Fundamentals of MIMO Wireless Communication
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Lecture – 25
Maximal Ratio Combining

Welcome to the lectures on Fundamentals of MIMO Wireless Communications. We are currently exploring diversity gain. We have seen the selection combining method. Now, we take a look at the next combining method know as maximal ratio combining.

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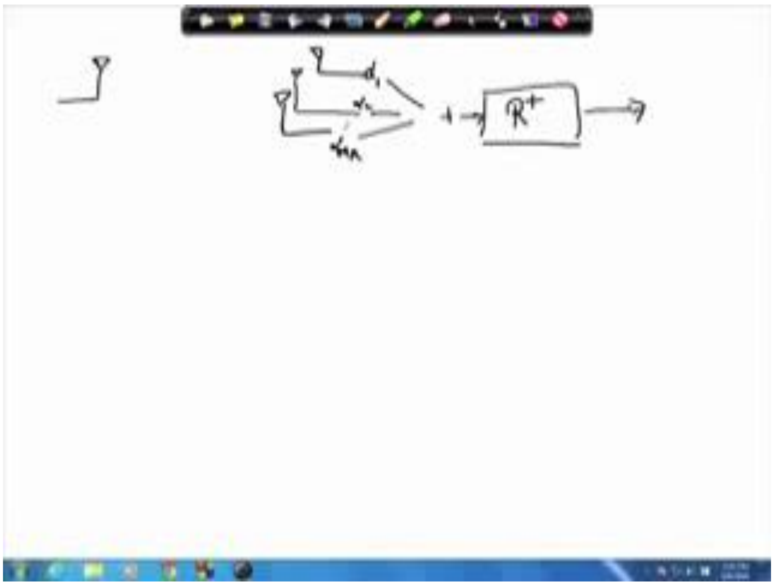

In maximal ration combining what we have is, again there is single transmute antenna at the source and there are multiple receive antennas and the receiver and these different antennal lines they would go to the combiner and there will be an output.

In each of the combining branches there would be some waving, but let us say alpha 1 alpha 2 up to alpha M R. So, this will be the different weight factors that would be given and the receiver they would be some kind of combining. So, previously there was a switch and that was going through the selection combining. But now we do not have a switch we have a weight and after the weight there is some kind of addition and it goes through. So, this is the different method of combining and we would term it the maximal ratio combining also refer to as MRC.

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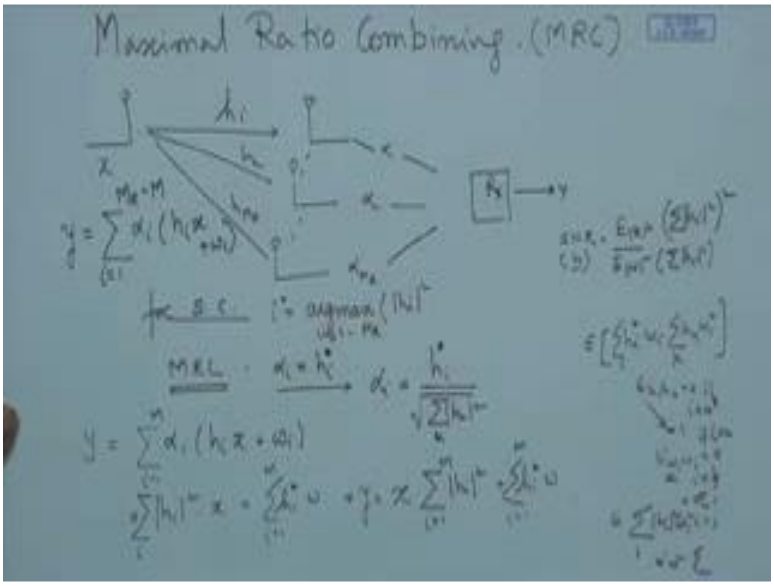

So, MRC is the common name for this and typically the diagram that was drawn x is transmitted about the channel which is having coefficient h, receive through one antenna who weight by a certain weight factor. The second antenna again weight by the certain weight factor and so on up to M R weight M R and then there is some processing at the receiver and then what you get is the output y. So, the output y can be written as some of I equals to 1 to M R it could also write it as M for simplicity in this particular case. Is alpha i times h i the ith link h 1 h 2 h M R times x, x is transmitted plus w i, w i is the noise due to that particular branch. So, this weight factor is getting multiplied with whatever it received.

