Fundamentals of MIMO Wireless Communication
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Lecture - 36
Capacity of Channel Unknown at Transmitter

Welcome to the course on fundamentals of MIMO Wireless Communications. We have discussed channel capacity and finally, we have arrived at the point where we I are able to understand the capacity of a MIMO channel for deterministic flat feeding channel. So, we need to now move forward and see different MIMO configurations and find what the expression of capacity is and what does that lead to.

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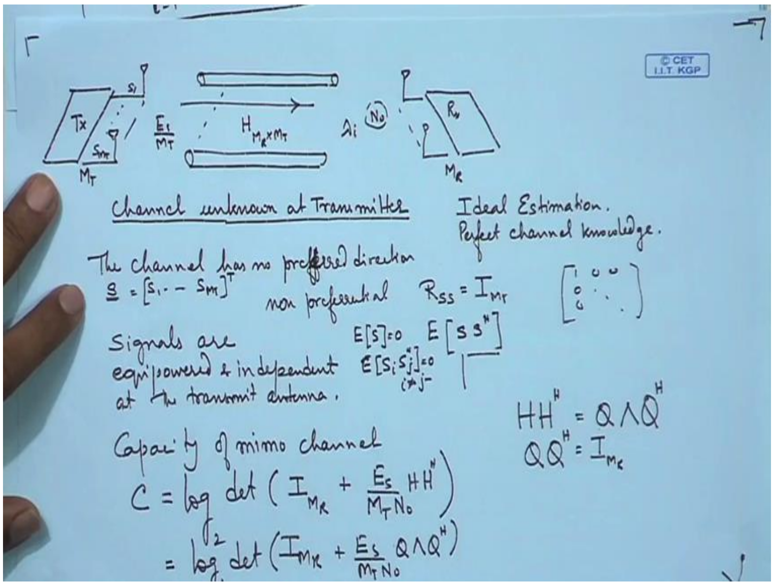

So, when we look at MIMO we have drawn these pictures several times before indicating MIMO system. That means, this is the transmitter and we similarly have receivers with multiple antennas, and we have indicated that there are MR is a variable use for indicating number of receive antennas MT is use to indicate the number of transcript antennas and H is the channel matrix which of dimension MR cross MT we have done on this expression we have also looked at the expression of capacity.

Now, we need to look at 2 different configurations one of the configuration is when there is no channels state information available at the transmitter and second situation is when channel state information available at the transmitter. So, the first case when there is transmission form the sourced to the destination it is all in this direction and as it was said in the previous lecture the channel H will be assumed to be ideally estimated ideal estimation, or perfect channel knowledge. So, this is one of the first situations and this is usually analyzed under the case know as channel unknown at the transmitter, the other situation which encountered after covering this is when the channel which is ideally estimated that the receiver is fed back to the transmitter.

So, that is the next situation which we going to analyze as of now let us begin with the case when this transmission from this source to destination in 1 direction. So, when this is the situation we have this signals s1 to s MT we transmitted from the source and since there is no knowledge about what is lying out there for the transmitter therefore, you cannot do anything with this symbols nothing can be done with it. So, since there is nothing that is known. So, in other words we usually say that there is the channel has no preferred direction and there is not specific to certain directions and is completely unknown to there is to the transmitter and therefore, the vector s which is composed of s1to s MT right s MT you cannot do anything with it you cannot specifically modify this.

So, the best non preferential the best non preferential thing that you can decide for this is to make R ss is equal to I of MT R ss is basically expectation of s times s hermitian. So, s times s hermitian s is a vector and s hermitian is the row vector. So, when you take a product of it becomes a matrix E of ss is a basically the expectation or the covariance matrix of s and we will ensure that e of s is equal to 0; that means, it is a 0 mean. So, when have E of ss hermitian, we also have e of s i, s j conjugate is equal to 0 or I not equal to j; that means, independent. So, we are taking independent symbols in these cases therefore, R ss is identity matrix that is it has 1 0 0 0 0 and like that. So, that will be the r s s; that means, the signals are eqipowered, and they are independent at the transmit antenna.

