Fundamentals of MIMO Wireless Communication
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Lecture - 30
MIMO Diversity – 2

We continue on the discussion that we have started that; that means, we have multiple input multiple output communication system, and we want to achieve diversity gain in the system. So, we have set of complex to vectors at the transmitter, we also have set of complex vectors at the receiver and we want to find out the best possible weights that can be put in order to maximize the SNR. So, that we can get diversity gain and we would like to see the performance of such scheme when we choose special weights and as we have already said we would to like to required some of the tools from the linear algebra, which we have summarized in one of the lectures and would require you to look at some of the fundamental principles and I have already specified one particular book for that this introduction to linear algebra by (Refer Time: 01:04) and many other books also, but up to your comfort ability you can choose the reference material.

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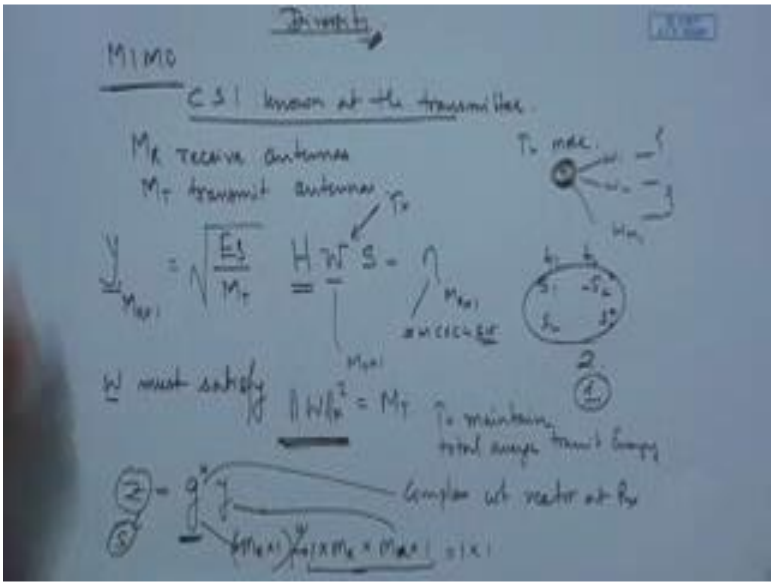

So, let us continue on what we have discussed in the previous discussion. So, we had this particular model that we were trying to set up and we have this received signal y is m r cross 1 transmit the power being divided equally amongst the transmit antennas, and we have already set this c s i known at the transmitter. So, there is weight which has to be decided and in case it is from the channel and this is also to be done at the receiver this equalization that we do at the receiver. At the end of it we get s which is 1, we did described how we get it in the previous selection.

So, we move on with this move on with this particular description that we have already started. So, what we do is we have to maximize the SNR as we have said.

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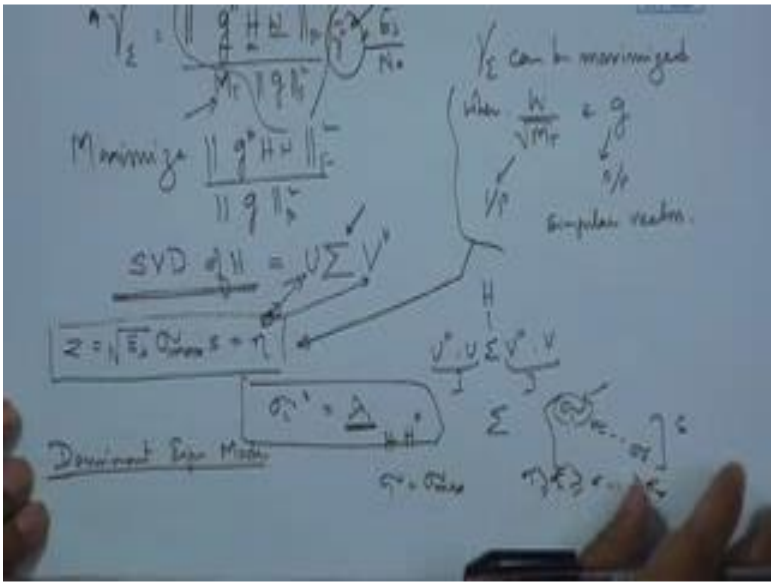