For selection combining; the combining was done by i star is equal to argmax of gamma i or argmax of mode h i square for I equals to 1 2 up to M R this is what we did for selection combining. But for maximum ratio combining there is MRC are we combine in a way which this name derived is we put the weight corresponding to the strength of each of the branch so that overall SNR is maximized at the receiver. In this case what is done is alpha i set equals to h i conjugate or other very precisely this is what you gone a fine in difficult books where they give you this particular expression which is useful for the DPSK or QPS modulation, but if you doing higher order modulation one has to choose alpha i is h i conjugate divided by square route of sum over k h k square. So, basically is a normalized coefficient.

We do this normalization so that at the output we get the values which can be easily decoded. So, when you are doing assimilations or when you having actually receiver this is what you have to use, but when you are doing error probability analysis you could do sufficiently with this for QPSK because SNR calculation remains the same when we do a either of these two things.

So, if we get back to y the received sigma y is equal to I equals to 1 to m we going to write m in sort of M R alpha i h i x plus w i w i is additive white Gaussian noise this is the channel coefficient this is Rayleigh faded you can assume it to be Rayleigh faded, but a it can be any general distribution which is equal to since alpha is equal to h i conjugate what we have is h i squared some over I times x plus h i conjugate w. Now this you could also write it as look at this x times I equals 1 to m mode of h i squared plus h i conjugate w. So, this is the received signal.

At the receiver what we have is sum of all the coefficients of the channel squared that means, each of the channel gain is getting weigh or when we are adding up the signal we are basically weaving it by the corresponding channel strength. In the previous case they would have been only channel gain, but here we are getting additional where all possible channel links that we have. So, compare to selection combining in selection all though we had all these branches one was taking only one of these and we are dropping the signal from the other branches, but what is being done by here all the branches are being used, but it is used in such a way that it maximize the signal to noise issue.

Now, if you look at it we are putting the weight of the branch corresponding to the strength of the branch compare to the other branches. So, it is basically the ratio of the strength that branch over the total signal strength. The advantage of this is if some of the branch is good you are putting more weight to that, if one of the branches is week you are putting less weight to that. Because, the error probability of this branch would be better than error probability of that branch, but I am not leaning out any branch I am taking it, but simply putting less weight to it and the weight is being made proportional to the channel strength.

So, following this if one has to calculate the signal to noise ratio from this one would get E of x squared by E of w squared that means, the x E of x squared by E of w squared. But at the denominator we going to get another term that is some over h i mode squared and in the numerator you going to get a square of this term that is some over h i mode squared whole squared, because you are basically collecting; sorry here is the sum of i equals to 1 to m you know some of I equals to one to m. So, when we are calculating the energy we are calculating basically E of you have to do E of h i w conjugate some over I times let say k h k w i conjugate because the conjugate of each this is what we want to give me the power. Since h i h k are independent, so we will assume independent that E of h i h k equal to 0 if I is not equal to k and is equal to 1 if I is equal to k. So, that is the relationship.

And again for w E of w i w j is equal to 0 if I is not equal to j and is equal to sigma square w for I is equal to j. This this is the assumption that we make over here. So, going by that the power is going to the added up only when h i is equal to h k that means, i is equal to k and since this is a function of h this is depended on h we do not actually operate the E on h so this particular part does not take into effect; only this comes into effect. So, we have h i h k part and since w i w j E f w w j is equal 0 for i naught equal j we basically have some over I mod h i squared w expectation of this E comes inside w squared w i squared. So, E f w squared times the summation that is that is the expression that we have over here. And the in the numerator in the similar manner we have this square of this.

