**Fundamentals of MIMO Wireless Communication**

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**Lecture – 20**
**MIMO Channel Characteristics**

Welcome to the lectures on fundamentals of MIMO Wireless Communications, we have moved into the special channel characteristics. We have seen how to write the expression for the SIMO, MISO and MIMO channels.

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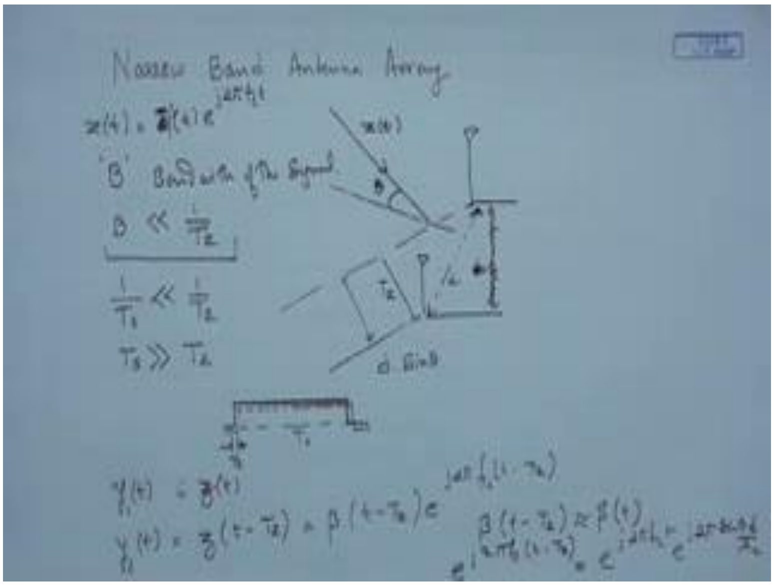

Now, we will look into the narrow band antenna array, which is one of the important assumptions we also look at the classical IID channel, which is the one we are going to use mostly and we will also look at the special correlation model which will be essential in rest of the course. So, with that we take a look at the narrow band antenna array, and in this case let us say that we have one antenna element here and we have another antenna element here, the separation between these 2 is d and we have the rather the diagram should be better if you special separation is d in this direction this is the proper access. So, let this be the a d separation between them and so, if this is the normal to the direction, let a wave front arrive and that wave front be the z of t and let it make an angle of theta with respect to the antenna array.

So, clearly what we can see is that, we wave to arrive at the first antenna it takes a certain amount of time, from the time it reaches the first antenna, to the time it reaches the second antenna and let this be denoted by t z. So, this is the extra amount of time that takes in propagating. So, this distance can be calculated to be d sin theta, you can do this expression if this is d then, this would be d of sign of theta as per this particular geometry. So, the this is the propagation is the distance between the antennas effective distance which aspect to the angle of the wave front and will also assume that z of t is represented by z s tilde t e to the power of j 2 pi x t. So, if we say that this is z of t in that case what the narrow band antenna array assumptions says that 1 by or sorry if this this particular signal has a band width of b lets b be the band width of the signal in in some cases you could also write it as beta of t times e to the power of j 2 pi of c t. So, if this is the band p band width of the signal, we would say that b is less and less than one by t z this is the narrow band antenna array assumptions. So, if you go by this the b is approximately could write it as one by t s, is less than less than one by t z that would mean that t z or t s is much greater than t z.

So, what would this mean? This would mean that if I would draw the symbol duration t s this is let this b t s that is this symbol duration corresponding to the band width of the signal this is much-much larger than t z. So, if this signal arrives at the first antenna the second symbol or the symbol would arrive at the second antenna as per this dashed representation and this gap in time is t z which is due to the propagation of the signal from the first antenna, to the second antenna with reference to the angle at which it arrives and the separation distance between the antenna. So, this is somewhat similar does not exactly the same, but somewhat similar to the assumption that we have made regarding a flat fading that is we said that the delays are between the multi parts are very, very close. So, they are almost the inter part delay is negligible. So, here what we are saying is that, the wave arriving between the first antenna element and the last antenna element are there is hardly any, any notable delay compare to this symbol duration. So, this is with reference to symbol duration.