So, the expression of SNR which is gamma combined we will put like this, would be the Frobenius norm of g hermitian, when you say it will be h times w right, frobenius norm square of that, divided by MT of course, there is MT times frobenius norm square of g and you can guess why I am doing this because, if you look at this expression we are multiplying g hermitian with y, see if you are multiplying g hermitian with y, we g hermitian is comes here this is on the left. So, we have a g hermitian which comes here, your g hermitian which comes here. So, g hermitian is h w frobenius norm of that divided by the frobenius norm of the weight that comes over here to the g hermitian and of course, e s by n naught comes in, yes why n not comes in which is gamma bar and MT comes in. So, MT is already here and e s by n naught that you are seeing would be gamma bar and g hermitian that gets multiplied here g hermitian h w is already in the numerator the g hermitian is there. So, I could have written this as e s by n naught. So, n naught times g hermitian from this product there would be g hermitian here, I could I could write it has g hermitian here, I could write it as g hermitian here, right I put it back and then it will be easier to capture.

So, this is what we have. So, therefore, we move on with our expression that we have here. So, it maximizes the SNR at the receiver. So, maximizing the SNR is equivalent to maximizing this particular expression maximizing this particular expression because this is e s by n naught do not have much to do with. So, we have to focus on this expression because this also we cannot do much this also, we cannot do much with. So, we have to maximize g hermitian h w frobenius Norm Square of that, by this would be done in many ways.

So, what we would be propose incorrectly instead of actually deriving through which maximizes we would like to use the SVD of h would like to use the SVD of h. So, SVD of h could be u times sigma some big matrix times v hermitian right this and these are unitary matrices and this is the single by diagonal matrix and of course, there is a certain rank of h. So, we will consider that at appropriate time and by doing this we could say that this sigma gamma can be maximized can be maximized when which use w by square root of MT and g as input and this as the output similar vectors.

So, if we choose them from u and v hermitian rather to be precise if you look at this I would choose w from v and g from u in that case u hermitian u would lead to identity and fee hermitian v would lead to identity so; that means, if w is chosen from v and g is chosen from u then, we would be left with this sigma matrix that is a see the sigma matrix that we have.

So, basically I would choose this, from this and I would choose this from this. So; that means, when we plug it back over here, we are going to get u hermitian. So, basically you would see this h is broken into u sigma v hermitian. If I multiply here v and if I multiply here u hermitian this would turn to identity, this would turn to identity. So, what this would be left with is sigma and sigma is basically sigma one sigma 2 up to sigma r the singular values and sigma one is greater than sigma two, is greater than or equal to dot, dot, dot is greater than equal to sigma r.

So; that means, we are left with Eigen values along with s and if I choose the vectors corresponding to the strongest Eigen values; that means, if I choose corresponding to sigma 1 and we do not put anything in these. So, then all the power would be in this particular mode only on the on the dominant singular mode. So, this singular sigma I we remembered is related to the Eigen values of h h hermitian sigma s square is equal to lambda i where lambda i are the Eigen values of h h hermitian. So, that that relationship is also useful to remember at this point. So, what we choose is w and g from these matrices and we choose them corresponding to the one which produces the sigma max. So, sigma one is basically sigma max and we do not use any another. So, you we put the total power into one of them and that is why this particular scheme is known as dominant Eigen mode transmission because, this is the strongest one we not using any others. So, we are using only the dominant mode and we choosing this input and output vectors in such a way that it maximizes and naturally from here, we can see the chooses the maximum Eigen mode and it can be shown that this particular choice maximizes their SNR.

If you choose that the received expression after all of this you could write as z is equal to root over s sigma max times, s times the noise this is the final expression that you are going to get at the end of it. So, this is the expression if we choose, if you would choose it in this way you are going to get the expression which is like this.

