Probability and Random Variables

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Lecture - 35
Spectral Analysis Contd.

So, last time we were discussing the spectral analysis and we defined what is called power spectral density. Of course, we are dealing with this continuous systems we are dealing with continuous systems you know analog system. So, you just start from there once again it will help us.

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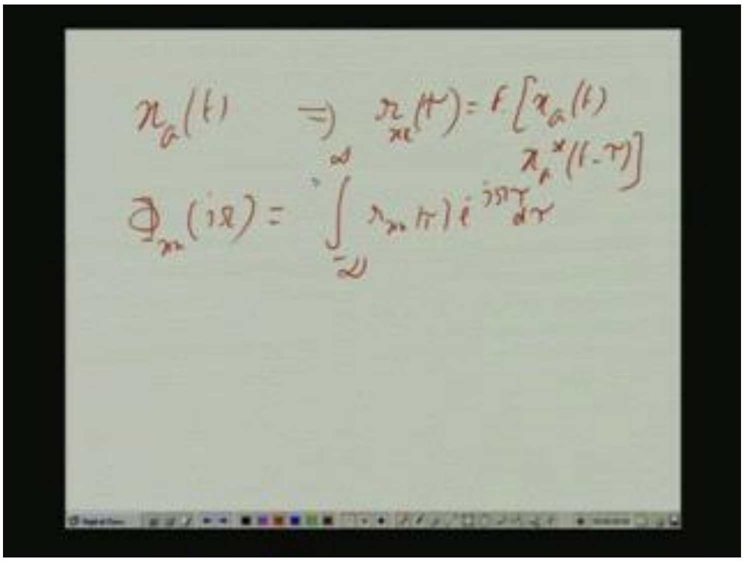

So, suppose there is a analog random process analog or continuous time. So, i just today i put a subscript a just to denote it is an analog signal. Xa t, let it be a random process analog random process and it has a correlation ra say t ra tau which i s. Then what is the power spectral density here? Phi a j omega this equal to maybe i change the notation a little here. So, i put rxx here it is assumed i will be dealing with only continuous time case or analog cases only.

So, even if i put rxx and do not mention any a here it is assumed that I am I am still considering an analog or there is continuous time random process. So, similarly here phi xx j omega which we know is a Fourier transform of the autocorrelation function rxx tau. Tt is e to the power minus j omega tau d tau minus i infinity to infinity. We have start we have seen that even though the random process is complex valued and therefore, the correlation is a complex valued function the power spectral density is a real valued function.

Then if the process is in fact, real valued then the power spectral density is not only real valued it is also an even function. In fact, it is a non negative function which we will show today. Then last time I was considering towards the end of last lecture last day's lecture. I was considering an analog system linear time invariant system with impulse response to ha t input was a process xa t output is a process ya t. First we saw that the output also is was if the input is wss and then we found out the output power spectral density. I just repeat it today in case there was some confusion left that time that we will clear up.

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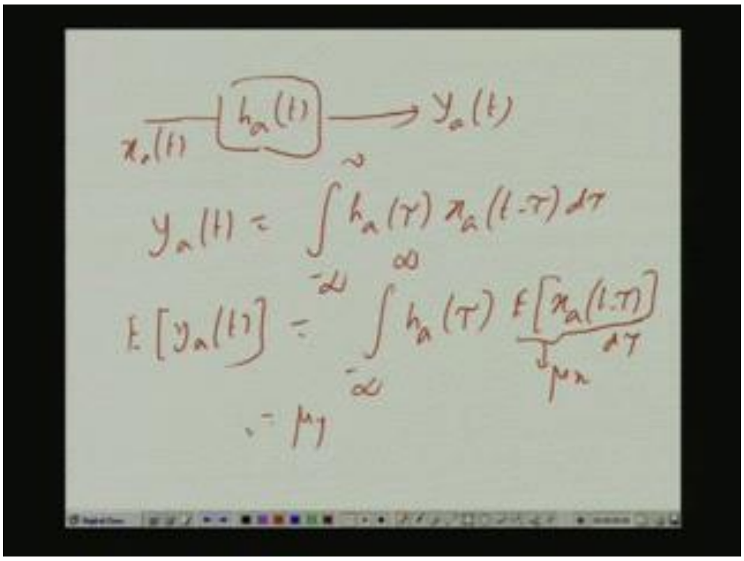

That is what i do now is that we have got a process xa t it is inputted toward analog system. System is ha t this is the impulse response. Once again i put a subscript a here to indicate there is an, it is an analog system hat is the impulse response and output is a random process ya t. Ya t is given by the analog convolution between these 2 xa t and ha t we all know. So, this is ya t which you can write either way may be you write this way xa t minus tau d tau minus infinity to in finity.

So, what is the output mean if input is stationary, now if what is the output mean ya t. That is you can take the expectation of y to add inside the integral because expectation is a expectation is a linear operator and then ha tau is not random. So, e operator will work directly on xa t minus tau and minus infinity to infinity ha tau E working on xa t minus tau d tau. But I told you that in put process that is xa t is wss. So, therefore, its mean is constant in dependent of the time whether it is t or tau or t minus tau.

