Fundamentals of MIMO Wireless Communication

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Lecture - 21
Statistical properties of H

Welcome to the lectures on fundamentals of MIMO wireless communications we are discussing the MIMO channel properties. So, we will quickly take a look at the correlation properties that, we have started with and then proceeded to see the statistical properties of the MIMO channel which is very vital for analysis of MIMO performance of MIMO communication systems especially the diversity or the capacity links.

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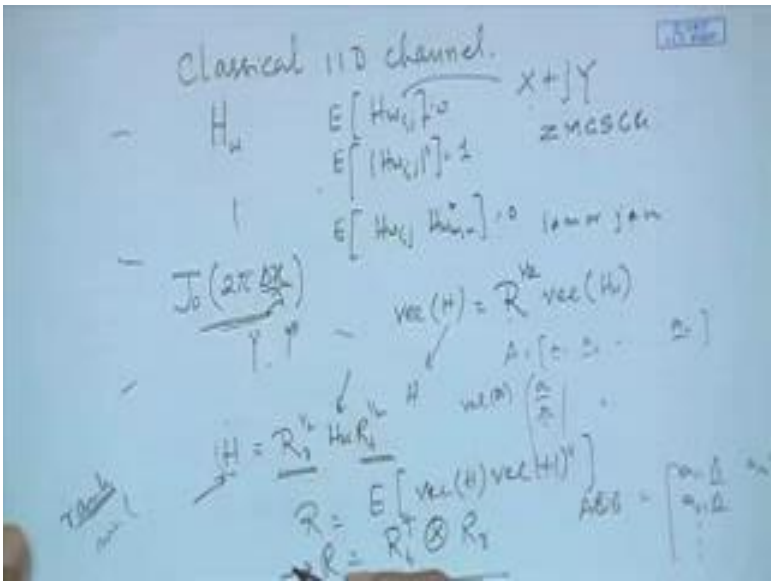

So, we have seen 1 of the most important thing are the classical IID channel this is 1 of the most important things that we have seen and it is basically the h matrix usually represented by the h w which means specially white and we have said that expected value of the elements of h w I j they are 0 and we have also said that the expected power of h w I j are equal to 1 and finally, expected value of h w I j times h w m n conjugate is equal to 0. Meaning they are uncorrelated switches specially white for I not equal to m or j not equal to n this is what we have said, we have also said that h w components are made up of x plus j y. Where x and y are independent and of 0 mean and same variants.

So, we have characterized them as 0 mean circular symmetric complex Gaussian. So, this is the characterization that we have given and we have also the special correlation the first kind of special correlation we studied was j 0 2 pi delta x by lambda, which basically shows that if the 2 antennas are separated by certain distant delta x and the correlation falls as a function of delta x with respect to lambda in this fashion and this is for uniform signals coming from all directions. If that is not the case then, you could capture the correlation in different way where, we could say that the vec of h could be written as the correlation covariance matrix raise to power of half times vec of h w. Where we have explained the vec operation means vectorized; that means, if there is a matrix a which is a 1 a 2 up to a n this is are all columns. So, vec of a - would mean you put a 1 a 2 and so on. And you can stack them up.

So, once you have got this vec of h then from this you can concern get h. So, that is the correlation is captured we where, also said that another simplified model to capture correlation is r r to the power of half times r w r t to the power of half; that means, this is the covariance matrix this is the specially white matrix which start with the classically classical white. This is covariance matrix of the receiver side this is that at the transmitter side and this is how you get it finally, if you would like to get r; that means, this r or this r you would have to get it by vec you would this r sorry this r is vec of h times vec of h our mission this is, how you would get it and you could also get back r from r t and r r as r t transpose chronicle product of r r and where we said chronicle product of matrix a cross a chronicle b would be given by a 1 one times the matrix b a 1 2 times the matrix b a 2 1 times the matrix b and so on.