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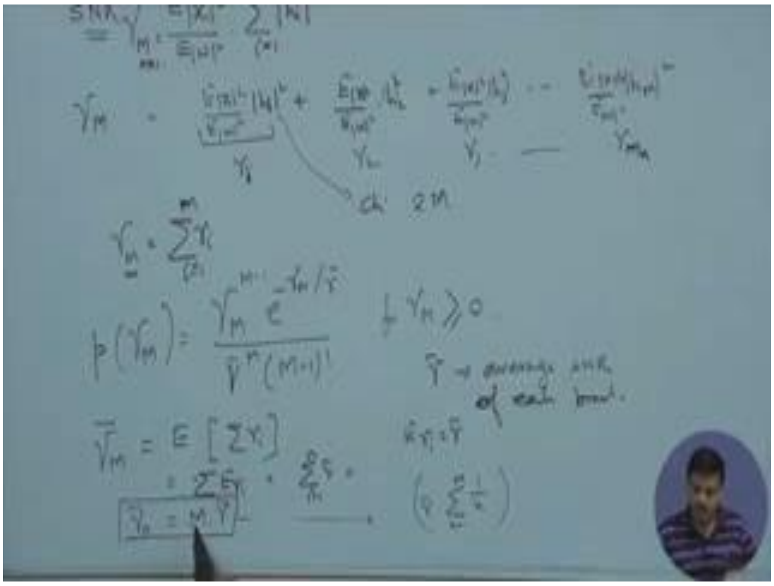

So, if you if you evaluate this are the SNR expression that one would get would turn out to be the E of w squared times sum of I equals 1 to m mod of h i square. That means this is the gamma bar that we are getting. So, the average power in the (Refer Time: 09:55) channel case times this sum over the channels strength over all the channels. In case of selection combining there was only one here we have added up all the channel strength. And if we are looking at gamma M, so basically this is gamma M the combined due to MRC. So, I could write it as E of x square by E of w squared times h 1 square plus E of h squared by E of w squared times h 2 squared plus E of h squared by E of w squared times h 3 squared mod of course up to E of h squared by E of w squared times h m mod squared. So this is basically gamma i gamma 1 this is gamma 2, gamma 3 up to gamma n.

So, what we can write this is equal to some over gamma i i plus 1 to m meaning that the SNR that is achieved at the end of the combining is the sum of the SNR of the individual branches, so the gamma 1 is the SNR of the first branch, gamma 2 is the SNR of the second branch, gamma 3 is the SNR of the third branch, and gamma n is the SNR of the mth branch or M R, MRth branch. So, by this particular method one is able to add all the SNRs for different branches.

Now again if we look at this, this particular one for Rayleigh distribution is chi square distributed with 2 degrees of freedom if there are m such terms, so basically it is achi square distributed with 2 m degrees of freedom and one could write p of gamma M is equal to gamma M raise to the power of m minus 1 E to the power of minus gamma M by gamma bar divided by gamma bar is to the power of m by factorial m minus 1 for gamma M greater than equal to 0. Where gamma bar is the average SNR of each branch; that means, expected value of gamma i is equal to gamma bar.

So, this way what we have seen is we have the combined as sum of all the SNRs and if you want to calculate gamma M bar the expected value this will be expected value of sum over gamma i which would be since all of these are independent we assume them to be independent it would be sum over E of gamma i which would be sum over gamma bar I equals to 1 to M.

So, that essentially turns out to be M times gamma bar. So, what we are seen is that the average SNR has become M times the SNR of the one of the branches. In case of selection combining just to compare it was gamma bar and k equals to 1 to capital M one by k this is how it was going, but here there is linear increase in the SNR because of the MRC combining. So, this is the best possible increase in the SNR that you can have. And hence this particular technique is quite famous in terms of combining.

So, this particular technique which is known as the maximal ratio combining it puts weights on to each of the branches and the weights are proportional to the channel strength. And in combining it actually uses the h conjugate so that when it gets multiplied we get mod h squared along with this particular term. And this leads to the situation when we try to calculate the SNR we get the sum of SNR of the individual branches yielding that gamma M is equal to sum of all the SNRs the average SNR. Hence is m times the average SNR of any one of the branches. This is the huge potential gain that we are having over here.