And you can call this thing as then mu x mu x can go outside the integral. So, you get again a constant which is in dependent of any time t you can call that constant mu y. So, output mean also is constant in dependent of time. So, it is proved that is first order stationary we will now see the case of output correlation. Again that will be function of only that we will see that is a function of only the lag, and therefore we do so that it will prove that output also is wss.

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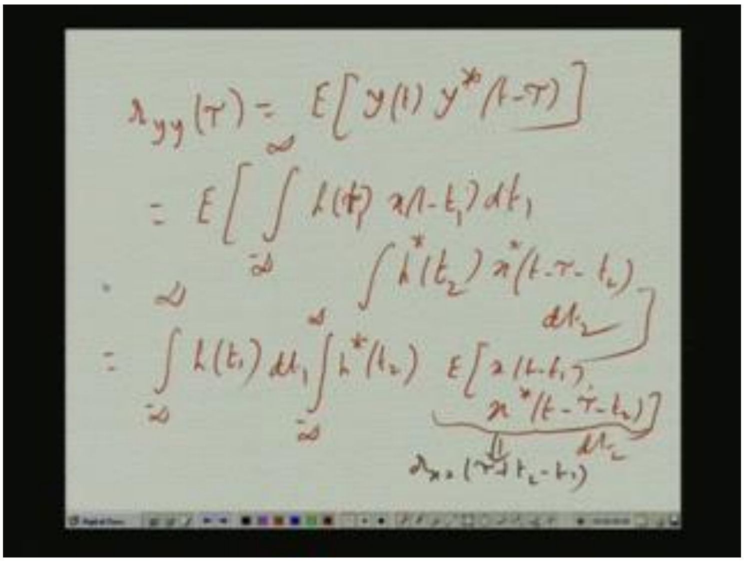

We did it last time. So, I am going little fast here. This is pretty straightforward because this whole derivation is pretty mechanical. So, you can afford to go fast. E double integral y t you know h tau x t minus tau this is with respect to tau. And then h say tau prime x sorry tau is already tau cannot be used, tau is already used outside. So, this cannot be tau tau is already a constant ryy tau. So, I cannot use tau as an in tegration variable.

So, let me call it let me make it t 1 d t 1 and t 2 this is y star so; that means, h star x star t minus tau minus t 2 d t 2. And then E will work directly on the product between x t minus t 1 and x star t minus tau minus t 2 because, other quantities that is h tau t 1 and h star t 2 they are not random. So, what you get and i can push the E operator inside the in tegral using the linearity of expectation operator. So, what you can do is this you have got h t 1 h star t 2. And then E working on x t minus t 1 in to x star t minus tau minus t 2 this is d t 1 this is d t 2.

Now, you see this quantity this quantity is given it is a autocorrelation function. And since in put xa t is wss the autocorrelation or correlation will depend only on the lag between the 2 time points. One time point is t minus t 1 another time in dex is time in stance is t minus tau minus t 2. So, it is nothing, but rxx tau plus t 2 minus t 1 and now you see if you in tegrate the entire thing with respect to t 1 and t 2. And get the result all t 1 t 2 disappear everything then the final result becomes just a function of tau.

So that means, the autocorrelation function as we wrote in the beginning as a function of tau is indeed a function of tau. So, there is lag it is indeed a function of lag only. So, this proves that output is wss if input is wss. And in put signal is passed through entire system input is wss output also is wss. The moment we say that output is a wss function wss signal immediately a question comes as to what is the power spectral density of the output then. That means you have to find out the Fourier transform analog. Fourier transform of ryy tau and there this expression will come handy and this integral.

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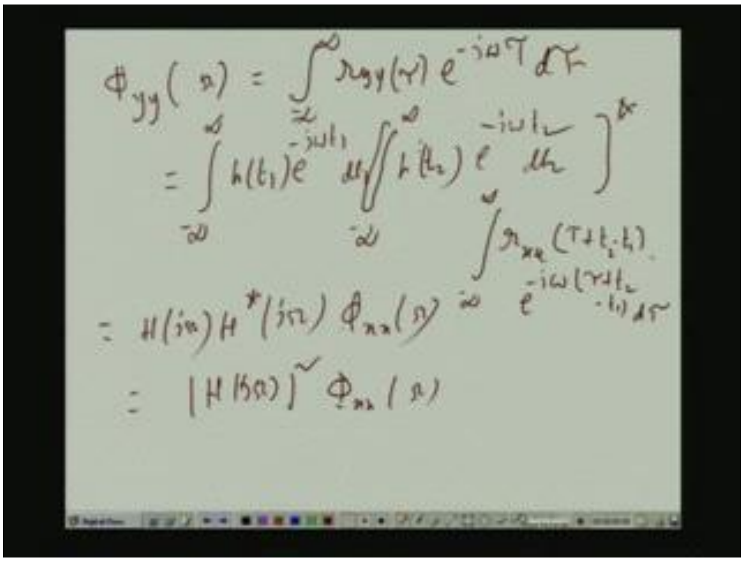

So, i will just keep it here. So, what is phi yy j omega? It is nothing, but ryy tau e to the power minus j omega tau d tau minus infinity to infinity there is a Fourier transform of output autocorrelation. Ryy tau e have just evaluated just a while back and this is a given by this bottom expression. So, this factor ryy tau has to be replaced by this. So, we basically have triple integral and you know this triple integrals if each of the integral even though in definite integral because, limits are from minus infinity to infinity.

