Fundamentals of MIMO Wireless Communication
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Lecture -12
Small Scale Propagation Received Signal Correlation (Contd.)

Welcome to the lecture on Fundamentals of MIMO Wireless Communications. We are currently discussing the signal correlation properties and we have gone half with through the derivation of signal correlation. So, we start from the point where we left in the last class.

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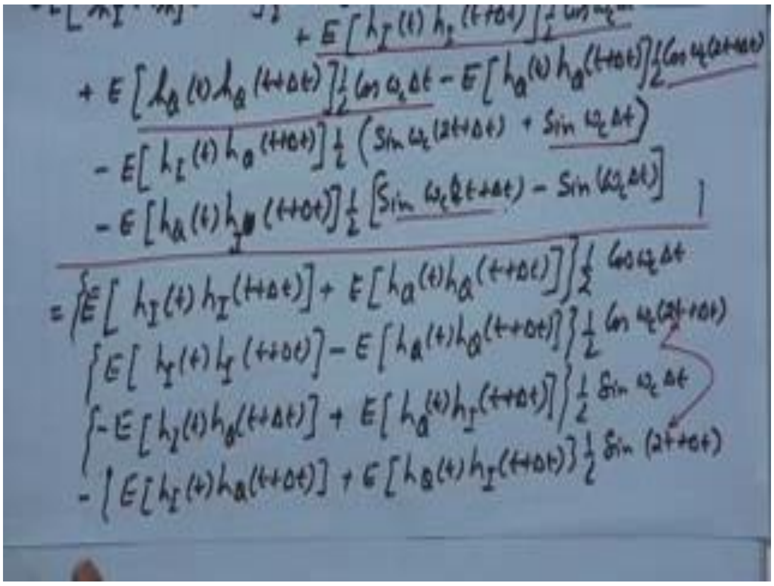

So, we had the expression as in this particular picture has this particular expression, and we started off with the expression here received band pass signal correlation as phi rr of delta t with r t and r t plus delta t.

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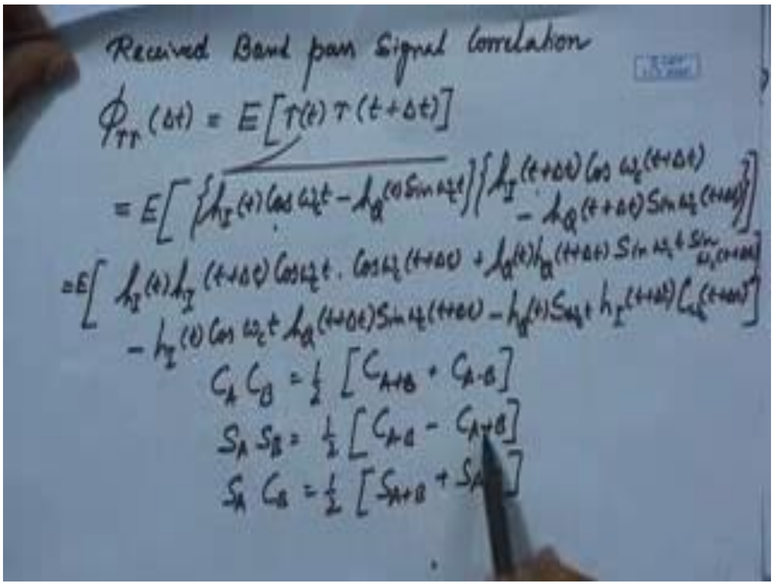

We expanded each of this expressions and it all multiplications use the trigonometry identities following which we collected like terms and finally, we have the expression where we have half cos omega c delta t along with the certain coefficient and half cos omega c 2 t plus delta t along with a whole set of coefficient. So, we have gathered the terms with t together and we have also gathered the terms without the t. So, we have this 2 sets of. So, we have basically four functions with coefficients of which 2 are with t and 2 are without t. So, at this point we will make very important assumption or and this is a very valid assumption details of which we may look into later lectures, but as of now we assume that this particular signal r of t is wide sense stationary.