So, if we say that the signal which is arrived in the first antenna is y 1 of t is z of t let say the signal which is arrived in the first antenna is z of t then the y of t is the signal y 2 of t which is arrived in the second antenna is z of t minus t z; that means, the signal which is here is as z of t minus t z or it takes extra time t z or the signal which is a started t z times a before. So, that is the delay between these 2 antennas, which would be beta t minus t z e to the power of j 2 pi f c t minus t z. We will also make the assumption that there is a ideal identical antenna pattern; that means, the antenna patterns of this and this are the same so; that means, we can make the assumption beta t minus t z is approximately equal to beta t; that means, the signal is almost the same or in other words you could have e to the power of j 2 pi f c t minus t z is equal to e to the power of j 2 pi f c t times e to the power of j 2 pi sin theta times d by lambda c; that means, the or d by lambda which is the wave length.

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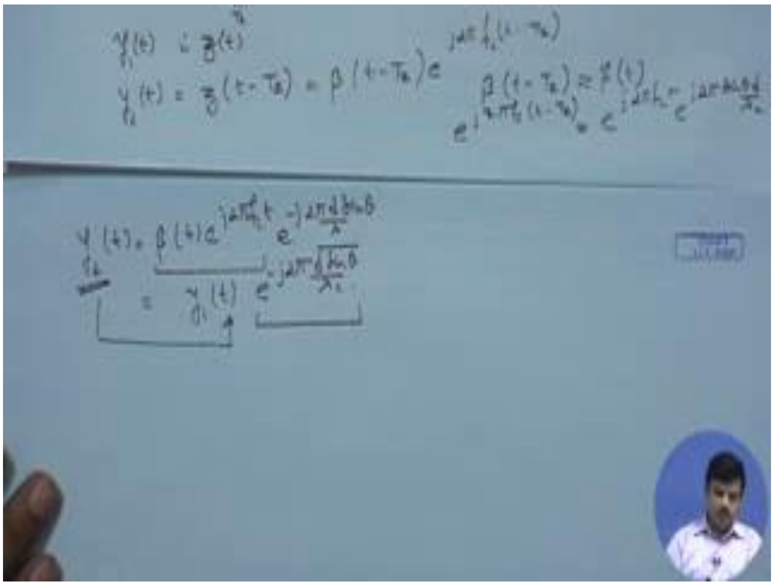

So, y 2 of t; that means, this is with this assumption we have y 2 of t is equal to beta t e to the power j 2 pi f c t e to the power of minus j 2 pi d sin theta by lambda which you could write this expression as y one of t e to the power of minus j 2 pi d sin theta by lambda c. So, if you look at the signal which is arrived at the second antenna is basically the signal at the first antenna along at which there is a phase delay due to the effective separation between the 2 antennas. So; that means, the signals which arriving at the different antennas are only differed from each other because of a relative phase delay and rest of the signal is almost the same.

So, this is one of the fundamental assumption when we make antenna narrow band antenna array; that means, we are saying that this distance is not large this distance is much smaller compared to the symbol duration. So, the signal is effectively the same, but there is a phase difference in the signal when it arrives at the antenna array from the first antenna to the, for this 2 most antenna otherwise, the signal is more or less the same. So, this is one of the important assumption that we make otherwise the system model would change if we do not have this narrow band antenna arrays option.

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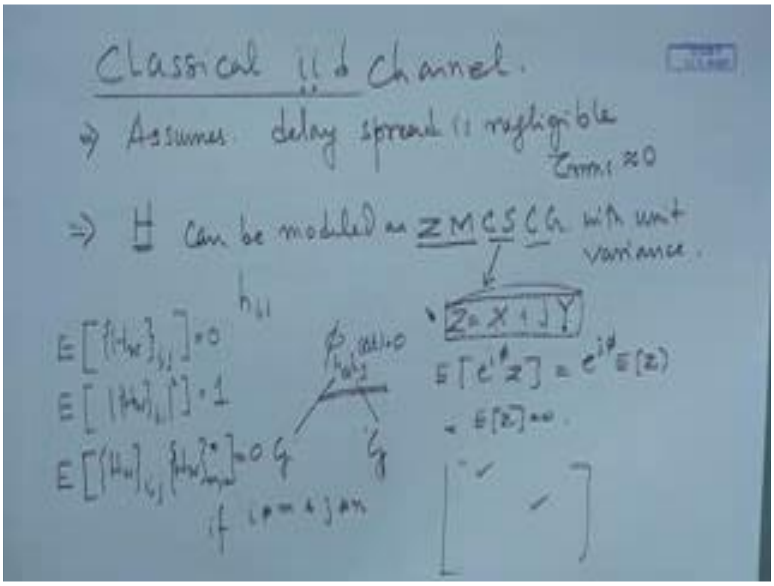