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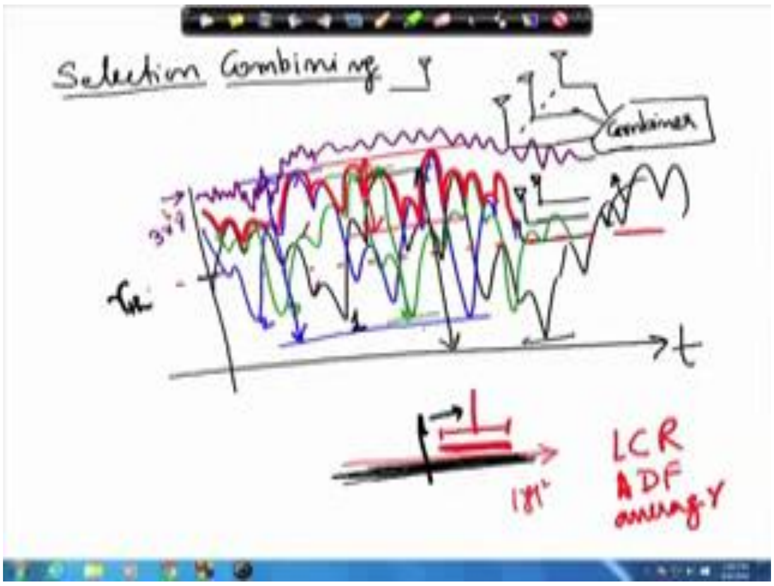

So, if we go back to some of the previous pictures that we had drawn. Let us take one of this, if we take any of this pictures which could be helpful. So, here we will always writing a crest, but if we now take the maximal ratio combining in this particular figure the curve would be this signal plus the green signal plus the black signal so it will be somewhere here and I will try to arbitrarily, but it is higher.

All I mean to say that at this point it was just the blue, but now it will this blue plus this green plus this black and so the signal strength would be there. It will be higher than that and the average SNR if it is here for all the branches the combined average SNR would be somewhere there which is in this case it is three times E gamma bar of each of this of the SNRs. So, this is very very interesting result that we have got.

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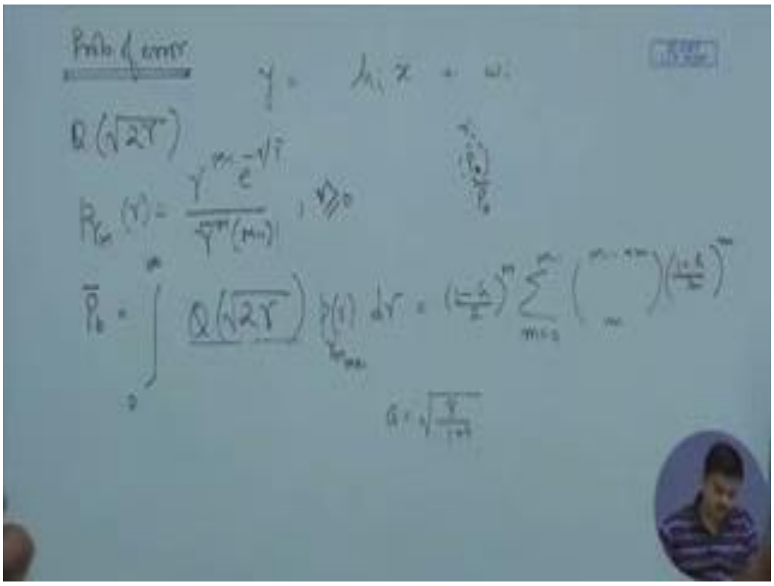

The next very interesting thing that we need to calculate is the probability of error. We have already made the assumption that for this particular course one has to do digital communication as a prerequisite. That means, it is expected that you have done some basic work in terms of error probability calculations, so we are not going to derive the error probability for a divide got noise in this particular course because that would be unnecessarily getting into some details. So, will assume the expression for error probability for certain consultations using which you can derive the probability in these particular cases.

Again reminder like the one we did in the previous case that if you have error probability expression in AWGN that means, when the signal strength is fixed you have fixed value of SNR. Whereas, in this particular case where your received signal y is h i times x plus w i and then we do certain processing here, so it is already variable and the SNR that is gamma i that I get is random. So therefore, the probability of error is also random and hence we would be interested in calculating average probability of error. We have told earlier in the previous discussion on selection combining how to calculate probability of error for average probability of error will use the same technique using the density function that we have derived to calculate the average probability of error in case of MRC and that will give us some good insides into the performance.