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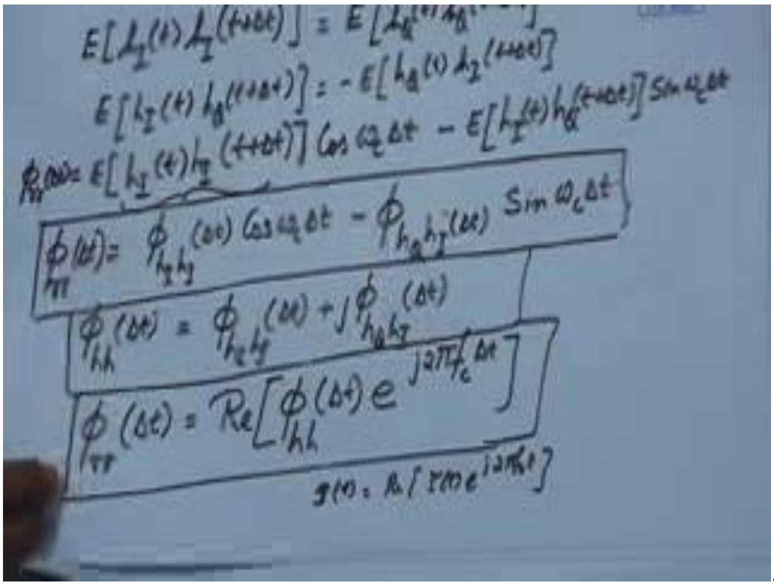
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We will assume a wide sense stationary for this band pass signal and that is quite valid and it is proposed by particular model known as wide sense stationary uncorrelated scattering in a famous paper by Bellow in the 60s. So, we will simply use that result over here and see what we get if you assume wide sense stationarity what we defining over here is phi rr of delta t. So, going by that this phi rr of delta t - if it is wide sense stationary process then it will be independent of t and will be dependent only on delta t. So, if we use this particular set then what we are left with is these functions which are function of t should have the coefficient which is 0. So, if we take that what we get is from this particular one we would have E of h I of t h I of t plus delta t is equal to E of h Q of t h Q of t plus delta t because if this has to be 0 then this must be equal to this.

Secondly, what we have is the other function of time is the sin 2 t plus delta t. So, if it is wide sense stationary this should have 0 coefficient and going by that we are going to have E of h I of t h Q of t plus delta t being equal to minus E of h Q of t h I of t plus delta t. So, in this expression we should have minus inside which was kind of (Refer Time: 04:25) in the previous expression. So, sorry this is minus over here if you set this whole equation to 0 we going to have this plus this equal to 0 or E of h I t h Q of t plus delta t is equal to minus of this then only the whole equation turns out into 0.

So, if we now apply this solutions into this expression what will be left with is this first equation there is a first expression; that means, this term as gone to 0 because this is wide sense stationary. Similarly this term as also gone to 0 and top of which we have results that h I of t h I t plus delta t h Q of t h Q plus h Q t plus delta t. So, with this 2 be equal we can use this in the first equation; that means, we going to use this particular expression in substitute over here. So, we are going to get 2 h I t h I t plus delta t because this is equal to this and this two will cancel out with this half. So, will be left with expected value of h I of t, h I of t plus delta t, 2 and half cancels out cos omega c delta t. So, that is one of the terms we are going to left with. Now this term is also going to 0 which gives these is equal to minus of this, if we take that and replace it over here then we are going to get h Q h I is equal to minus of this term. So, will be getting 2 times minus of this term, minus. So, again the 2 and this half cancels out minus E of h I of t h Q of t plus delta t with sin omega c delta t.

So, now we have the expression for phi rr of delta t. So, we would have this as - this particular expression we could replace it as auto correlation of h I h I parameter as by delta t cos omega c delta t minus similarly we could write same for this. Now instead of writing h I h Q one way of writing this would be h Q h I delta t indicating correlation of h Q with that of h I we could also write it as h I h Q it is a matter of notation we would be following this notation. So, where ever you are writing this things you should follow the same way of representing this notation. It is not a uniform notation or in some cases people would write h Q h I some cases you would get it as h I h Q. So, we would follow this particular notation as used over here.

So, this is phi rr delta t. So, looking at this expression or if you would derive the expression for phi hh, you would also get it as phi hh delta t is equal to phi h I h I delta t plus j phi h Q h I delta t and then we would write phi rr delta t is equal to real part of phi hh delta t e to the power of j 2 pi f c delta t where 2 pi f c is omega c. So, we have this three important expression which describe the signal correlation.

So, through the last expression what we see is that the signal correlation in the band pass of the pass band can be related to that of the signal in the base band equivalent through the carrier frequency, and the real part of it just like we have earlier written that r t or s of t is equal to real part of s tilde t E to the power of j pi f c t. So, this is the expression is similar and what we can see is that again the translating from pass band to base band and base band to pass band is identical and hence it is related through the f c and the corresponding functions of base band or pass band. So, we can study the base band equivalent or the complex envelope and we could still characterize the channel.