So this is the primary thing which we take into account. So, when we look at the expression of capacity which we had done in the previous lecture capacity of MIMO channel. Of course, we are doing in the case were channel knowledge is not available at the transmitter we would write c is equal to log of the determinant of I M R plus E s up on MT n naught HH hermitian. So, when we write a small H on top it is the hermitian and not the channels it is not H raise to the power of the h. So, this expression we had derived in the previous lecture. So, whatever things we do it will be based on this particular expression. So, now, we say that HH hermitian these hermitian matrix it is a square matrix it could be factorized into Q number of Q hermitian where Q are orthogonal matrixes and lambda is the diagonal matrix made of the Eigen values of HH hermitian. And we also have the property that Q Q hermitian is of I M R right. So, in that case you could write re write c as log determinant of I M R plus E s upon MT n naught Q lambda Q hermitian. So, this is just doing substitution of HH hermitian is equal to Q Q hermitian using that we have got this expression.

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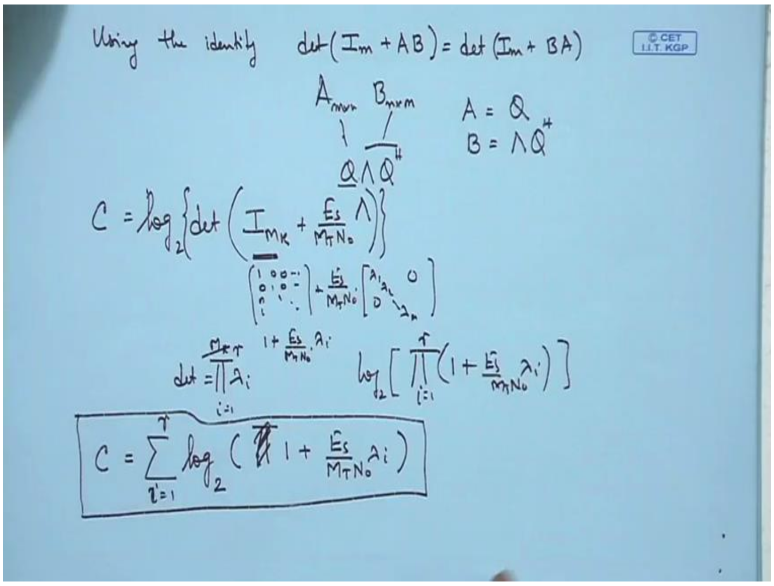

So, now will be using the identity the matrix identity, at determinant of I m plus the product of the matrix a and b could be set equal to determinant of I m plus product of the matrix b times a and you can take a to be m cross n and b to be n cross M Right because we have HH hermitian. So, if we have it in this format a then yeah. So, we will take this and we also have Q Q hermitian as I MR. So, this is already given over here Q Q hermitian is I M R and the if we have a is equal to Q suppose I said a is equal to Q and I said b is equal to lambda times Q hermitian. Suppose I use this in the expression above what I would get is a Q lambda Q hermitian could be written as a lambda times. So, we could write it this Q lambda Q hermitian we could write it also as lambda, because if this you would identify to be a and this you would identify to be b. Then you could write b times a inside the determinant. So, if it is inside the determinant b times a. So, you going to get lambda times Q Q hermitian. So, if it is lambda times Q Q hermitian Q and Q hermitian is equal to i. So, basically it will be let with the lambda.