So, when we have it here we would like to proceed further we would like proceed further.

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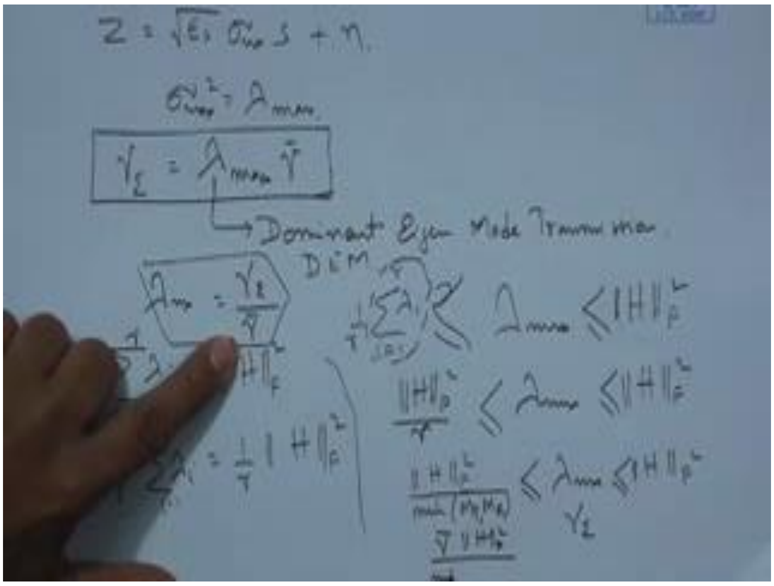

What we have is the received signal z is equal to root over e s sigma max s plus the noise and of course, we remember that sigma max squared is equal to lambda max which we have already said lambda is the Eigen value of h h hermitian, this is evident here. So, when we calculate the SNR the SNR given by gamma combiner is equal to lambda max because this is squared times gamma bar gamma bar is e s by n naught. So, that is the expression of the combined SNR at the receiver and since we are using this lambda max in the SNR therefore, the main dominant Eigen mode is even more evident.

So, dominant Eigen mode transmission right, so this is or in short form d e m. So, with this you can carry on and we could say that this gamma sigma is equal to lambda max times rho. And of course, if we see look at this we can say that this lambda max is equal to this combined by gamma bar this, we will use at a appropriate time. So, a since we have i equals to one to r lambda i is equal to the frobenius norm of f squared we could also say that one upon r sum over lambda i-i equals to one to r is equal to one by r h f squared. So, this again we are going to use. So, from this we could say that lambda max would lie between 2 values the highest value would occurred when there is only 1 lambda all others are 0 in that case this is equal to h f squared and the other case when all of them are equal or we take the average value. So, lambda max is within i equals 1 to r lambda i 1 by r. So, lambda max lies between the maximum value and the average value.

So, this is the range of lambda max. So, if this is the range of lambda max we could say that the average error or yeah. So, so basically we would take one more step this is equal to h f squared by r because this whole thing is equal to h f squared we have already seen that. So, basically lambda max lies between h f squared and h f squared by r.

Now, r is the rank of the matrix. So, rank of the matrix means number of independent columns that is that is there in the matrix or the minimum of MT and m r. So, we would write this as h f squared by minimum of MTMR this is the rank of the matrix times lambda max is less than or equal to h f squared. So, this is what we have and we have already seen from this expression from this expression we could write this as the gamma sigma, but the gamma bar. So, gamma bar can go on either side and we would have gamma bar h f squared by minimum of MTMR. So, basically this, we have saying that the combined SNR lies within this range. So, this is the range of within which this gamma sigma lies and then we can find the range of probability error from this expression.