So, preceding further for DPSK or QPSK you could say that; before Q the error probabilities Q function of route of two times the SNR. In this case the combined SNRs PDF turns out to be gamma raise to the power of m minus 1 E to the power of minus gamma by gamma bar gamma bar raise to the power of m m minus 1 factorial for gamma greater than equal to 0. Now this is true because gamma is E of x square by E w square multiplied by mod x square. Now mod x square is greater than 0, E x square is greater than 0, E w x square is greater than 0 so on all are greater than equal to 0, so we have this gamma to be greater than equal to 0. And if one has to calculate P t bar is average (Refer Time: 18:26) one of to integrate from 0 to infinity d gamma p gamma because of gamma M MRC right the error probability at a particular SNR.

So, what is meant by this is this functioning this expression gives the probability of error when SNR is this. And we are averaging it using this p gamma because the probability of the SNR been a certain value is given by p gamma d gamma and we are saying that gamma arranges to 0 to infinity in this whole range of values we going to get the average P b. This expression is bit (Refer Time: 19:10), if we define g to be square route of gamma bar gamma gamma 1 minus one plus gamma bar we going to get in a term which is pretty complex looking expression m equals 0 to m minus 1 (Refer Time: 19:29) of m minus 1 plus m m 1 plus g by 2 raise to the power of m. So, this this gives the average probability of error for MRC combining.

Now, this way to calculate the probability of error is not so straight forward, but other techniques by which we can do it. So, let us take one particular method of doing the error probability calculation.

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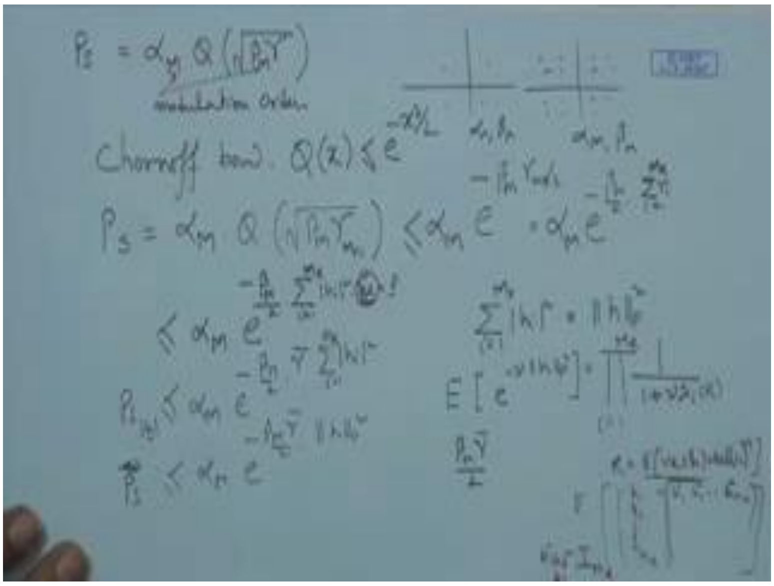

In general the probability of symbol error, I am talking about general expression for probability of symbol error can be approximated as alpha M Q function of square route of beta m times gamma the particular gamma that we are using. This is the probability of error expression for particular modulation. Let us take a rectangular square modulation at depending upon the size of m. So, this alpha M is not the m that we are using for a MIMO communications this is different; this m is the same as this related to the modulation, this m is indicating the modulation ordered this is the modulation order. Please do not get confused with this alpha M and beta m. So, by varying this co efficient different size of m could be use like one could be QPSK another could be sixteen form or any such thing. Depending upon each here alpha M and beta m would be different here alpha M beta m would be different. So, we have to be careful with that.