So, this is how the chronicle product is that you could relate the covariance matrix. So, you start up with a classical IID and from which you generate using this covariance matrix you can generate h which is of interest. So, we will be getting most of the results for h w, but we will be interested in h as well because this is going to bring in changes to the system, but both of these are important this going to give us is very important insights. Whereas, this is going to give us practical result in many cases and while doing this we would also referring to the rank of a matrix. So, rank of matrix would mean if it is rectangular matrix the minimum of numbers of rows and columns if they are independent columns and otherwise, it is the number of non-zero rows or the number of independent columns. So, the number of independent linear equation that can be form would give the rank of the matrix.

So, what we will be using is a lot of matrix operations and we will just take a look at some of them at an appropriate time. So, what do you need at this point of time is to give some reference on linear algebra we will summarize some of the important results which will be used throughout the course.

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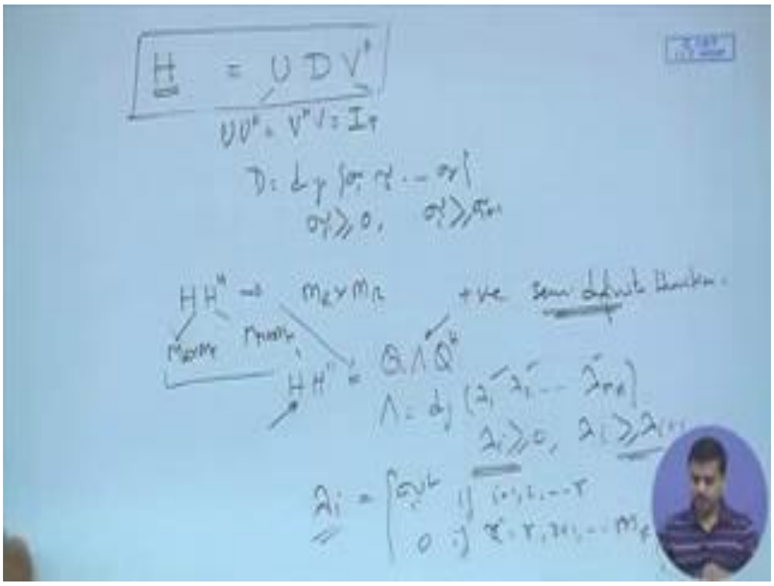

So, what will be look at the statistical properties of h if h let us write h as the matrix of our concern then, we could break h or factorize h into a singular value of u d and v are hermitian. Where u and v are united matrices and d contains the diagonal of singular values with the properties that u, u hermitian is equal to v hermitian v is equal to I of size r; that means, that r is the rank of h. So, if it is a square matrix and the columns are independent then r is equal to size of the matrix otherwise it is less than the size of the matrix.

So, this length is not equal to m in that case this length is equal to r there are r components 0 components we do not write we can always make a bigger matrix of the full size as h, but mostly we will have it with respect to rs and d is basically the diagonal containing sigma 1, sigma 2, sigma r and we have said that we will make the criteria that sigma I is greater than equal to 0 sigma I is greater than sigma I plus 1 that this is the ordered matrix and these are the singular matrices containing the singular vectors and h h hermitian; that means, h h hermitian product this is m r cross m r clearly because this is a m r plus m t and this is m t cross m r. So, basically this whole product is m r cross, m r positive semi definite hermitian positive, semi definite hermitian matrix and it can be decomposed into q lambda, q hermitian, where q are orthonormal matrix and lambda are the diagonal lambda 1, lambda 2, lambda m r, because there are m r and also it is true lambda I greater than equal to 0 positive.

So, semi definite means each of the Eigen values are greater than or equal to 0 lambda I greater than or equal to lambda I plus one; that means, this is ordered and we also have lambda I is equal to sigma I square if I is equal to 1 to up to r and is equal to 0 if I is equal to r r plus 1 up to m r. So, this is what is going to be used. So, we will be mostly using this lambda in our expression and we will be using this h h hermitian in most of our work. So, as we go ahead.