So, as of now what we have over here is the pass band char correlation function is a related to the I component of the base band correlation function with cos and the cross components. So, this auto correlation component this is a cross correlation component along with the cos omega c delta t and corresponding sin omega c delta t.

So, with this we would now proceed to study this phi h I h I and phi h Q h I because they characterize phi hh because only after understanding, only after expanding this components we will be able to completely characterize phi rr, because phi rr characterize in terms of phi hh. So, from this point we proceed with our description of phi hh. So, in order to study phi hh; that means, auto correlation of the complex envelope what the base band equivalent we would first begin with the auto correlation of the I components.

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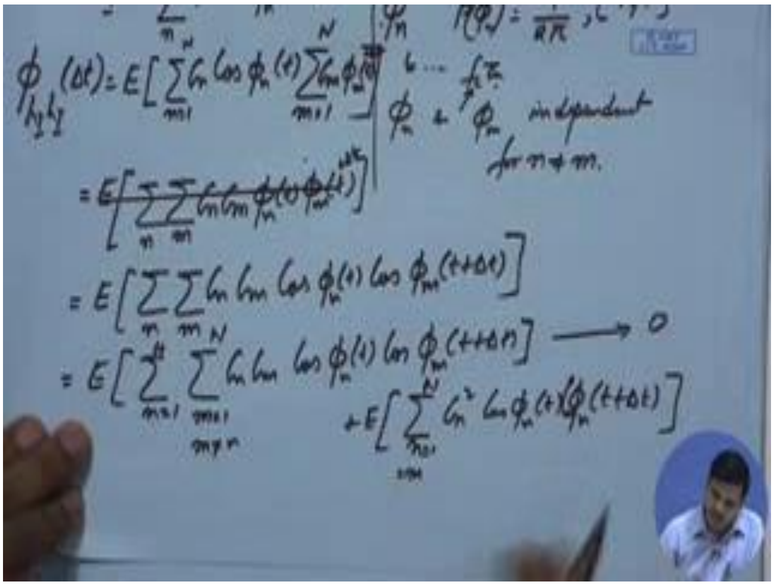

So, what we have as h I - if you remember it is n C n cos phi n of t. At this point we would iterate the assumptions which we have made earlier and they were phi n is uniformly distributed; that means, p phi n in the range of 2 pi; that means, pi 2 pi why this was so because phi n had a term amongst others which is f c tau n and we said that since f c is very very high when it close to gigahertz or few 100s of omega hertz small change in tau n would result in big changes in phi. So, we would make. So, it is a reasonable assumption to make phi to be a random distributed we can make the assumption is uniformly distributed in the range minus pi to pi.

We would also make the assumption at phi n and phi m is independent for n not equal to m. Now, this is again part of assumption that we made wide sense stationarity because there is full model is wide sense stationarity uncorrelated scattering which leads to basically that it appreciates in the different direction or the nth paths are independent of each other. We will try and see them later lectures. As if now we will use these assumptions which are common and which are fundamental in helping us derived these expressions.

So, we try to calculate phi h I h I delta t which is equal to expected value of sum of n equal to 1 to capital N C n cos phi n of t times m equal to 1 to n C m cos phi m of t this expression and again we will follow the same techniques; that means, we will expand it. So, we could write sum over n sum over m C n C m phi n of t phi m of t sorry it is t plus delta t t plus delta t. So, that would be E of; we have a cos sorry excuse me we would write it as C n C m cos phi n of t cos of phi m of t plus delta t. So, we have a cos a and cos b we will expand that, we will have n equals to one to n and we will break it into 2 parts.

So, the first part we will break where m is not equal to n C n C m cos phi n t cos phi m t plus delta t and we have the other part which is sum over n equals to 1 to n and n, n equals to m equals to 1 to m, C n square cos phi n of t and phi n of cos t plus delta t this is cos. So, we have this 2 terms we break we broken into 2 terms for a ease of our steps that which are supposed to take.

And if you look at this particular first term, if you look at the first term we have made some assumptions that phi n and phi m are independent for n not equals to m. So, we have created that set of summation where n is not equal to m and if they are independent we have expectation of cos phi n times expectation of this and if phi n is uniformly distributed between 0 to 2 pi we can say that this whole term expectation of this term goes to 0 and that term goes to 0. So, this term would finally, end up with 0 and will be left with only the second term. So, if we proceed with the second term what we are going to have is again this cos of phi n and cos of phi n t plus delta t. So, this could be again broken up into 2 parts; that means, cos of phi n t plus phi n t plus delta t and one which is the difference.