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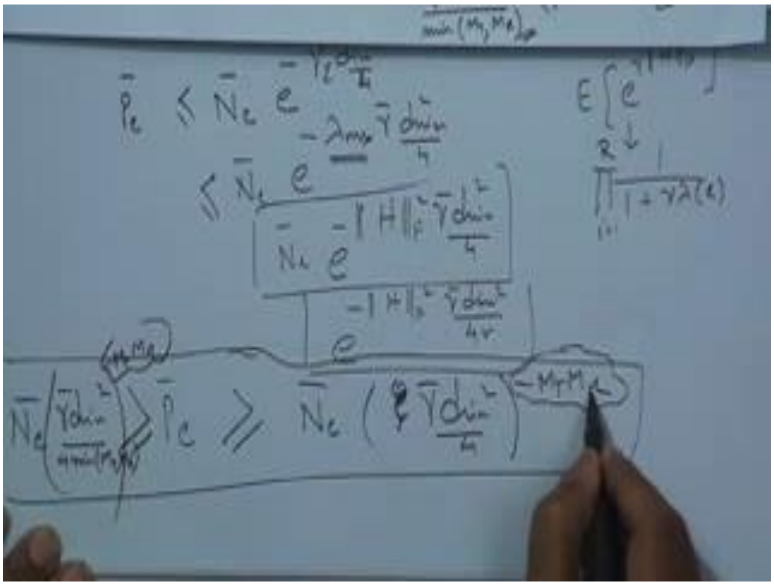

In case this is full rank in case is full rank we could say this is gamma bar times max of MTMR in case of full rank this is MT times m r this MTMR in case of full rank. So, if it is MTMR minimum when divided by minima of MTR MR, we have the max of MTMR and this is the range of gamma sigma MTMR. So, when it is full ranked this is MTMR. So, when h is full ranked is equal to MTMR right times gamma bar. So, if this is the case we could say that p e bar which is less than or equal to n e bar e to the power of minus gamma sigma d min squared by 4 this is what we have and we could say that yeah n e bar it is less than lambda max gamma bar d min squared by 4 and it has 2 range, we have already said the 2 range values of it that is one of the values is h f squared gamma bar d min squared by 4 e to the power of minus n e bar.

So, this is one limit the other limit is e to the power of minus h f squared to gamma bar into d min squared by 4 r this is the second limit which, we have seen over here the 2 limits this is one, of the limit this is the other limit or yeah. So, basically this R1 is this one is this is the range of lambda max. So, with this range we can calculate the average probability of error or we can say that the average probability of error would lie in the range of gamma bar d min squared by 4 raise to the power of minus MTMR for the full rank h and n e bar times gamma bar d min squared by 4 times minimum of MT for m r or you can bring them to the numerator by the maximum of MT or MR. So, what we have saying is that the average probability of error lies in this range and which gains in this whole raise to the power of minus MTMR.

So, in either of the case exponent we have minus MTMR minus MTMR comes from always remember we are using that e of e to the power of nu frobenius norm of f squared this is expression is what we are using is 1 by 1 plus mu lambda r this what we are using i equals to one to the size of r, basically it is MTMR in that case. So, so the order of diversity that is gained it is MT times m r so; that means, a full order of diversity is achieved in this and there is a certain range of SNR which is available and the SNR as you can also see has a range which varies from MTMR times gamma bar to gamma bar times max of MTMR. So, so this is the gain that you can get from dominant Eigen mode transmission which is again another way of extracting diversity when, you have channel straight information available at the transmitter.

Ah we continue with our discussion on dominant Eigen mode. So, we stopped at a point where we could see that the error probability expression for dominant Eigen mode has in the exponent MTMR; that means, the full order diversity is achievable by using channel straight information at the transmitter while having multiple antennas at the receiver as well. So, we with this brings us more or less to the conclusion of the different popular schemes of diversity gain that can be used there are many other schemes of course,, but we would restrict our self to this few things that we have discussed. So, with this we move on to consider the effect of correlation on probability of error this is very, very important because we studied correlation we have seen how correlation can be captured in the channel model so.