But, if we assume that each in tegral is converging then edges in that case the in tegrals can be in terchanged like in terchanging double or triple summations you know. Provided each sum converges they can be in ter changed. And we are assuming that you know we are dealing with functions where all the in tegrals are converging and all that. So, we are not looking for those exceptional cases. So, in that case i can interest the integrals and then what we have that is the integral with respect to tau will be the in ner most integral.

First you replace ryy t tau by these 2. So, there are 2 in ner integrals 1 with respect to t 1 another with respect to t 2 then I inter change the integrals, the integral which was outside. So, long which is the outer most. So, long that is with respect to the tau that becomes the inner most. I am leaving a gap deliberately we will see why and then the inner integral rxx tau plus t 2 minus t 1 then e to the power minus j omega tau d tau. This I write as e to the power minus j omega not just tau, but i bring in those 2 t 2 minus t 1.

So; obviously, you have to cancel out those extra terms that I have brought in 1 is e to the power minus j omega t 2. So, therefore, I have to have e to the power plus j omega t 2 which I bring in here. And another term it should be e to the power plus j omega t 1. So, I should have e to the power minus j omega t 1. First one is obviously, another Fourier transform of H which is the frequency response of the system. Second one you can take you can take the star out from here put a minus and the entire integral you can bring a star on.

So, this is nothing, but H star j omega and here tau plus t 2 minus t 1 can be called tau prime everything is remained same. So, this becomes the Fourier transform of rxx tau prime which is nothing, but in put power spectral density phi xx j omega. I remember last time I did it in a hurry and in fact I should not or and it... So, that is why I am doing it again. Now, remember 1 thing I am writing phi yy j omega. In fact, it will be better if I remove j, because these are all real functions there is no place of j there.

So, permit me to remove the j from here initially, I put a j because I was carrying out Fourier transform. You know and normally when you take a time function and take a Fourier transform the Fourier transform is denoted as a function of j omega. So, keeping that in mind I put a j, but now I know that power spectral density is a real function. So, there is not point in keeping the j there. So, I remove the j similarly from here also. So, you see in put power spectral density is multiplied by the mod square of the or that is the magnitude square of the frequency response.

So, you can design a system where for certain frequencies the magnitude may be high of the transfer function or the frequency response and in certain another frequencies magnitude may be less. And therefore, at certain frequencies you can boost up the power spectral density and at certain another frequencies you can suppress. So, you can give a shape to the input power spectral density which gives rise to the idea of filtering.

That is you can generate function whose power spectral density is mostly a low pass type which is that is plotting versus omega. You get its function having nonzero and appreciable values only around omega equal to 0 close to. And as you go far away from origin it becomes less and less that kind of signal is called low pass opposite of; it is called high pass and all that which you know. So, all this can be obtained by just designing for a given specified phi xx omega you can design a filter to have a output power spectral density of some specified shape, this gives rise to the idea of filtering. This I will now this, I will now use to show that any power spectral density is not only a real function it is also a non negative function.

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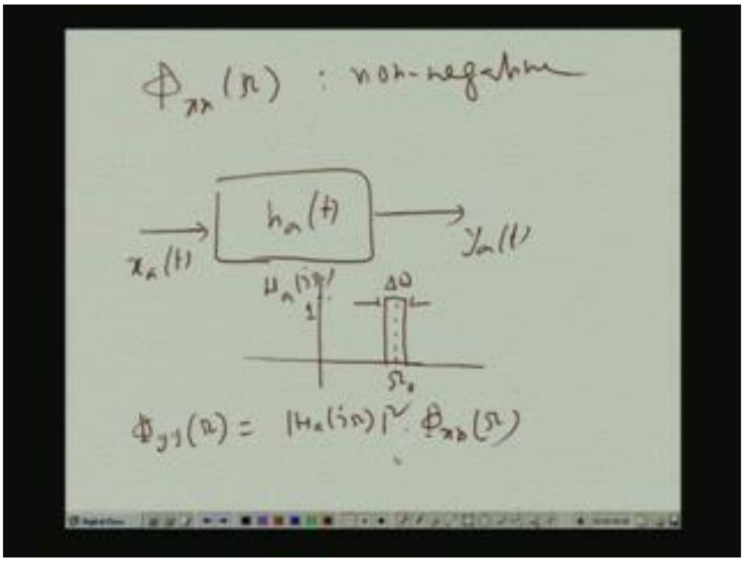

That is to show this what I do let xa t pass through a system ha t. Xa t can be complex valued ha t can be complex value, but remember if you call the output ya t then power spectral density of input that is xa t or power spectral density of output. That is ya t they will always remain real irrespective or whether xa t is complex ha t is complex or ya t is complex.

Now, I design the system ha t such that it is just a band pass filter very narrow pass band I am I am not showing it to be that narrow here, but actually very narrow in finitely small bandwidth. Centre frequency is some omega naught, capital omega is my notation for analog frequency that is radian per second. So, centre frequency is omega naught and this bandwidth is say delta omega this is my Ha j omega. Obviously, ha t in this case is complex because that you can see Ha j Omega is 1 sided I do not have a similar thing on the left hand side. And therefore, if you take the Fourier transform it will be complex valued. So, Ha t is complex valued here, but I do not care. What is that is simply height may be equal to 1 in to we have just seen this theory few minutes back we have we have. In fact, proved it.