With this we move on the next important description of these kinds of MIMO channels, and we have the classical IID channel. This is one of the important channels which we will use in most of our work, the classical IID channel assumes the first assumption it makes is that the delay spread is negligible. The delay spread is negligible in other words tau r m s is approximately is equal to 0. So, that is what we have already said in the previous lectures that in rest of the course we going to make this assumptions. So, this is basically you are referring to one of the conditions of the classical IID channel. The second one it states is that the H with this 2 underline indicates the MIMO channel can be modeled as 0 mean z m is 0 mean circularly symmetric complexes Gaussian with unit variance. So, this will be with a 0 means each of the components of H r having a mean of 0 we have already seen.

Let say we have H i comma j as one of the component we have seen the situation, where H i j is the mean of H i 0 and; that means, there is scattering from all directions. So, that is the one of the cases circularly symmetric would mean that it is complex Gaussian; that means, if it is z is represented as x plus j y these are real Gaussian and they have 0 mean as well as they have the same variance. So, that is the circular symmetricity leads to us and they are independent of each other and complex Gaussian is of course, these particular expression, circular symmetricity usually would mean that expectation of e to the power of j 5 of real 5 of z is equal to e to the power of j 5 and e of z which leads to e of z being is equal to 0 so; that means, effectively we have this as the channel coefficients where, x and y components are independent. We have seen that 5 H q H i delta t they are uncorrelated equals to 0 this is what we have seen and we have seen that H q and H i are Gaussian both are Gaussian. So, for 2 Gaussian random variables which are 0 mean and they are uncorrelated would mean that these are independent.

So, in our case we have H i and H q components are independent. So, if they are 0 mean and they is complex. So, basically it is a 0 mean a circular symmetric are complex Gaussian representation. So, this this condition satisfies and in other words we are basically stating that expectation of H w, we will describe what is H w where basically talking about H w in terms of specially white. So, w indicates white you know about white Gaussian noise. So, white means in this phase dimension there is probability of rays arriving from all directions with equal probability. So, H w i j; that means, for this matrix ith i comma j element; that means, ith row j column element expectation of that is 0 this is one of the first thing the second thing is once we have said the mean is 0 would say that H w i comma j the mod squared value of at is 1; that means, it is unit variance and it is surplus symmetric; that means, uncorrelated or specially white. So, we have we are talking about this classical IID; that means, it is independent identically distributed independent and identically.

So, what we have is H w i comma j H w m comma n conjugate if you take this product, is equal to 0. If I is not equal to m and j is not equal to n so; that means, if I take this matrix and if I take any 2 elements  and we are takes their products with the conjugate of 1; that means, projection of 1 on the other and take the expectation or we are taking the correlation of them they are uncorrelated with each other they are being Gaussian they are been independent.

So, basically we are saying the spatially white and that is being indicated by the H w that we have over the. So, this describes the classical IID channels, which is one of the very important channels that we use in describing many of the results.