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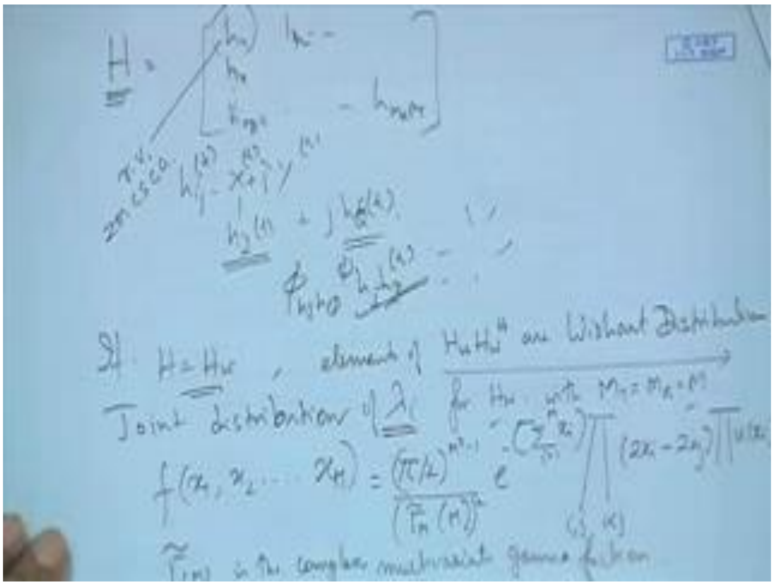

Further what we have is h if you take a look at h the h contains elements h 1 one h 1 2 and so on h 2 1 h m r 1 up to h m r m t right. So, these are the contents of h now these are random variables we have already said them. So, these are 0 mean circular symmetric complex caution so; that means, h I j could be written as x plus j y we have seen that and these are varying with time we have seen them. So, basically this is h I f t what we have studied before and this is h q of t.

So, we have studied the properties of h I and h q we have studied phi h I h q phi of h I h I we have studied all this properties and we have got this 1 follows spectrum and equal angle of arrival otherwise there is other spectrum rations spectrum and so on so forth. So, since what we essentially want to say that this contains random variables which have been very well studied. So, this itself is a random matrix. So, if h is random what we have seen before these lambda is which is Eigen values or the singular value they must also be random because whenever we look at this particular expression or we take a look at this expression that h h hermitian can be factor into this the moment h is different of course, it is going to have a different set of Eigen values of course, it is going to have different set of Eigen values and every realization of h will give us a different realization of the matrix lambda.

So, the matrix lambda contains these different Eigen values and hence all these would be different and they would follow and they would also be random. So, what we are going to look at is some properties of this lambda and properties of h h hermitian. So, if we take this we will let study the case; that means, the h is equal to h w; that means, I am talking about the classical IID channel in which case I am referring to this conditions classical IID channel; that means, each element expectation is 0 the power is 1 the elements are uncorrelated and they are complex caution. So, this is the matrix that we are taking for our study. So, if h is equal to h w then, elements of h w h w hermitian are Wishart distributed this is a particular distribution which does not look very simple, but this is useful to know and the joint distribution.

The joint distribution of lambda i; that means, we are talking about the Eigen values for h w channel; that means, we looking at the h w channel with of course, we are making 1 assumption that m t is equal to m r is equal to m; that means, you are making a square matrix assumption and this will be will be using square matrix for multiple reasons again I arbitrating that will be using h w will be using square matrix the reason is we are going to get a direct inside into things. If we do not do that then, if we take a different values of h; that means, let us say we take h which is not classical IID channel or we take the 1 which is rectangular we definitely get the results the results would not, look as straight forward or simple if we take h as classical IID or as square they would be cumbersome and they would not give us a direct insight into what the expression look like where is this you make a assumptions, we going to get a picture we going to get a view of how it looks like that is very, very useful at certain times and since we are try to get an insight will make this assumption.

So, all the other case is derivable from this assumption. So, need not worry that we are using a square matrix assumption, we will be limited in our understanding our understanding will be. In fact, better when we have this all the results were derivative of this it will be some modification on top of this, but the fundamental result would give us given by these things we will see the variation. When we take h instead of h w; that means, we are taking correlation in the particular channel. So, we proceed with this and with this particular situation the joint distribution of the lambda is; that means, f of x 1 x 2 up to x m we are not writing m r or m t, we are writing x m is given by complex expression pi by 2 raise to the power of m square minus 1 divided by gamma tilde m of m is a gamma function square e to the power of minus summation I equals to 1 to n x I product of pi indicating product over I and j, but in the order I is less than j 2 x I minus 2 x j unit step function u of x I indicating positive values and gamma tilde m of m is the complex multivariate gamma function.