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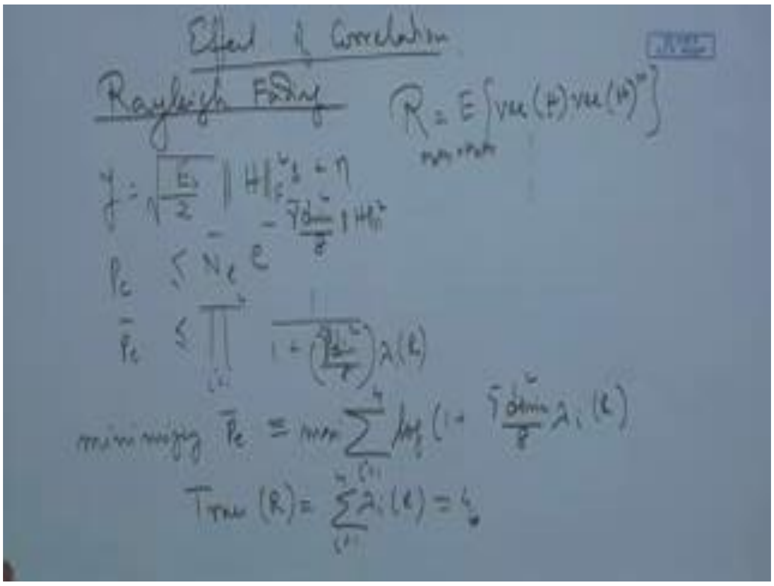

In h w channel, we have already seen the performance most of the schemes that we have seen are for h w channel and what we will now see is, if it is not h w. If there is a certain amount of correlation involved then, how does these expressions of error probability change, when there is correlation involved? So, let us move forward and we will take Rayleigh fading, we will take Rayleigh fading into account and we remembered that r is basically expectation of vec of h times of vec of h hermitian. So, a vec operation, we remembered with this is h it stack pictorially. So, this r that is of the size MR MT cross MR MT or MTMR cross MTMR.

So, this is the size is the big matrix typically we could write expression of y as root over e s by 2, frobenius norm of h times s plus n this is usually for the alamothi scheme as we conceive that this is typically for the alamothi scheme and the probability of error expression is given by thus p e is less than n e bar e to the power of minus gamma bar d min squared by eight times h f squared this, we have come across and p e bar is less than or equal to pi equals to 1 to 4, 1 by 1 plus gamma bar d min squared upon 8, times lambda of r lambda i f r and this is a product this is a 2 cross 2 alamothi scheme.

So, what we are interested in is minimizing the probability of error expression that is what our interest is in. So, if you have to minimize the error probability expression this is as good as you maximize the log is sum of one 2 4 i-i take the log of this instead of operating this if i take the log of this it is easier this product becomes a sum. So, it is because this it is in the denominator we would like to maximize the denominator that would minimize this. So, i take the log of it and i minimize this particular term. So, we have log of 1 plus gamma bar d min squared by eight lambda i of r this is what we have and we also have trace of r is equal to sum of 1 to 4 lambda i of r is equal to 4 again with total bar constant. So, this is the total bar constant of the channel that is that is required.

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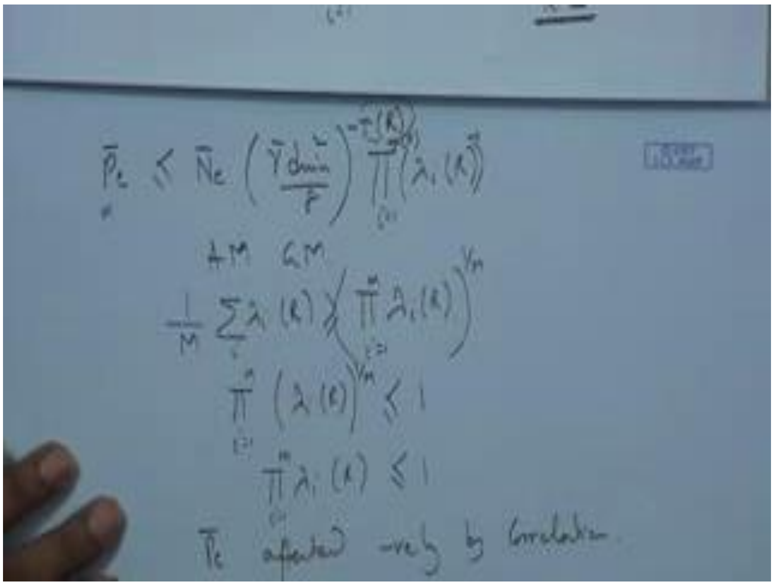