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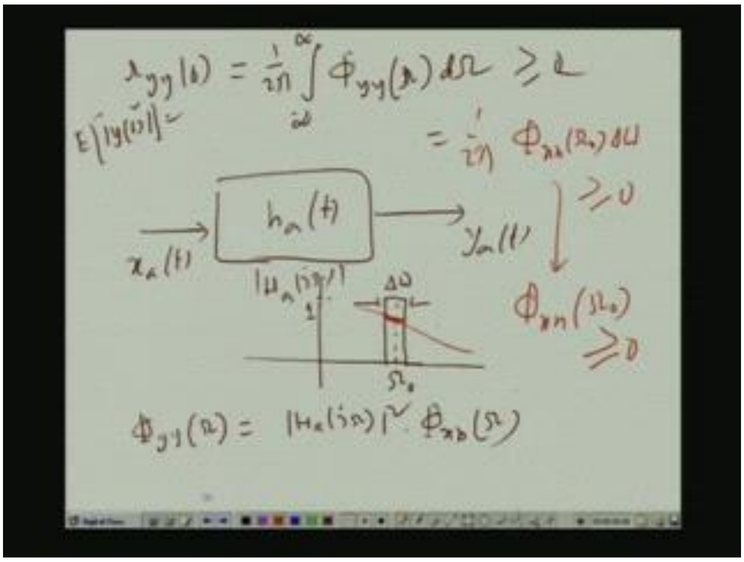

Now, now what is ryy 0 that is autocorrelation at lag 0. We have seen that this is given by 1 by 2 pi minus in finity to in finity phi yy omega d omega. In fact, that is why this is called power spectral density. Quickly how just to recall how it comes we have seen that power spectral density is the Fourier transform of autocorrelation function. So, therefore, autocorrelation function or may be correlation function here is the inverse Fourier transform of power spectral density.

So, if in the inverse Fourier transform ryy tau if you put tau equal to 0 then e to the power 0 is 1. So, you get this integral only which also shows how the shows that the power average power ryy 0. Average power of the process is given by the integral that is the area under the power spectral density. So, by looking at the power spectral density function you can see which part of the power spectral density function contributes most to this area and which less and thereby... You get to know how is the average power distributed along the frequency axis where it is high where it is low. That is the reason why it is called power spectral density function.

That is beside the point we have already discussed this last time. Now ryy 0 you all know that it is greater than equal to 0 because, it is what is ryy 0 it is nothing but E mod y t square variance that can never be negative; that is real and non negative. Now, in this case what is ryy 0.I have to in tegrate phi yy omega from minus infinity to infinity. But, there is no point in in tegrating from minus in finity to in finity because it is 0 everywhere. Because, input power spectral density whatever it is that is multiplied by this narrow band band pass filter.

So, output power spectral density will be confined only around omega naught with this band of delta omega elsewhere it will be 0 because H j Ha j omega i ts magnitude is 0. You can then we can take it according to the magnitude. So, therefore, what I have to do if the suppose input power spectral density is like this some function then at what is approximately the area. Area will be I mean if delta omega is very small you can take the centre I mean value of phi xx omega at the centre point that is phi xx omega naught.

You can assume that to be constant over this zone of delta omega because delta omega is very small. So, whatever be the value at the centre frequency omega naught for input power spectral density you can take that to be flat because delta omega is very small. So, in that case very simply this product. If you take the product and then compute the area that area will be what phi xx omega naught in to 1, because this height is 1 in to that is this height in to delta omega.

Phi xx omega naught in to delta omega that is the area and multiplied by 1 by 2 pi and this is greater than equal to 0. Now, delta omega is always positive this only shows phi xx omega naught is greater than 0 greater than equal to 0. But which was omega naught purely arbitrarily. So, this is true for any omega naught you ask me to you ask for phi xx omega at any omega naught. I can design a band pass filter with centre frequency at that omega naught and do the proof and; obviously, shows that power spectral density at that omega naught is greater than equal to 0.

So, it is a non-negative function it is proved. Another thing I forgot to mention is a concept of white process. There is 1 class of random process which is called white process. In fact, you come across this term in the context of communication where you deal with the noise called additive white additive white noise. White process is where the signal is absolutely uncorrelated.

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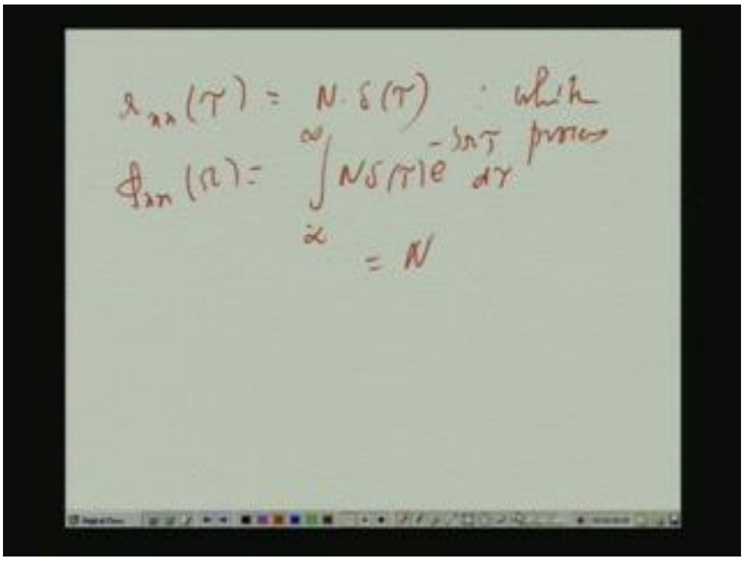