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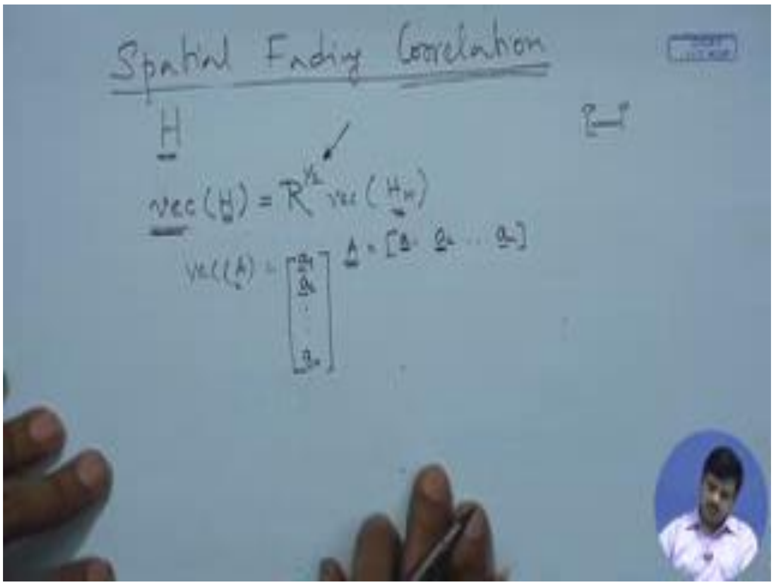


So, we have seen two very important characteristics or two very important description of the MIMO channels which we have going to use. Next we are going to look at the spatial fading correlation. So, we have described is a white channel; that means, it is uncorrelated all the points are independent. So, now, we have to look at the correlation because we said if antennas are spaced. So, that this is less than the co variance distance there will be correlations in this signal, independent of H are correlated. So, basically we are talking about H the channel matrix which is having correlated elements.

So, this is modeled has vec of H this is the vectorization of H H is a matrix will describe how to vectorize it is r which is the covariance matrix a square route of that times vec of H w in other words it says that if I can generate H w; that means, a spatial white channel and I have this correlation model then using this or rather if I have this I can generate H w and these would bring correlation amongst the elements I would generate a channel matrix which should be correlated. So, this is spatially white channel using the covariance matrix i could generate a spatially correlated channel coefficients which will be useful in performance analysis.

The vec operation it basically stabs the columns of a suppose vec of a is what we are interested in and if a is equal to a 1 a 2 this is the column, this is the column, this is a column i put it as an underline underscore indicating it is a vector up to let us say a n. So, this is a matrix this is a column, this is a column, this is a column, these are all vectors. So, these matrix vec of a would be the operation a1 a2 so this is the whole column this is a whole column up to a n. So, a 1 stacked above a 2 stacked above a 3 and so on and so forth. So, this is how you do the vec operation. So, with this you could generate this vec this particular H and from vec of H you could get back to h.

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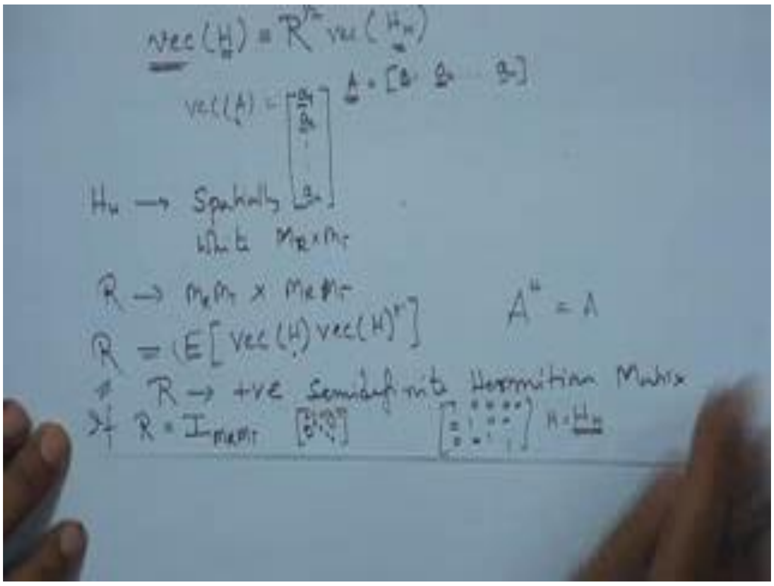