So, basically this HH hermitian is converted to Q lambda Q hermitian which can be written as lambda because we have finally, taking the determinant of it E s by MT n naught remains there is no change to it plus I M R remains there is no change to it and you have determinant log is 2 as the expression of capacity. So, this is the a new expression of capacity this is nothing but from HH hermitian we have taking this factorization, used in identity and modified in this form. So, now, a when we focus in this particular expression a, what do we see? This is a diagonal matrix 1 0 0 0 0 1 0 and so on. What is this is you could write it as E s by MT m naught plus lambda 1, lambda 2 up to lambda r or lambda n, if we take MT equals to MR. So, so; that means, we have sum of 2 diagonal matrices which will again be a diagonal matrix with the elements has E s by MT m naught times lambda i. So, basically this will be a new diagonal matrix with these as the diagonal entries.

So, we are taking the determinant of matrix the determinant of matrix is basically product of the diagonal values a product of a Eigen values determinant is equal to product I equals to 1 to MR times lambda i or rather you should say r where r is the rank of the matrix. So, we could rewrite this expression as c is equal to sum of the I is equal to 1 to rank of the matrix log 2 of, So you could put it as a product there is an intermediate steps. So, we have already skip that step because it is a product of these basically the intermediate step we have a log 2 times product I equals to 1 to r 1 plus E s by MT n naught times lambda i. So, this is what we have. So, log of product is equal to sum of log of individual terms1plus E s by MT n naught times lambda i.

So, where r is the rank of the matrix and lambda i's are the Eigen values of HH hermitian and these entire lambda i that we have taking are greater than 0. So, we have this as the final expression of capacity for MIMO system where channel is not known at the transmitter and we have arrived at by making certain assumptions that R ss is equal to I of MT and H of course, it is standard notation that we have used.

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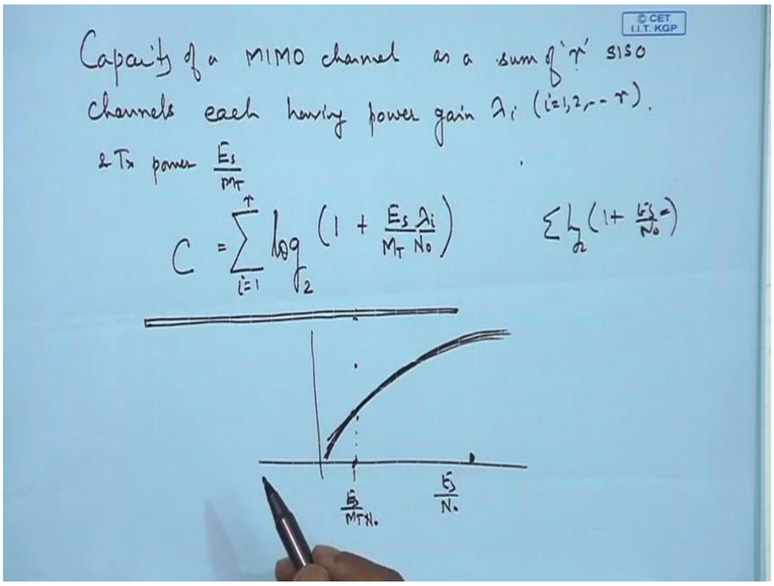

So, if we have if we take careful look at this particular expression what can we say about it. So, when we take a look at this expression a we can say that the capacity of MIMO channel this is very interesting capacity of a MIMO channel. You can say as sum of r number of SISO channels each having power gain lambda i equals to 1 2 up to r and transmit power E s upon m t. So, when you look at it you see this once again very, very carefully we could write it again log base 2 1 plus E s upon MT by n naught times lambda i.

So, this really is expression of capacity of single input single output system where E s upon MT times lambda i you can says is a single power effective single power e n naught is a noise power; however, when we have MIMO links these becomes a sum of I equals to 1 to r; that means, it is a sum of R SISO channels having a power gain of lambda i this is a very, very fundamental. So, what basically MIMO gives you a MIMO gives you a r parallel SISO links as if the between these antennas as if when we have these 2 antennas when we have these 2 systems we can say that, as if there are parallel SISO links and how many of them are there are r number of parallel such SISO links between the transmitter and receiver and each having a gain of lambda i and the transmit power of each of the links is E s upon MT and each of the links is experiencing a noise power of n naught.