That is say rxx tau it is value is nonzero only when tau is 0 and otherwise 0.That is if you take 2 samples; however, close they are their correlation is 0. Only when tau is 0 that is the sample i mean you take it on itself and then find the correlation which in fact, gives rise to variance then it has got some value. So, it is basically N in to some delta tau where delta tau is the. So, called direct delta function or impulse function unit impulse function this kind of process is called is white process.

In this case power spectral density why white we will see will be what if, you take the Fourier transform you basically get N. Delta tau e to the power minus j omega tau integral that will be 1 this is the N which is constant in dependent of frequency. So, it is flat that is why it is called white that is all frequencies are present with same you know I mean power it is like white like. It consists of. So, many frequencies in uniform proportion that is why it is called white noise.

I will consider 1 or 2 examples of how to compute power spectral density and all that for the analogue case. But before that, let me now consider discrete sequences random sequences. And similar treatment is valid there also. So, then we compute that part first and if then if time permits I will go come back to this example otherwise I will take up this example in the next class.

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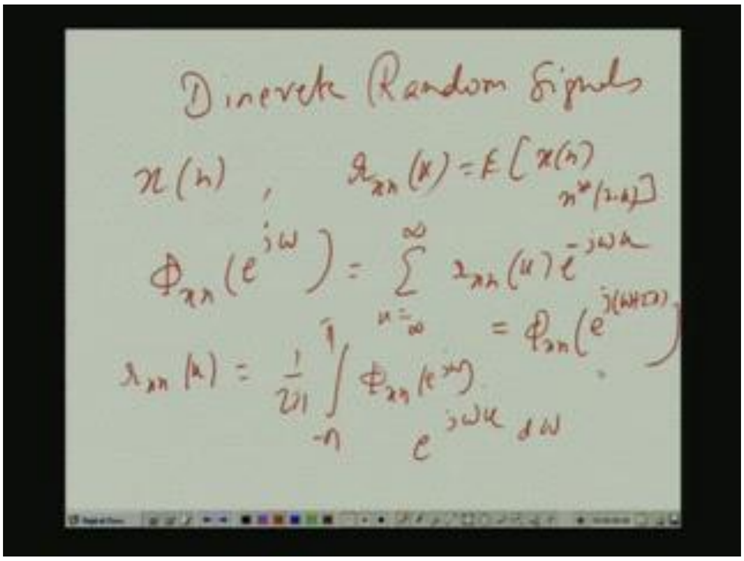

Discrete that is so long as dealing with analogue that is, as putting subscript a. Now, i will be dealing with a sequence that is discrete random signals. Here I am giving xn when n is an integer. So, I have got a sequence x 0 x 1 x 2 this is random sequence here correlation r rxx say k k is an integer is what. Just a minute give me a minute i hope you did not get distracted any way. This is a function of k because this is wss this is a wss process I mean I will carry out the same treatment and same interpretations we will follow.

So, let me not spend too much time on this. I will take the just discrete time Fourier transforms. You know this is a sequence. So, here the Fourier transform will be not the analogue Fourier transforms. But dtft discrete Fourier transform and that is your phi xx and instead of writing as a function of j capital omega it is a function of j omega, omega is the digital frequency its unit is just radian. I could have called in fact I should have review I can write as a function of just small omega, but again you know in I follow the convention in dsp.

They write dtft function and e to the power j small omega though e and j are constant they are not variable only omega is variable. But it is written in the same in this way. So, that, you can easily differentiate it distinguish this differentiate it from analogue Fourier transforms. And also it is always reminds us that this dtft is actually a power series e e to the power j omega after all what is dtft. The dtft is the this is the formula giving rxx k if you take its dtft it is this. So, it is actually power series of e to the power j omega k from minus infinity to infinity.

Then what is rxx k it is the inverse dtft similar treatment you see minus pi to pi remember the dtft this is periodic. I mean this is part of dsp I presume that you already know it, but in case you do not know please see certain things. What is the unit of omega after all omega in to k has to be an angled radian k is just integer dimensionless which means omega has to be radian omega is called digital frequency.

What is the relation with analog frequency capital omega we will come to that later it is just omega radian. And 1 more thing if instead of omega you can see and replace omega by say omega plus 2 pi. What do I get? I get 1 term as e to the power minus j omega k as it is. What is the I will get another term e to the power minus j 2 pi k. Now, e to the power j 2 pi k or e to the power minus j 2 pi k is always 1. That is if you take any angle 2 pi and take any integral multiple of that in the positive or negative direction and take e to the power of that, that is always 1.

So, e to the power j 2 pi is 1 e to the power j four pi 1 e to the power j six pi is 1 e to the power minus j 2 pi is 1 so on and so forth. So; that means, if you replace omega by omega plus 2 pi still you get the same expression that is this is same as e to the power j omega plus 2 pi. If that means, so that means, this function is periodic function of omega over a period 2 pi.