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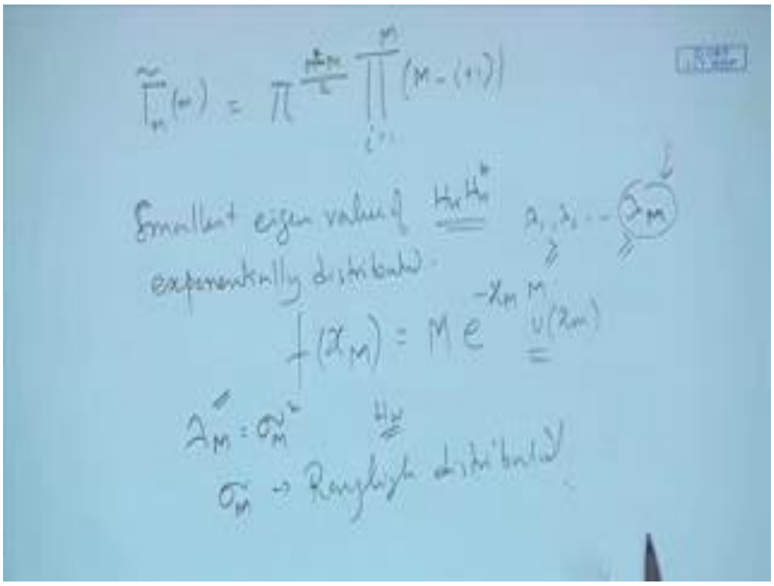

And it is defined as it is defined as gamma m of m is equal to pi to the power of m square minus m divide by 2 m square minus m divide by 2 pi of I equals 1 to m m minus I plus 1 factorial.

So, this is together. So, these 2 expression are together would define the joint distribution of lambdas. So, from the joint distribution lambdas of course, you can calculate the things that will be required or the other important thing that we have is the smallest Eigen value of h w h w hermitian. What is that we have lambda 1, lambda 2 up to lambda m? We have said lambda 1 is greater than or equal to lambda 2 greater than equal to lambda m therefore, lambda m is the smallest Eigen values. When we get this Eigen values they are ordered in this manner. So, this is exponentially distributed this is useful. So, f of x m is given by m e to the power of minus m times m u of x m this is greater than equal to 0 this is the indication.

So, what we have is this kind of thing this is smallest Eigen value this is useful in calculating the outage probability because when we talk of lambda which basically the Eigen values of the h hermitian. So, basically contains some kind of strength information about the channel. So, the if we take the minimum of; that means, we are talking about the minimum mode of the channel and what we are saying here is that this is the strength of the minimum mode of the channel is distributed exponentially. So, if we have to calculate the outage we could probably, we use this expression in order to calculate the outage probability of the minimum mode of the channel and so, basically what what it means is that the h w singular value.

That means, singular we are taken the lambda as the Eigen value of h h hermitian and lambda m is basically the minimum Eigen value. So, lambda m is equal to sigma m square right going by our earlier description what we have said here you look at this lambda I is equals to sigma I square for I equals to 1 to r and equals to 0 for up to m r and here we are said we are taking h w. We have said h w is the especially white channel of full rank with probability 1. So, therefore, r is equal to m r we have said m r is equal to m t equals to m so; that means, this equation holds for I equal to 1 to m. So, lambda I is sigma s squared. So, that is what it is. So, this is exponential distributed in in that case sigma m would be Rayleigh distributed.

So, this is again another important result which can help us in understanding some of the properties of performance of MIMO communication system. So, these are some of the important results we move on further.