So, that could be the description of what could be the MIMO of capacity of a system. So, what we have fundamentally achieved is whatever a capacity of a SISO link, if we add r of them. Where each of them has a single strength or a strength of lambda i corresponding to the Eigen value of a that particular link and each of the channel is excited by E s by m t; that means, a certain fraction of the total transmit power and we add up the capacities we get the capacity of a MIMO link. So, if we a try to understand this expression implications of this will get a better feeling of it. So, if we do not have this summation we would have log based 2 1 plus E s by n naught for a SISO link and of course, that the channel coefficient would be there.

So, such a link if I would make a E s upon n naught on this axis the capacity increases like the function and we are operating at some point e s, but what has this done is it has make E s upon MT at this point. So, this is let us say E s upon MT it has reduced the s n r however, it is added. So, if this is the point it has added r of them. So, if I roughly speaking if I would say this is the logarithmic curve not done to scales. So, there are 2 units1plus 2 plus three so and so on so forth. So, what we are clearly see is that in a SISO link there was logarithmic increase in capacity log base 2 1 plus E s upon n naught, as E s would increase the logarithmic growth.

Whereas here the capacity has grown several folds because, we have adding it up although we have reduced it over here in in this part of log function there is a linier line approximation that can be made for the log and we adding the up. So, there is significant improvement in capacity and there. So, basically there is some kind of a linear increase in a capacity that you can say and as you increase a e s. So, E s upon n t it is not simply logarithmic growth. So, this grows along with that there is a summation, which basically describes that there is a linear a growth in capacity for success steps.

So, now, a what will be doing is a such a channel a would like to see the given such a system we would like see that what is a configuration of a this channel coefficients or what kind of a channel properties would lead to the best capacity that can be achieved. So, we have to try to understand that this expressions include lambda i s right and basically lambda i s from HH hermitian. So, what we are trying to says what properties of H would lead to the best value of c because this is for a given set of H if it is for a given set of H then c is a function of h. So, if H is random c is also random we will see those things in details, but as of now we are trying to find out what property of H would lead to a maximum value of c.

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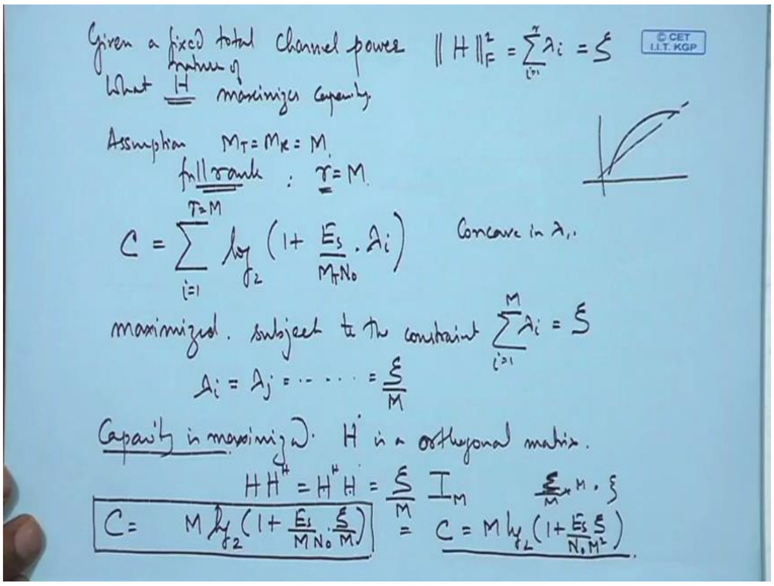

So, to do that will considered that let us be given a fixed total channel power indicated by norm squared of H this we have studied that is the sum of the alimental squared values and this could be set equal to trace of HH hermitian which is sum of all the Eigen values and let us say this is equal to sum constant zeta and that is the given total fix channel power so; that means, channel has a total power a which is not changing. So, I would like to find what H what is the nature of H what is the nature of H that maximizes capacity?