So, H w that we have over there is a spatially white m r sorry it is m r cross m t channel matrix and r is m r m t cross m r m t covariance matrix and r can be obtained as e of vec of H time's vec of H dimension look at this. So, basically if you have H from H you could get r. So, this is another way of stating it. So, if I have r, if I have r, then I could get H if I have h, then I can get r. So, this is either way of going from H w to H or from going from H to r. So, this is a expectation means over several realization of H you could get the covariance matrix and this being a hermitian matrix you can clearly see this is hermitian matrix. So, r the covariance matrix is positive, semi-definite hermitian matrix. What is it mean it means that it is hermitian; a hermitian means the hermitian of A would be equal to a. So, that is hermitian matrix positive semi-definite would mean that all the Eigen values of H are all greater than or equal to 0. So, that is the, that is what it means of course, there are more-more description.

So, from this point onwards as you can clearly see we are using some of the results of linear algebra. So, again at this point it would be very crucial if you could revise some of the x important formulas from linear algebra which will be used quite often in this case. So, now, you can see is that if r is equal to i m r m t i m r m t means it is an identity matrix of dimension m r mt; that means, identity matrix is the one where only the diagonal elements and one rest of all them are zeros. So; that means, what it is says is that it is correlated only with this with itself and the result of correlation is 1 which is pretty natural. When you correlate with any other element the values are all zeros; that means, it is 1 1 1 1 0 0 0 0 all zeros except the diagonal element all are zeros so; that means, any cross correlation term is 0. So, if this is the case, if it is uncorrelated with any other element naturally it means that what we have is H w, clearly if this is identity matrix if you even if you go by this, if it is identity matrix then H is equal to H w straight forward from this also. So, in other words it is basically bi directional that we have trying to explain if you get r as identity that would mean that you are dealing with n a spatially white channel.

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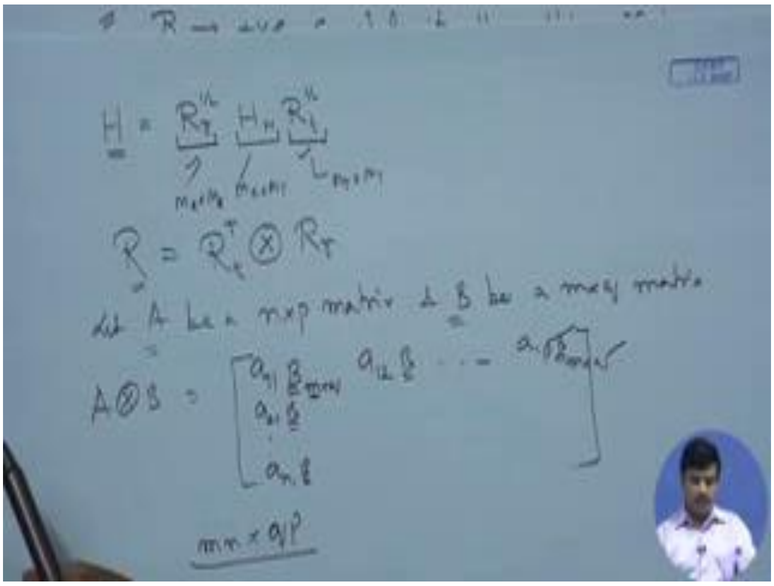


Ah instead of this sometimes another way of describing the correlation is also followed where it is simplified in term of H is equal to the receive covariance matrix is to the power of half H w the transmit covariance matrix raise to one half; that means, you have H w one has measured to receive covariance matrix or the inter antenna correlations because of the spacing and stating environment and the transmit antenna covariance. So, if I have these 2 measures separately again using H w you could generate the H matrix which has correlation. So, r t is m t cross m t covariance matrix this is m r cross m t and this is m r cross m r covariance matrix. So, both since these are covariance matrices these are again positive semi-definite matrices. So, we can use this method or this method to arrive at H which would be having spatial correlation.