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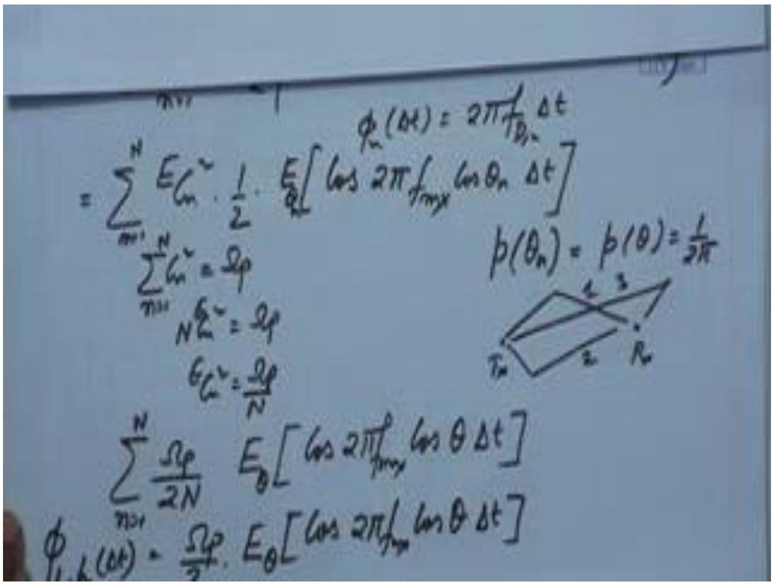

So, when we have that will be getting E of sum over n equals 1 to capital N, C n square and of course, there is this half cos of phi n of t plus phi n t plus delta t plus cos of phi n times delta t because when we take it out phi n of t minus phi n of t plus delta t. So, will be left with this term and if we look at phi n of delta t this is 2 pi f d n times delta t. So, that is what will be left in this particular expression.

And if we look at this term which is phi n of t and phi n of t plus delta t you are left with phi n of 2 tin inside this and again as we bring expectation operator inside this term would be going to 0. So, what will be left with is this particular term. So, if we take this term then we could write it as summation of expected value of C n square times half times expected value of cos 2 pi. So, for our case what we have is f max times cos theta n times delta t. If you refer to the earlier expressions of phi n f d n, d n is basically max times cos theta n you have to get this expressions. Now we are going to make some assumptions again or if we look into the details, so that we are able to derive the expression correctly one thing we would remember that sum of C n square is equal to omega p and n equals to 1 to n n equals to 1 to capital N. So, if we have the expectation operator on this, so what we would be getting is n times C n square expected value of C n square is equal to omega p or in other words expected value of C n square equals to omega p by n.

So, we can say this over here. Now on this expectation operator, at this expectation is to be taken up across theta n and to do that we basically need p of theta n. So, p of theta n is basically probability of ray the nth path arriving at a particular theta and we make again this assumption that means, what we are talking about is there is a transmitter there is a receiver. So, rays are coming from different directions is the first path, the second path, the third path. So, the first path what is the probability that first path comes to particular direction.

Now we have been saying the probability of this ray coming or the nth path is 1 by 2 pi is the distribution and that will be same for all the paths. We do not know whether the first path comes from this direction or that direction or the second path comes from this particular direction. So, we would say that p theta n is equal to p theta is equal to 1 by 2 pi for all paths and that would help us in evaluating this particular expression. So, when we take a look at this first part what would we get is omega p by 2 n and of course, n equal to 1 to n and this expectation operator would still remain expectation operator cos 2 pi f max instead of cos theta n write it as cos theta times delta t.

So, if we now look at this, this expression is independent of small n this expression is independent of small n and this gets n times added. So, we would have n multiplied and this would result in omega p by 2 times expectation over theta. So, this was expectation over theta n this was expectation over theta times cos 2 pi f max cos theta times delta t this is basically phi h I h I delta t. And we have seen earlier that we have got phi h I h I delta t is equal to phi h Q h q delta t. So, we have evaluated this which is equal to this one we only have to remind to calculate h I h Q and it is kind of from this derivations you can almost guess what would be the h I h Q; so, phi of h Q h I delta t would be this you can work it out expectation over theta instead of cos there will a sin 2 pi f max cos theta delta t.