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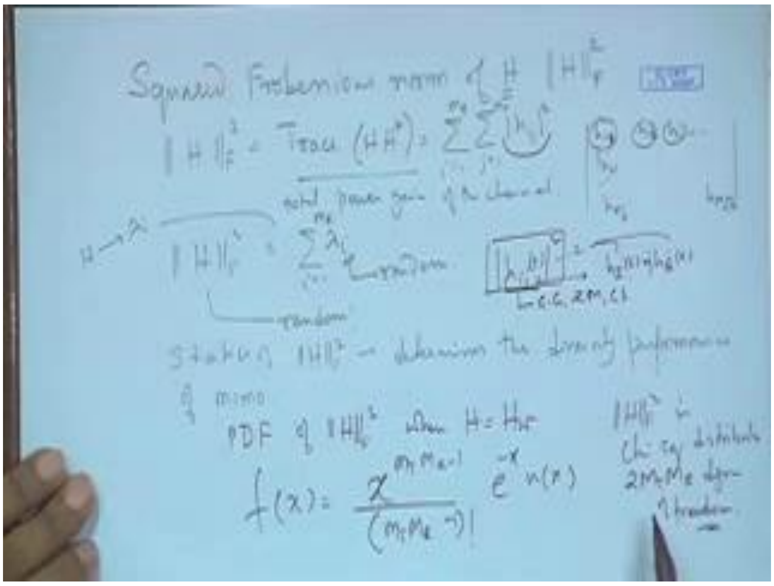

And we take a look at this squared Frobenious norm of h. So, it is indicated by h with double lines on both the side square on top and below f indicating Frobenious norm squared Frobenious norm of h which is defined as h of squared is equal to trace of trace is a matrix operation of h h hermitian we have explain what is hermitian operation hermitian operation is the transpose with the complex conjugation which is equal to I is equal to 1 to m r j equals 1 to m t and h I j mode squared. So, if you take a look at this j is going from 1 to m t I is going from m r and we have h I j; that means, we are covering all the channel coefficient and if you take a look at this h I j squared is basically the strength of the I j component.

So, h 1 1 h 1 2 h 1 3 and so on, h 2 1 up to h m 1 finally, h m m in case of h w channel otherwise m r 1 m r m t, this squared plus this squared plus this squared plus this squared. So, all they squared. So, what we are getting is the strength of the channel. So, this is in other words this is can be interpreted as the total power gain of the channel this could be 1 way of expressing this and we also have the expression h f squared is equal to I equals to 1 m r lambda i; that means, you are adding up all the Eigen values . So, that is again from the trace of h h hermitian trace of h h hermitian is the sum of all the Eigen values where lambda I are the Eigen values of the h h hermitian. So, again just have a look at this since lambda is are random as we have said previously this is also random naturally because h is random h leads to lambda is being random if lambda is are random this whole sum is random and therefore, this is also random.

So, this is also random variable and the statistical properties this statistics of h f squared determines the diversity performance of MIMO systems we will see how I mean this is very, very important random variable. And if you take the p d f of h w; that means, if you take the p d f of h f squared when, h is equal to h w right. So, we are taking for the especially white channel we could write f x as x raise to the power of m t m r minus 1 divided by m t m r minus 1 factorial e to the power of minus x u of x. So, this is always positive 0 greater than equal to 0, 0 only when this is equal to 0. So, the expression for the p d f is this and therefore, we could say that h f squared is chi square distributed and with 2 times m t m r decrease of freedom.

Why do we say so? Because if we take a look at this let us take a look at this 1 h I j squared. So, h I j squared is basically we could write it as h I j of t if you remember mode square. So, this h I j of t is complex caution it is 0 mean and it is circular symmetric complex caution because h I j could be written as h capital I of t plus h capital q of t indicating that is I component there is a q component of course, there is j we have studied the statistics 0 mean we have seen the Rayleigh distributed circular symmetric because these are independent of equal variance. So, if this is Rayleigh this whole thing is chi square distributed with 2 degrees of freedom first degree and the second degree and how many of them are there m r and m t are there. So, m r times m t times 2 so; that means, 2 m r m t degrees of freedom of chi square distribution.