So, it is enough that you plot it from 0 to 2 pi or minus pi to pi i prefer minus pi to pi and whatever you see there that just gets repeated. These are all facts from dsp, but in case you do not know it is worth mentioning here. Coming back to rxx k this is called in verse dtft relation. And I suggest that in case you do not know please read the first chapter of Oppenheim and Schafer’s DSP book and review the topic of discrete time Fourier transform.

The inverse Fourier transform is not easily than analogue by the way. What is the inverse dtft? This is given by this omega k d omega. So, you want to find out the autocorrelation function at an in dex k. So, you carry out the integral with respect to omega you bring the same k here in side. This also means rxx 0 that is E of mod xn square which is rxx 0.

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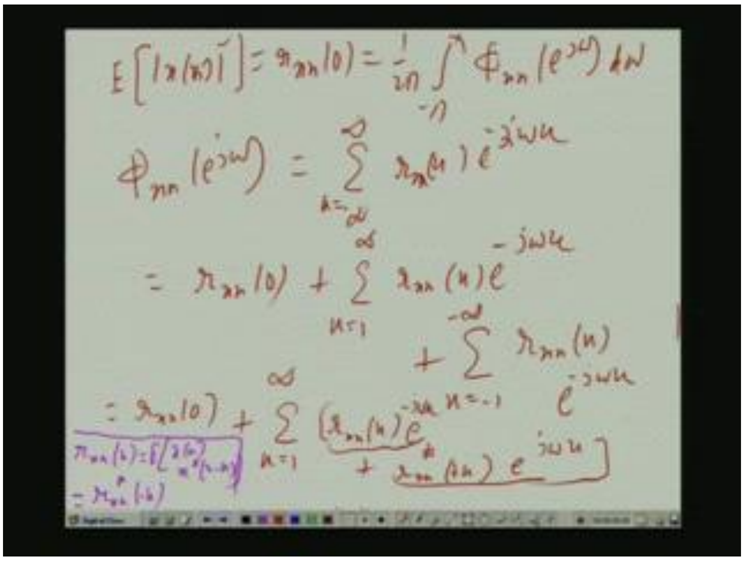

That is nothing, but e to the power j omega 0 this k is 0 here there is 1 and therefore, you get just this integral same interpretation follows d omega. So, you see phi xx e to the power j omega is such a function of omega which when integrated from minus pi to pi then that integral. That is the area under this function phi xx from minus pi to pi that gives you average power. That is why it is called power spectral density in the same way as phi xx j capital omega was called the power spectral density of the analogue signal xa t earlier.

So, Same treatment I am just giving the new definition that is all physical interpretations and all they remain same. As before this is again a real function same derivation I mean same proof similar proof all the integrals are to be replaced by you know this discrete sums that is take k equal to 0 out separately rxx 0. Rxx 0 is what for irrespective of whether xn is a complex valued process or real valued process rxx 0 is as is the variance E of mod xn square. That is E of xn in to x star n minus 0 lag 0 autocorrelation for lag 0 which is always a real quantity, so, rxx 0.

Then other summation here it was from minus infinity to infinity you can take this way 1 summation from k equal to 1 to infinity another part is k equal to minus 1 to minus infinity same thing. Now, here k is if you replace k by minus k then what is a. Firstly, what happens to the summation in dices k by say minus k prime in case if I get confused k by minus k prime say to start with. That means when k is minus 1 k prime is 1 and when k k is minus infinity k prime is infinity. So, summation ranges from 1 to infinity.

So, rxx minus k prime e to the power plus 0 omega k prime and now call k prime as k and 2 summations can be combined in to 1 because, limits are same plus rxx minus k e to the power j omega k. Now, remember 1 thing i am doing it separately here. Do you remember rxx k what, what was it e of xn x star n minus k right this according to me is also rxx star minus k why first what is rxx minus k. You have got x star n plus k xn and then if you take star over it the star star cancels what you and a star comes on xn.

So, you have x of n plus k in to x star n, n plus k if equal n 1 it becomes x n 1 x star n 1 minus k expected value of that which is again rxx k. That means what is rxx minus k this is nothing, but rxx star plus k. See you see whatever you have here and these 2 1 is the conjugate of the other rxx k rxx star k e to the power minus j omega k e to the power plus j omega k. So, real number z sorry a complex number z and then its conjugate summation will be real and rxx 0 the outside quantity is also real. So, more or less real it is a real function.

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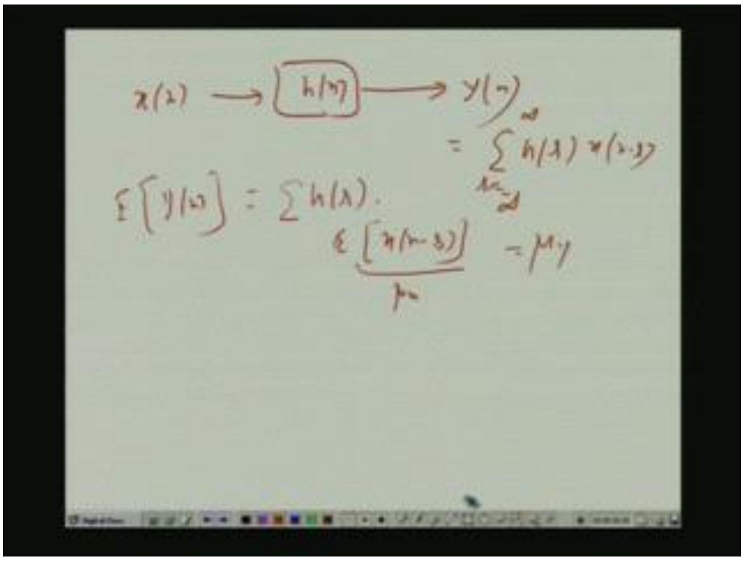