Now, the relationship for connecting r t and r would be r is equal to r t transpose chronicle product with r of r where a chronicle product could be defined in a way that let a be a n cross p matrix and b be a m cross q matrix if a and b are 2 such matrices then the operation a chronicle product b would be defined as a 1 1 which is the first element of the matrix a times the matrix b then a 1 2 times the matrix b and so on up to a 1 p times matrix b here it will be a 2 1 matrix times b up to a n 1 times matrix b matrix b is of the size m cross q this is n. So, definitely this whole product is m n you can clearly see m number of elements here n number of elements here m n cross you can guess the next one this is p this is m cross q. So, p and q would lead q p. So, this is the dimension of the chronicle product that we get. So, here what you would get is m t cross m t m r cross m r. So, basically m t m r cross m t m r is the size of r which is already described as the 1 of the r particular matrix.

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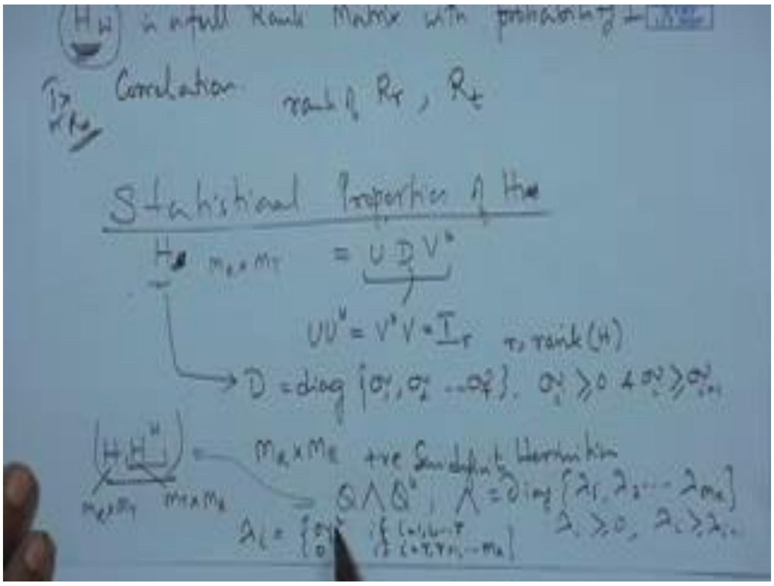


So, this way we could add correlation properties into the channel matrix which is otherwise spatially white now is, this H w channel that will be using in our case is having the property, that H w is full rank matrix with probability 1 is the full rank matrix with probability 1. So, this is very, very important to consider. So, this, this we should remember that H w is random. H w is random because it contains m m t cross m r number of elements each of the elements are complex Gaussian 0 mean and circularly symmetric. So, that is H w entries. So, it can form matrix because it is a random matrix which can have certain amount of correlation the moment there is a correlation rank is no longer a full rank matrix. So, what we are saying is we are talking the channel the channel contents or the channel elements of H w in such a way that it is full rank matrix with probability 1. So, if we are using H w it will be the 1 which is full rank.

In presence of if there is any correlation. If there is any transmit or receive correlation the rank is modified and will constrain by the rank of r r or that of r t. So, it is constraint by the rank of r r and r t and then the rank of the matrix is less than the full rank of H H w will see many thinks in details later on. With these we would move on to sum of the statistical properties of H w, this is somewhat important. So, if we have are talking about the statistical properties of h; that means the channel matrix.

So, H is m r cross m t matrix. So, you can breakdown H H into u d and v hermitian by means of singular valid decomposition. If you do a singular value decomposition you going to get d as the diagonal matrix and u and v are such they are inventory matrices such that u times u hermitian is equal to v hermitian times v which is equal to I of ordered r where small r is the rank of h. r is equal to rank of H this is how the relationship holds. So, if you do singular valid decomposition you are going to get d as the diagonal elements you write it as the sigma 1, sigma 2 up to sigma r with sigma I greater than equal to 0 and sigma i greater than or equal to sigma i plus 1; that means, the diagonal elements are arranged in such a way that the first element is greater than the second element and dash-dash up to the last element. So, the last element is the lowest singular value and all of them are greater than or equal to 0. So, they are non negative that is one of the outcomes of this particular process.