So, when we talking about of this to make like simple we make the assumption MT is equal to MR is equal to m we will we will keep this assumptions. So, that we get sum easy results and we have said in the before in the beginning that we are interested in a in you know, getting in insight in to this system to getting insight it is not necessary that we look at very, very complicated expressions a, but it is important the expression gives us a meaning. So, to get some easy understanding of things we would like to have MT equals to m I equal to m for MT not equal to MR a results could be slightly different, but the meaning would still remaining the same a enhance we take a MT equals to MR equals to m. So, that are expressions are more meaningful.

So, with this and therefore, we would I shown that we also assured it is a full rank this also helps us in our looking through the expressions. So, therefore, the r is equals to m. So, we use the expression we arrived at in the previous discussion is that capacity equals to I equals to 1 to r which is equal to m times log base 2 1 plus E s upon MT n naught times lambda i, right and this is the expression we have. So, this is concave in variables lambda i because we have described a concave in convex function. So, this is concave function the value of the function is about called. So, this is concave in lambda i and therefore, we can say that this is maximized subject to the constraint which is given here subject to the constraint I equals to 1 to m lambda i is equal to zeta. So, we want to maximize this subject to this constraint and it is a well known result, 1 can prove it also will not digress to that particular outcome right now is that lambda i when lambda i is equal to lambda j; that means, all the lambdas are equal that is each of them is equal to zeta by m this expression is maximized.

So, that is a well known result which will take advantage of will not go in to further derivations because that is going to take some additional few pages and I would recommend you take a look at it otherwise this can be shown. So, this is the result which we have. So, this intern indicates that this is maximized that means, capacity is maximized. When all the lambdas are equal and that happens when H is a orthogonal matrix will show it very quick steps that why H orthogonal should lead to all of this because if H is orthogonal HH hermitian should be equals to H hermitian H should be equals to a identity matrix.

So, I have I of m a, but since there is a certain amount of power associate with it which is given by this particular expression we are going to have sum coefficient outside it because, if you take a trace of this HH hermitian it is the sum of Eigen values. So, sum of the diagonal elements over here a would be zeta by m plus zeta by m m times which will be equal to zeta. So, a trace of it I going to get zeta upon m times all of these are 1 m which is zeta. So, this condition will be satisfies. So, only when these a diagonal matrix and a diagonal or identity matrix and that is when, HH hermitian is identity; that means, HH hermitian identity would in other words mean that H is an orthogonal matrix.

So, if these in orthogonal matrix then we consider capacity is maximized and then the expression of capacity can be if you look at the inside the term log is 2 1 plus E s upon m times n not. So, MT I have written as n naught n naught lambda i, where all equal. So, that is zeta upon m. So, inside there is no dependence on this sub index i. So, therefore, the summation a each of the components of the summation are all the same and therefore, we could simply write it is m times log base 2 1 plus E s by m times n not m squared or we could simply write it as c is equal to m log base 2 1 plus E s upon n naught m squares times zeta. So, 1 tiny step left after this we will move on to a new page with this, what we have is now if we put the constraint if we have mod or the mod square of I j element of H squared to be 1.