And once again you can see this response of linear systems and all that we we will do it very quickly. Suppose, x n goes through h n it is a linear shift in variance say we get y n. Y n is nothing but the discrete convolution h r x say n minus r r from in general minus infinity to infinity. So, what is E y n. Input is wss E will go inside what even it will work directly on x in to E of x n minus r in put is wss.

So, expected value of any sample of x be it of n minus r sample at n minus r that is constant that we can call mu x. So, this summation is again a constant in dependent of n. So, we can call it mu y which is in dependent of n.

So; that means, output mean is constant. So, in the output is stationary in the first order we will now see that output is stationary in the second order also you know the frequency you can see is absolutely similar to for you had there.

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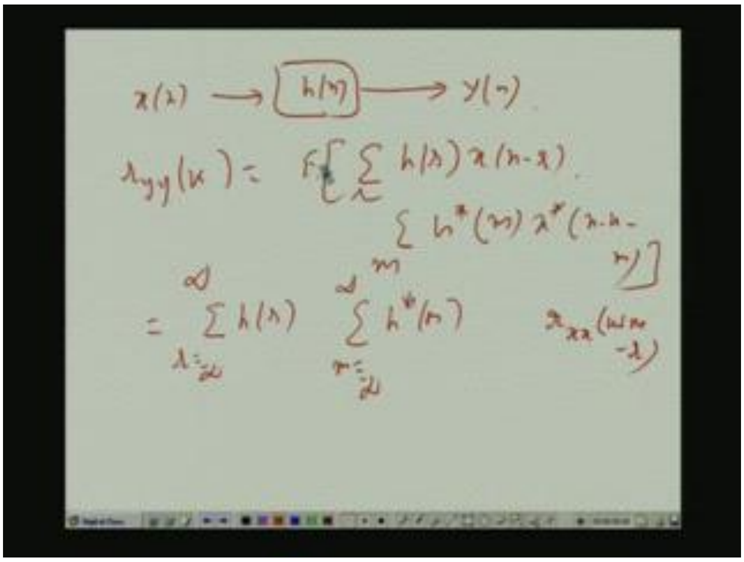

What is E of yn? Yn is what? This yn in to y star n minus k y star n minus k, y star means h star I will separate variable m and this is r all from minus infinity to infinity. Y n and then y star n minus k isn’t it y star n minus k means convolution evaluated at not at n, but n minus k. So, n minus k minus m star because it was y star. Actually, I am skipping 1 step because I know assume that you yourself can work out that step.

The first summation gives y n second summation gives y star n minus k you can verify I suppose. And then E will work directly on wherever you have x n x star and all that. Suppose E in side using the linearity of E h r. And then we have got E working on this x n minus r in to x star n minus k minus m. Input is wss. So, this called be autocorrelation will be functionally of the lag, lag is what n minus r minus within bracket n minus k minus m which is r. And you see if you carry out the whole summation m and r disappears everything becomes a function of just k.

So, output correlation is indeed a function of k. So, this proves that output also is wss. Obviously, this leads to the following question if it is wss then what power spectral density is. Finds we have to take this Fourier transform.

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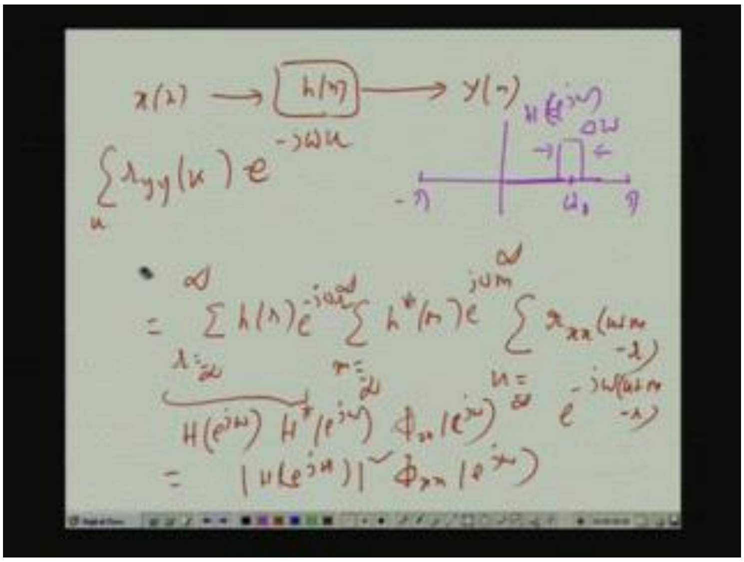

So, I remove this, this intermediate step now. I take the Fourier transform this is over k. So that means, this entire thing has to be sum multiplied by this entire thing has to be multiplied by minus j omega k and summed over k that sum over k will be the outer sum. But I can interchange the sum make it the inner most sum j omega. Just a minute j omega k, but instead of writing k I call it k plus m minus r.