If you look at H H hermitian this is m r cross m t this is m t cross m r this is m r cross m t size, this size is because of the hermition operation m t cross m r. So, naturally this whole product is a m r cross m r this is a hermitian matrix and is positive semi-definite. So, it being positive semi-definite again the product of the or the Eigen values are greater than equal to 0, and these H H hermitian you could decompose into H H hermitian you could decompose it into q lambda, q hermitian where lambda is diagonal matrix consisting of the Eigen values H one sorry lambda 1, lambda 2 and up to lambda m r. So, these are the Eigen values of the H H hermitian and lambda is r greater than or equal to 0 and lambda is greater than or equal to lambda i plus 1, again similar thing like the singular values. So, with this we further have the description that lambda i is equal to sigma i squared if i is equal to 1, 2 up to r n is equal to 0, if i is equal to r r plus 1 up to m r.

So, we have said here is that whether, we look at the singular values of h; that means, here we have H from which we have arrived at the singular values or we take the H H hermitian and we arrive at the Eigen values of H H hermitian, they are related to each other in the form that lambda i the Eigen values the ith Eigen value of H H hermitian of H H hermitian is equal to square of the ith singular value of the h. So, this is the relationship that we can use in all our expressions that will encountered in future and when we talk about the singular values or singular value decomposition or the Eigen values in this when you do the Eigen value decomposition will assume the first Eigen value is greater than the second Eigen value. So, on up to the last Eigen value.

And they are all greater than or equal to 0. Now this positive semi-definite of this H H hermitian and H H hermitian you can or these r r matrices or this r matrices; that means, r r or r t matrices, you can easily verify from the first principle. That means, if you set out you set out with let H be equal to H 1 1, H 1 2, H 2 1, H 2 1. Suppose you set up certain with H you will you will arrive at conditions where you going to get that this particular product H H hermitian this particular product, is having only terms which are like mod squared terms and mod squared terms are greater than or equal to 0 the minimum value that mod squared term can get is as 0. Otherwise I mean it cannot take negative values and that is that is another reason why you call it positive semi-definite.

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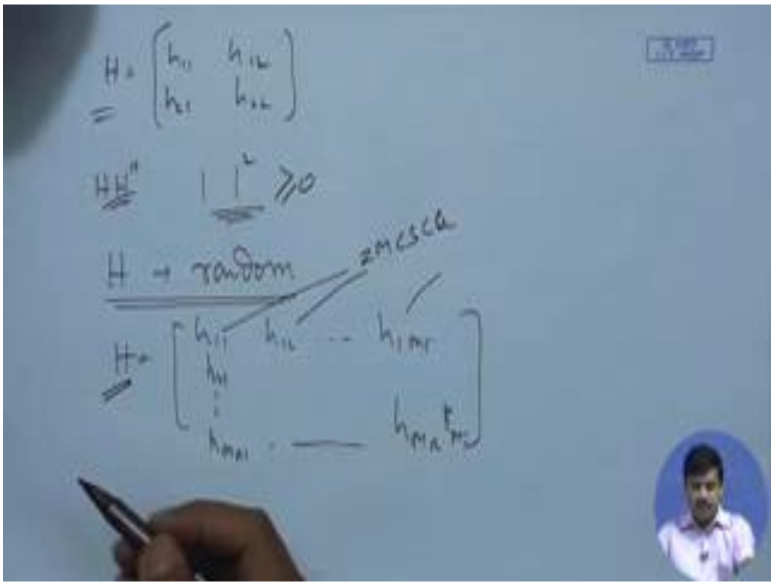

Now, one more thing to remember that since H is random because H is made up of H 1 1 1 H 1 2 up to the H 1 m t H 2 1 up to H m r 1 and H m r m t each of these are random that is what we said these are 0 mean circularly symmetric complex Gaussian. These are all random what H is random if H is random so are the singular values and so are the Eigen values.

So, we should remember that these are all also random values and what we can what we will briefly look at in the next lecture is about the distribution of this Eigen values and that of the minimum Eigen value of h. So, that would give us a description about the H channel which is the MIMO channel coefficient. So, that is all for this particular lecture, in the next lecture we going to see the distribution of this lambda and also one particular MIMO (Refer Time: 31:10) functions for (Refer Time: 31:12) H which is also very important which will be using in the (Refer Time: 31:16) array calculations.

Thank you.