So, moving on forward we would like to see that the error probability expression p e bar is given as less than or equal to n e bar times gamma bar d min squared upon 8 raise to the power of rank of r right because you have this pi incase of correlation would going to 4 assuming there are 4 independent branches; that means, these values are all non 0 values, now if there is some kind of correlation it would not be ah 4 order matrix the rank would be less than that or less than that of MTMR. So, we want the rank of r, it could be less than that if, there is certain amount of correlation and if we look at this expression we have taken the case where, gamma bar is stronger than one much-much stronger than 1. So, we are neglecting one in this expression and we have already raised this to the power of minus r we have taken at out of this product sign.

So, we have the product of lambda i of r to the power of minus 1 i equals to 1 to rank of r right this is in the denominator. So, we have basically neglected 1, that is over here to simplify our, results that we have here, now we should remember that the this thing is maximized now this thing is maximized when, if you look at this either this expression or from this expression it can be achieved from both the cases this thing is maximized you want to maximize this when all lambda i is are equal to lambda. So, when they are all equal then this gets maximized this can be verified for all i.

So, this can be verified separately so; that means, this is maximum when o lambda is are equal that is for h w channel this is true on the channel is h w; that means, specially white channel now it is not specially white channel r is not equal to i matrix; that means, the co variance matrix is not equal to identity matrix this is not true, if this is not true then this is not maximize. So, that clearly means that when we do not have a specially white channel the error probability is not minimized. So, something else happens. So, that means, from this side also we can see it.

Now, let us see further what we have in this particular case. So, we would like to recall the arithmetic mean geometric mean in equality from which we can say that one upon m sum over i lambda, i of r is greater than or equal to geometric mean i equals 1 to m lambda i of r raise to the power of 1 upon 1 m. So, this is the geometric mean this is arithmetic mean and, we could say that pi i equals 1 to m. Now since this is should be equal to m lambda i of r raise to the power of one m is less than or equal to 1. This is what we have we are just bring into the left and side because this thing is equal to m because of total part constraint. So, this thing should be equal to 1 this less than is equal to 1. So, in other words pi i 1 to m; that means, product lambda i of r is less than or equal to 1.

So, if that is less than or equal to 1, what we have over here is this particular product is less than or equal to one. So, there is a minus one in the numerator, if you would see so; that means, the denominator is less than 1 this whole product. So, this whole term is increasing. So, it is it is modes of we are multiplying it by a term which is more than one. So, that that clearly indicates that this is this is increasing the probability of error and this is with equality this holds with equality. When one all lambda i is are the same, so from this what we can also gathered is that the error probability p e bar is affected negatively by correlation. So, so what you get you get the double affect the error probability increases because of this is less than or equal to 1. Secondly, what you get is this is raise to the power of r the rank of r so; that means, if rank is less than MTMR then this exponent is also less is exponent is less as well as this term adds to it, overall p e bar increases when compared to a specially white channel. So, in summary we can say that if there is correlation present the probability of error is affected in the negative way.