We will use the Cherroff bond. So, by the Cherroff bond we have Q of x can be said to be less than or equal to E to the power of minus x square by 2. This is an approximation and that we can use, but the n result is if you use this particular method the answer is that we have to get out of this are a pretty nice. So, we can get good result out of this, so that is why we will be using approximations and they are not very very wrong they hold quite well. So, we will proceed with this particular expression. So, we can say that P s is equal to alpha M times Q of square route of beta m times gamma, in this case it is gamma MRC. This is less than or equal to alpha M E to the power of minus beta m gamma MRC by 2.

So, this this this is the expression that we are using, the Cherroff bond we are applying here now gamma MRC we will be using so that is basically this is equal to alpha M E to the power of minus beta m by 2 times sum over i, i equals to 1 to let us use to M R indicating M R number of received branches gamma i. So, this is the particular expression that we need to use. Remember that we could also write it as this one, so basically p s is less or equal to alpha M E to the power of minus beta m by 2. Instead of writing gamma i we have I equals 1 to M R mod of h i squared times E s by n naught or p s by sigma squared omega. This could also be written as row, this could also be written as row or one could also write this as alpha M E to the power of minus beta m by 2 times gamma bar times some of h i square I cross 1 to M R, so one could write this as p s is less than or equal to this thing. And this as a function of h vector; that means, all the all the channel co efficiency in case of MRC meabs is channel coefficients, this conditioned on this particular channel coefficients.

So, that the whole that vector depending upon, so what we are interested to find at this stage is P s bar. So, if we have to find P s bar what we need to do is we have to take the expectation of this over this particular think. So, if we if we look at this mod of, if we take this expression I equals to 1 to M R sum h i squared for MRC this we could as could as write it as h f square (Refer Time: 24:43) gamma h because it is contains the sum of square of all the channel coefficients. This is that we have here and that that is very very interesting. So, p s we could write it as the alpha M E to the power of minus beta m gamma bar by 2 h f squared. And now this it is going to turn handy. So, if we have to take the expectation of this, so remember E of E to the power of minus nu h f square is given by; this think we had derived earlier is given by the product of I equals to 1 to M R.

Look at this in our case we have M R in the previous case we had M R m t when h was MIMO, but in this case there is one input multiple output is the vector and M R number of contents in it so the R matrix; that means, the covariance matrix is of dimension MR cross M R. So, this goes to M R otherwise it would have gone to M R m t 1 plus nu times lambda I of R, R is the covariance matrix of the vec vec hermitian; R comes from expectation of vec of h times vec of h hermitian. So, this is what we have to use over here. So, nu in over case is a basically beta m gamma bar by 2; that is what we have to use. In case of identity that means, in case of h being independent that means on h i are all independent if I do the separation vec of h could be h 1 h 2 up to h M R times h 1 h 2 up h M R of course conjugate and then an E operation.

So, what we going to get is h 1 1 squared h 1 h 2 conjugate and so on and so forth, basically diagonal would be h 1 1 squared h 2 squared h 3 squared and h M R squared. Whereas, cross diagonal elements would be cross terms for uncorrelated or independent case E of h 1 h 2 conjugate is 0 and so other coefficients excepts for E of h i h i squared.

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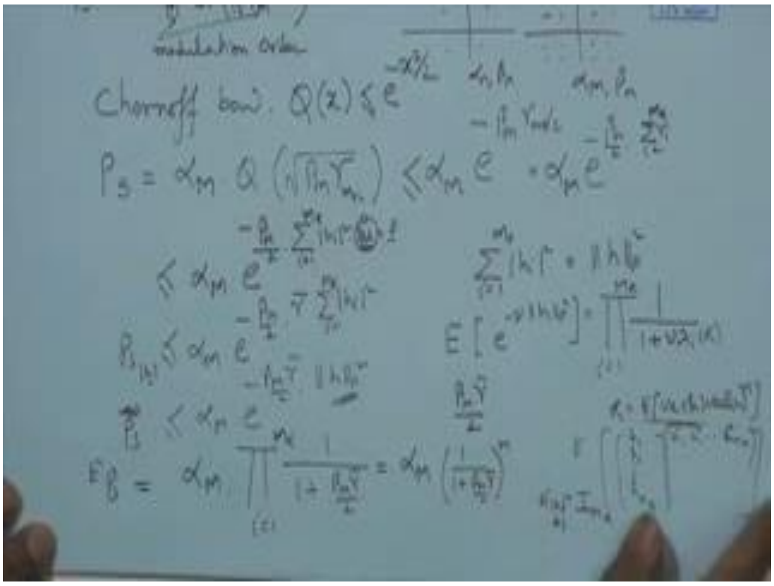