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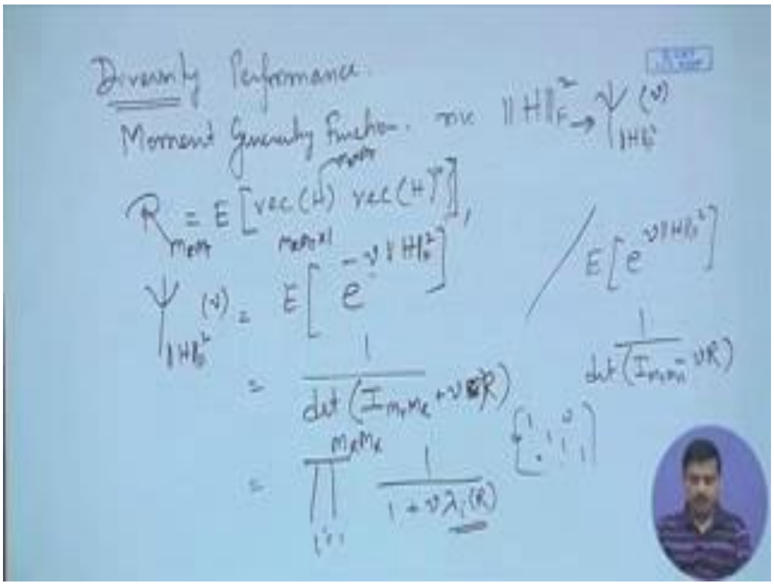

So, this is this is 1 of the important things that we need to know and will be we may be using them and the quantity of interest to calculate the diversity function since when you calculating the diversity performance of MIMO communication systems. We would be using the moment generating function this is quite useful especially when you calculating the b r. So, there are different ways of calculating the b r, but 1 particular result is exist which is often quite useful in giving this performance.

So, we do not it I mean if you divert the function of the random variable we interested in h f squared and if we write it as psi of h f squared of let say, nu this is how we are going to present it. So, assuming Rayleigh fading we assume Rayleigh in h I j. We could say r covariance matrix is expectation of the vec of h this we already seen vec of h hermitian this we have already seen this expression right. So, this a covariance matrix of h and psi of h f squared of nu is given as expectation of e to the power of minus nu h f squared now in some cases you could write it as e to the power of nu h of squared it is all the same it depends on what we what we do at the end. So, this has been evaluated it is not very strict forward evaluation. So, if this is evaluated for the Rayleigh fading this 1 turns out as 1 by determinant of I matrix I is the identity matrix of size m t m r plus nu times r if it is plus over here you would get 1 by determinant I m t m r naturally a minus nu times the lambda I sorry yeah sorry sorry this is times nu of r.

Now, if you look at this the determinant we could write it as product of I equals to 1 to m t m r 1 by since is the I matrix 1 plus nu times lambda I of r the size of r is m r m t because this is this is vec, vec means h is m r cross m t. So, vec of h is m r m t cross 1 this is 1 cross m r m t. So, this is basically m r m t of the size m r m t. So, the determinant is the product of the Eigen values and product of the Eigen values would mean that lambda I is at the Eigen values this matrix is an high matrix; that means, it is 1 one 1 one and 0s elsewhere right. So, basically the diagonal entries get this 1 added to it. So, determinant can be formed in this fashion. So, once we are writing this particular expression is going to help us in calculating the diversity performance because will be often getting expressions of this order which will be very, very useful.

So, what we need to note is we have to use the Eigen values of the r and we have to find this variable are nu or v which will be very much helpful in calculating the psi of nu and they will be displacing nu with the appropriate value and will be able to calculate r error probability expression. So, to summarize what we have done is primarily the most important thing we have looked at in this particular session is the squared Frobenious norm. And how we have derived how we have found what squared Frobenious norm is, squared Frobenious norm is basically the total strength of the channel.

And if we look at the moment generating function of h of squared it turns out to be of this structure which is product of 1 over 1 plus nu times lambda I r, where nu is given in the psi is given by this expression. So, I have already said this could be a minus this will become a plus if there is a plus over here it would become a minus over here, it does not matter basically because this whole expression will be used. So, the sign is unimportantly in our case, will be using this for calculation of probability of error which is again very, very important expression for use.

We will be using this in the future lectures on (Refer Time: 27:50) performance of calculation of diversity schemes in MIMO.

Thank you.