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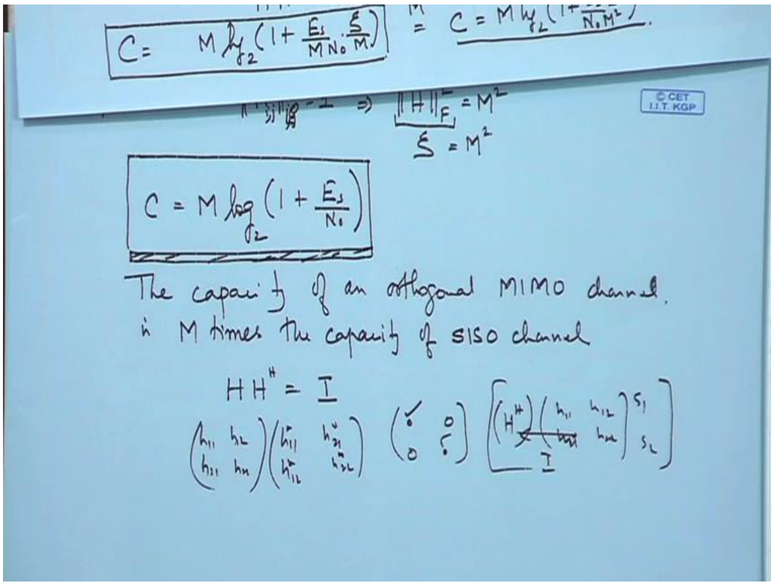

That means, we are putting the constraint that each element of H has a power of unity right. So, this would mean that Frobenius norm squared of H which is sum of all of these values and we have said initially that this is a MT equals to MR equals to m. So, we have m squared such elements. So, this would be equal to m square. So, we have said that this is indicated in our work by zeta which is zeta is equals to m squared. So, if we use this result there we going to get a will be using zeta equals to m squared. So, zeta equals to m squared this cancels out we could write c is equal to m log base 2 1 plus E s by n naught.

So, what do we have over here this is a beautiful result that we have in front of us this result tells us that the capacity, we could rather write it this would be very use full the capacity of an orthogonal MIMO channel. So, what is an orthogonal MIMO channel an orthogonal MIMO channel is1where HH hermitian is identity and zeta up on m indicates the strength of the channel, but the total strength is zeta. So, that is what it is. So, we could have taken this could be m or anything else again result could be the similar. So, this is the capacity of orthogonal MIMO channel n is m times the capacity of SISO channel this is the wonderful result. So, look at the inside of it log of 1 plus E s by n naught this is the capacity of a SISO channel.

Whereas, since we are MIMO we have m times of the capacity of SISO channel. So, this is true when it is orthogonal and to get an understanding what does it mean is that we have said HH hermitian is equal to some kind of identity matrix. So, a; that means, the components of H the of the crossed terms do matter they cancel out each other. So, components cancel out each other means that anything a transmitter on this antenna is not reflected over there because that you can clearly see when I write HH hermitian if we take a small H 1, H 2, H 1 1, H 1 2, H 2 1, H 2 2 and multiply this by H 1 1, H 1 2 contribute H 2 1 contribute H 2 2 contribute you want to get H 1 1 squared plus H 1 2 squared it is a diagonal element as the cross diagonal element H 1 1, H 2 1 conjugate and H 1 2, H 2 2 contribute.

So, basically cross terms cancel out they are 0. So, there is no cross component. So, when you are transmitting H 1 one H 1 2, H 2 1, H 2 2, s1s 2 you have H 1 1, s1h 1 2 s 2. So, when you transmit at the receiver when, you process it; that means, I would multiply H hermitian at the receiver for recovering it this would result in an identity matrix there by indicating that s is decoded without having any influence of s two. So, this is the meaning of orthogonality and we can also remember that a when we talked about alamouti scheme a we said that the scheme basically orthogonaleses the channel. So, there also, if would have done H hermitian H would have arrives at diagonal or an identity matrix with certain amount of power in that matrix where non diagonal elements are 0 indicating that a single transmitted in any one of the channel modes are not having any influence on any other a of the received channel modes.

So, this a this is a very, very interesting and a very, very important result that, we have arrived at for us to remember and one of the most interesting things that, we could learnt in this course is indicated in this particular expression we will move on further in the next few lectures and talk about the capacity of a MIMO channel. When channel is known at the transmitter in the affect of correlation we have already studied correlation and would also look at a few other transmitter receiver architectures.

Thank you.