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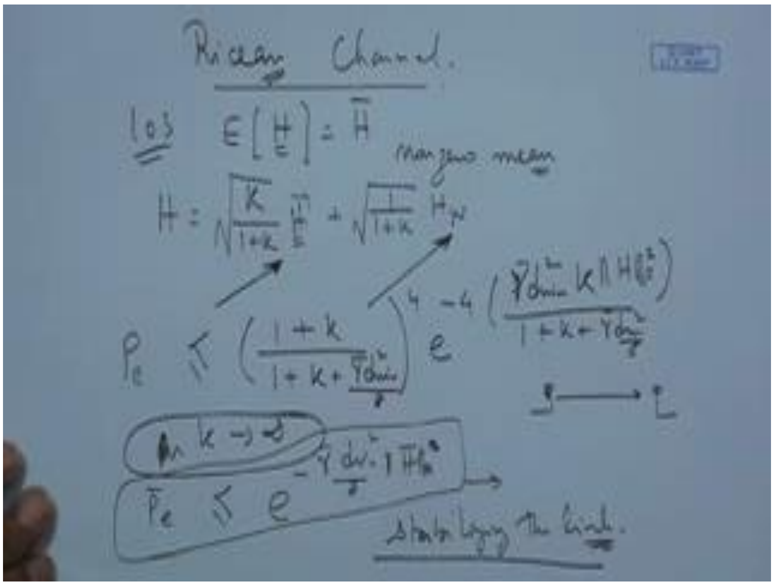

Moving on forward what we would like to add is, suppose we have Ricean , effect or Ricean channel. So, if you have Ricean channel; that means, there is an l o s or expected value of the channel coefficient there is some, mean value of h; that means, non 0 mean this we have describe earlier for the flat fading now we have the matrix; that means, each element of them they are non 0 mean, and we could say that h is equal to square root of k the Ricean component which we have described thirdly in our description of channel coefficient and one plus k times h w. So, we have line of sight component and we have a specially white component together forming a Ricean channel in this case we have the probability of error expression given as less probability of error expression less than, 1 plus k divided by 1 plus k gamma bar this is a per branches SNR divided by eight raise to the power of 4 e to the power of minus 4 gamma bar d min squared k mode h f squared divided by 1 plus k plus gamma bar d min squared on 8.

So, this is the expression for it. So, as k tends to infinity; that means, as you get more and stronger line of sight component and this is hardly anything. That means, we have an antenna and there is only direct line of sight there is no scatterer around. So, you can imagine a satellite communication or micro wave length in that case what is happening is of course, we are talking of multiple antennas p e bar is less than or equal to as k tends to infinity this converges the expression minus gamma bar d min squared by 8 mod h f squared right and; that means, it is as tending mode towards mode towards line of sight on mode towards a non fading condition and so; that means, when you have Ricean channel the moment you have Ricean channel; that means, you are you are stabilizing the link.

So, as your k component increases as your k component increases there is hardly any component over here and as there is line of sight component this component is the mean of it. So, this component is the mean of it is. So, what you see over here is the mean mean component over here which is the fix number everything is the fix number. So, error probability expression that you get is becomes a fix number so; that means, the link is stabilized there is no variation in the link and it is almost like a w n change channel.

So, what we have seen is that when there is diversity even without as you increase the diversity towards infinity your performance [bec/behaves] behaves like a w gn or it means a w g n and the last case what we saw suggest now is when you have Ricean component then as the Ricean components becomes stronger and stronger you link stabilize and that is also logical straight forward you receiving direct signal from the antenna and there is no fluctuations no fluctuations at all. So, this co variation, link is stable we have also seen that if, there is correlation in the in the in the channel then, also then the effect is that error probability increases in 2 effects; one is that the product of Eigen values is less than one which is in the denominator and therefore, it increases the error probability. Secondly, the exponent of average SNR is also not MTMR or not the full rank for capital r which is the covariance matrix it is rather the rank of capital r.

So, if rank is less than the full rank your exponent is less, your less diversity gain as well as you have shift in the error probability expression as well. So, to summarize we can say that diversity gives you improvement in error probability correlation decrease the gain of diversity or it increases the probability compare to a h w channel. And if there is line sight Ricean component you can also say that the link, become stable and there is no fluctuations. So, it becomes more or less like h w channel with an appropriate gain. So, i would like you to summarize these important conclusions and we will make this as one of the last lectures on diversity after this, we will move on to look at special multiplexing or the channel capacity of MIMO systems.

Thank you.