is having over a amount of sand that bucket is having. So, rest of the bucket is filled with water and the water across all the buckets is the same that is why, it is as if you are pouring water on sequence of buckets can imagine it to be sequence of buckets on which you are pouring the water and they are connected at the bottom.

So, these are connected at the bottom. So, when I pour water and suppose we have some amount of sand filled him them in that case the level of water would the same in all the buckets, but the amount of water that will present in the buckets would be the level of water taken away the amount of space which is not available. So, the amount of space which is not available is inversely proportional to the SNR or the amount of water that goes into is proportional or it is related to the SNR.

So, we can clearly see as lambda one is bigger than lambda 2 the SNR is a to noise is bigger or the relative noise is smaller the amount of power you are putting in this is more. So, more the channel strength more the amount of power that you put in and this has to be done in the interactive manner i will give the algorithm very soon. So, this is1of the well celebrated water pouring algorithm as we have already indicated in this particular expression over here or what we have written over here and in order to arrive at this there is a particular sequence of steps there is usually to be taken. So, the weight is done is you would set the iteration count p is equal to one and one has to calculate capital k given by m t upon gamma minus or m t upon r minus p plus 1 plus 1 plus n naught upon E s sum over i 1 upon lambda i i equals1to r minus p plus r r is the rank of the matrix r is the rank of the matrix.

The power is allocated to the ith sub channel and calculated as gamma i the power you have to give is equal to k minus m t n naught upon lambda i times e s. So, this is the amount of power that is given and this is the expression of power which we are taking about and the explanation of this we have already given and ones you do this. So, basically it is in plus. So, if it turns out that gamma r minus p plus 1. There is a particular iteration is less than 0 if it is negative you reject this solution and go to i plus 1. So, you increment to the next iteration and you carry on repeating this. So, basically you calculate this constant after you calculate this constant use this constant in this expression in calculating gamma i. So, once you have found the values of gamma i then you can put them back into the expression of capacity that is over here which is going to give you the maximum value of capacity that we can achieve yes.

So, this is what we need to arrive at and yeah. So, the basically the capacity expression when channels to information is available would be driven by this, where you depended on lambda i and lambda i the optimum values of lambda i we are going to obtain as a solution to this particular expression.

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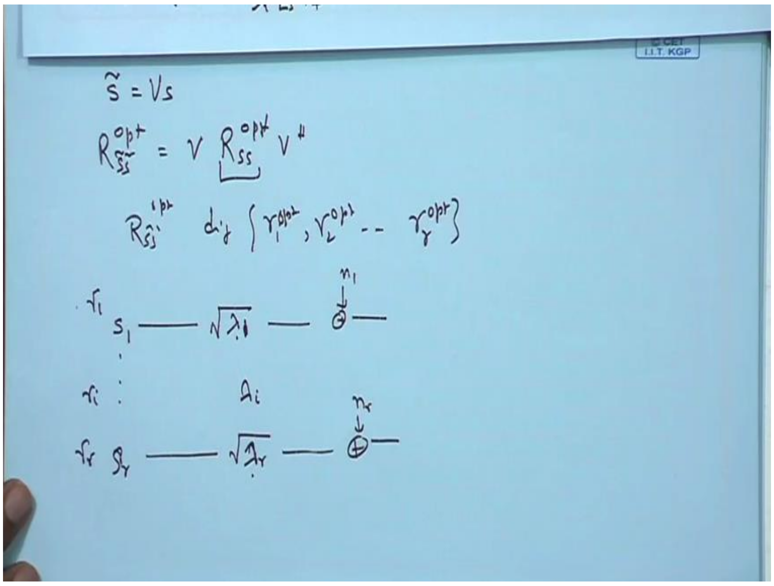

So, basically what we can say at the end is suppose we have s tilde is equals to v s right we could say R ss optimum is equal to v times R ss optimum. So, basically r s tilde optimum times v hermitian this is the diagonal matrix because, s are independent. So, you have R ss optimum would again be a diagonal matrix of gamma i gamma one optimum gamma two optimum up to gamma r optimum. So, one could view this as you are sending s1through a channel whose strength is lambda one and you are adding of course, noise n one to it, but the weight you are giving to this is gamma, gamma one like that you have s r you are giving the weight of gamma r to it this is lambda r.

So, as if you have r parallel channels in r parallel channels the gain of the channels are proportional to lambda one to lambda r or proportional to lambda i and you are providing amount of power gamma i to that particular channel which is related to gamma i by the expression which is given here which in other words means that higher the value of gamma i more the amount of power that you should be put in. So, with this we have covered two very important class of systems 1 when channels state information is not available at the transmitted for that particular case we do equal power transmission for all of them where we said R ss is equal to i m i m of t where as in the case where channels study information is available at the transmitter we do not say r s s; that means, the covariance matrix of the source symbols is equal to i, but rather it is equal to the diagonal matrix formed with the gamma optimum as given here. So, this is what is finally, radiated although R ss is identity, although this is an r plus r diagonal matrix, because s r 1 independent; however, what is actually radiated is having this kind of power.

So, if you can allocate power in proportional to the channel strength, we can achieve the maximum value of capacity when channel state information is available at the transmitter. On the other hand when channel state information is not available to transmitter the best you can do is provide equal power and leave the covariance matrix of the transmitted signals to be identity matrix. We conclude our discussion on channel capacity for these two cases here, we would move on further to discuss the effect of correlation and some other MIMO transmission schemes in the remaining few lectures.

Thank you.