That means, this would turn not to be E of h squared times and identity matrix or size M R and this is equal to 1, so basically the eigen values of them would be 1 so basically lambda as the 1. So, in our case you can apply this, so basically when I take expected value of p s is basically expected value of this over h f so that we could write it as alpha M and then this one pi of i equals to 1 to M R 1 by 1 plus beta m gamma bar by 2. So, this particular expression if we look at it whole this is the same, so this you could also write it as alpha M times 1 by 1 plus beta m gamma bar by 2 raise to the power of m.

And we could proceed further that means, we are assuming that all gamma i bars or gamma bar that means, the average is strength is all of the branches are the same that is we have made the assumption gamma i bar is equal to gamma bar.

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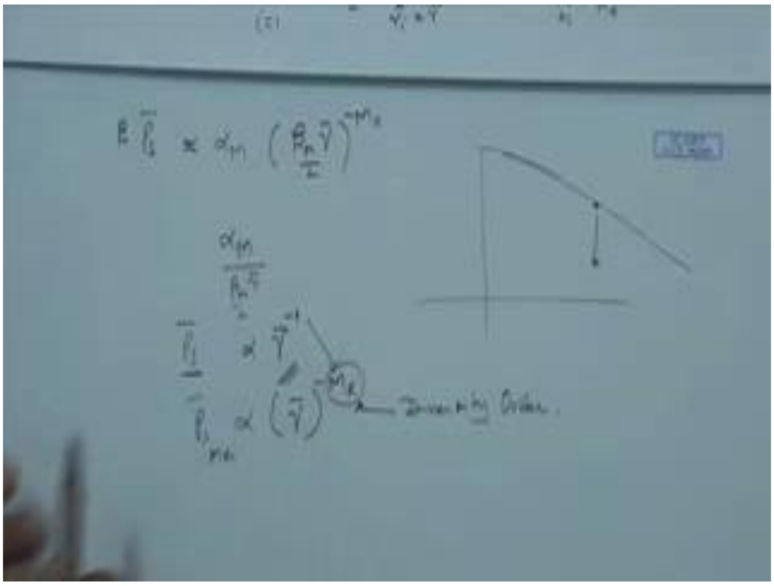

And if we assume that gamma bar is pretty large that means for large SNR approximation we can say that expected value of P s or P s bar probability of symbol error we could approximate it as alpha M rewriting the approximate expression times beta m this increases very very large that means, larger compare to 1 gamma bar divided by 2 raise to the power of minus m.

What we can compare this is if we have m is equal to 1 that means, M R is equal to 1 there is only one branch we would have alpha M divided by beta m gamma bar by 2. That means, it is proportional to gamma bar raise to the power of minus 1 and which is true for a see so case. So, what we are seeing from this expression because we made certain approximation the first one being Cherroff bond that we made over here and second thing that we used is this expected value of all the m g of the movement genetic function of (Refer Time: 29:48) norm of h that we have used over here that is helping us to get this particular expression.

So, what we are saying from a see so where it was proportional to the average probability of error this proportional to inverse of average signal strength now it is instead proportional to incase of MRC is roughly we can say it is inversely preoperational to the average signal strength raise to the power of minus M R that means, as M R goes the probability of error decreases and the order by which is decreases is given by M R which is number of received antenna branches. And we will see later that this exponent is known as the diversity order.

In summary what we see is because of this kind of combining the probability of error expression is which is inversely proportional to the average SNR raise to the power of 1 or inversely proportional to the average SNR, what we see it is now inversely proportional to the average SNR raise to the power of M R that means number of received antennas. That means, if probability of error previously was something like this in the d b d in the log scale now the probability of error at the certain value would be much lower at that particular SNR and that value would be dictated by this particular term.

Will see more details of this particular expression in the following lecture in which we are going to analyze the impact of m and will see what is meant by coding gain and diversity gain in more details.

Thanks.