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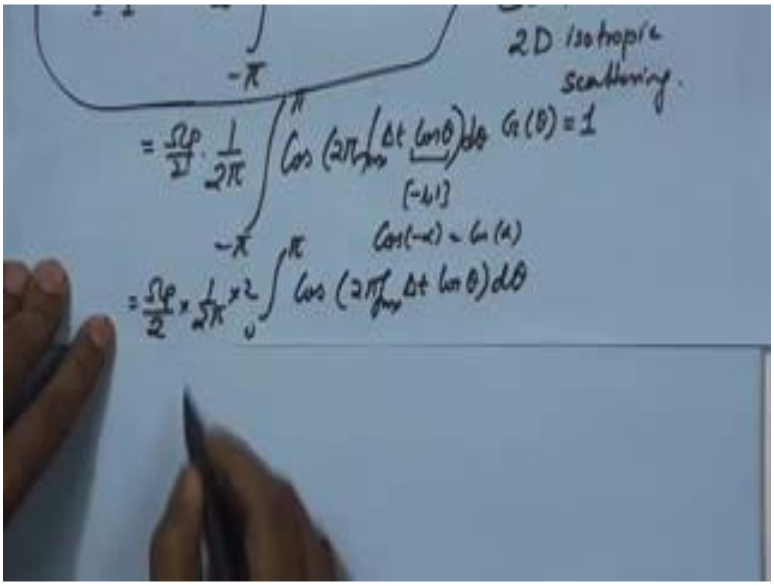

This is a kind of clear because the coefficient that you have with this h I h Q is sin omega c delta t. So, I mean that is kind of a indication, but if you derive through the steps you have going to arrive that this. So, what we have finally, is the expression of phi h I h I delta t and phi h Q h I delta t which would together help us in defining the expression of phi hh delta t, because phi hh delta t is in terms of phi h I h I delta t plus j phi h Q h I delta t and here we have phi h I h I delta t defined and phi h Q h I defined.

So, with this we have been able to define phi h I phi h I delta t. Now since phi h I delta t is defined we can then define phi rr delta t. So, what remains for us to do is to evaluate this expressions for E theta. So, basically this expectation if we can work out this expectation over theta then we have solved the complete expression of phi hh. So, if we go ahead and again our comes toward (Refer Time: 22:59) is p theta is 1 by 2 pi and theta in the range of minus pi to pi. So, this is basically the 2D isotropic scattering model that, that is what we referring to continuously and since this was first proposed by Clarkes we would usually call this Clarkes model.

This is how well known that means the probability of rays coming from all directions that is the probability of rays coming from different directions is uniform now this is very, very important. So, we just need to remember this is the Clarkes model of 2D isotropic scattering is a very, very famous model and if we follow this and we try to evaluate phi h I h I delta t what we get is omega p by 2; that means, we are evaluated from this point and to do this E of theta we get minus pi to pi cos 2 pi f max cos theta.

Now, we have the term that this has to be calculated. So, to calculate this we need to have 2 things - one is p theta and also we need to identify what is the gain of the antenna as a function of theta. Because sometimes there are directional antennas and sometimes Omni directional antennas again in the 2D isotropic scattering model we will assume that the gain of the antenna with respect to direction g theta is equal to 1; that means, it is a the same gain in all directions. So, ideally speaking we have to have g theta p theta d theta. So, this actually gives us a general method of calculating the correlation function. So, you will be using p theta equals to 1 by 2 pi and will be using g theta is equal to 1 in our case, but this expression would be used for other situations p theta is not equal to 1 by 2 pi, g theta is not equal to 1. From this point the results would be deviating. So, we could get omega p by 2 into p theta 1 by 2 pi, 1 by 2 pi minus pi to pi cos 2 pi f max delta t times cos theta I have moved delta t on this side d theta.

So, if we look at cos theta, theta has a range of minus pi to pi. So, cos theta has a range of minus 1 to 1 and cos of minus alpha is equal to cos alpha. So, using this we could also write this as omega p by 2 multiplied by 1 by 2 pi, 0 to pi because of this will be integrating only in one range and we would multiplied by 2 cos of 2 pi f max delta t cos theta d theta.

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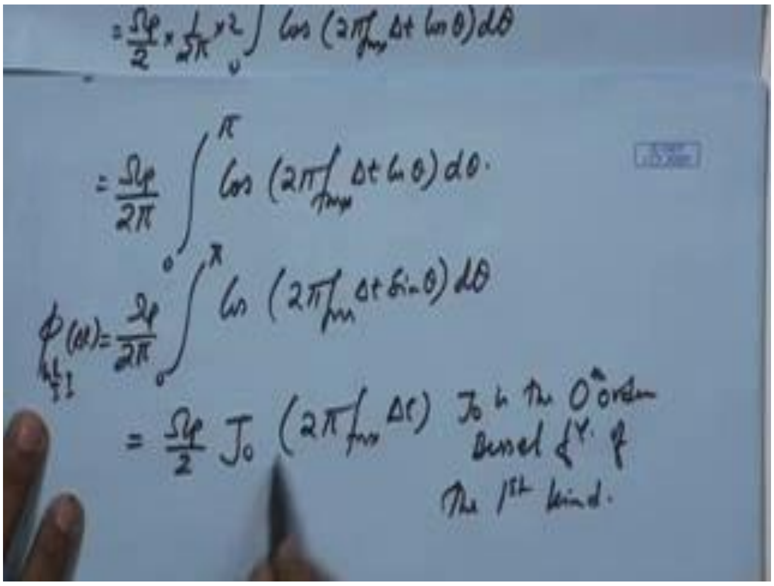

So, this would cancel out, 2 2 would cancel out. So, what we have this as omega p by 2 pi integral 0 to pi cos 2 pi f max delta t cos theta d theta and this could because again of cos theta being in the range of plus minus 1 and cos of minus alpha being equals to cos of alpha you could also write this as 0 to pi, cos of 2 pi f max delta t sin theta d theta without any change in your expressions and then this is the well known Bessel function of the 0th order of the first kind f max delta t. So, where j 0 is the 0th order Bessel function of the first kind.

So, what we have derived to this point is the auto correlation or the phi h I h I component of delta t which has come out to be Bessel function for the first kind of 0th order and the expression would look like, the figure would look like the one we have here.

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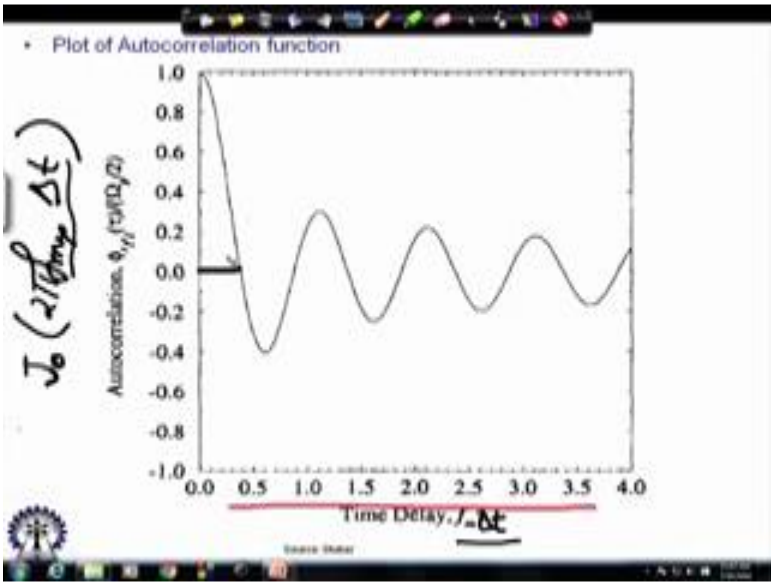

As you can see clearly this is the x axis this is the x axis which is indicating in this particular figure this figure is taken from (Refer Time: 28:23). So, this notation for our example will be having f m delta t on the x axis. So, that is basically normalize because what we have on the x axis is omega p by 2 is here j 0 2 pi f max delta t. So, if you look at this expression and we take f m or f max delta t then we could chose different values over here and we could plot it and they all to map to one figure. So, this is x axis is having f m delta t on the x axis that is f max delta t this is auto correlation function which is normalized by omega p by 2. So, that the max value is at 1, we could clearly see that what we have in the x axis is just j 0 function. This curve is very very important because at certain point you can clearly see that the correlation which is 0 and from this we will be defining what is known as the coherence time of the signal that is the time for which signal is correlated to itself.

We stop our discussion at this point and we will continue the next lecture and describing what is the coherence time and also we look at the correlation of h Q and h I components. So, that we are able to define the auto correlation or de-correlation of the received signal which would help us in understanding or characterizing the channel partially for small scale propagation models.

Thank you.