So, I brought some extra term that we cancelled. So, I should have e to the power plus j omega m that I write here and i should have e to the power minus j omega r that I write here. First quantity is; obviously, the frequency response the tft of h n there is a frequency response second quantity you can take the star out. So, e to the power minus j omega m and it becomes. You can easily see that this is nothing but H star e to the power j omega and here k plus m minus r can be called k 1 everything else will be same.

So, this is the tft of discrete time Fourier transform rxx k or rxx k plane rather which is the input power spectral density which is phi xx e to the power j omega. So, again I get that filtering expression H square. This is the discrete time version of that previous equation. Again using this you can show that phi xx e to the power j omega is also a or its not only a real function its a nonnegative function. How? This time we will just consider this and H e to the power j omega I will not prove it which is again a band pass filter.

The range is from minus pi to pi this, this is a band pass filter centred at say some omega naught of with delta omega. If the xn is passed through this you know xn is a complex valued and of course, Hn is complex valued yn is complex valued. But, the power spectral densities are real that you have already seen. And where is the output power spectral density that will be given by this figure this thing mod H e to the power j omega square phi x e to the power j omega.

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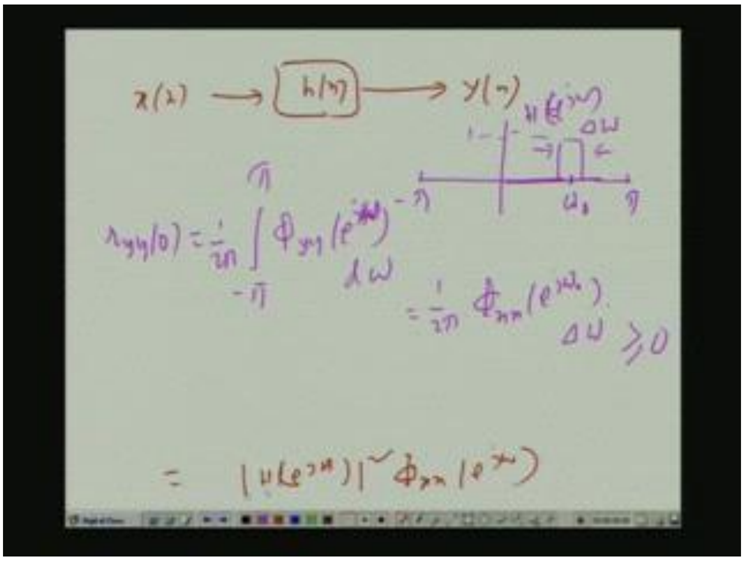

So, what is ryy 0 as before this should be integral of output power spectral density from minus pi to pi multiplied by 2 pi minus pi to pi. Phi yy e to the power j omega d, d omega e to the power j small omega, but this integral you know is localized only with around omega naught within a band of delta omega and filter height is say 1.

So, it is a question of finding the value of input power spectral density at the centre point omega naught assume that remains constant or flat over the inter band. Because band is in finite simply very small so, overall area becomes nothing but 1 by 2 pi times phi xx e to the power j omega naught times delta omega. And this must be greater than equal to 0 which means power spectral density is nonnegative function so on and so forth.

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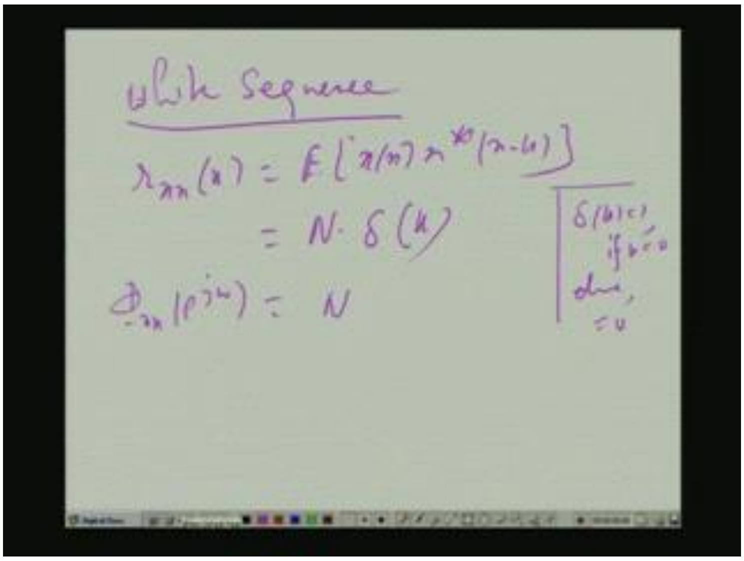

Unlike analog case here also you can define white sequence. If rxx k which is nothing, but if it is some constant say n times delta and this delta is not direct delta function it is just a sequence it is called unit impulse sequence delta k. What is delta k? Delta k equal to 1 if k equal to 0 else equal to 0; that means, xn is uncorrelated with any sample any of its neighbours far away or nearby ones. And when k is 0 rx is 0 which is variance. Variance is given by N because delta 0 is 0 1. For such a process the power spectral density phi xx e to the power j omega; obviously, if you do the dtft you can see this constant N. So, it is flat that is why it is called white fine.

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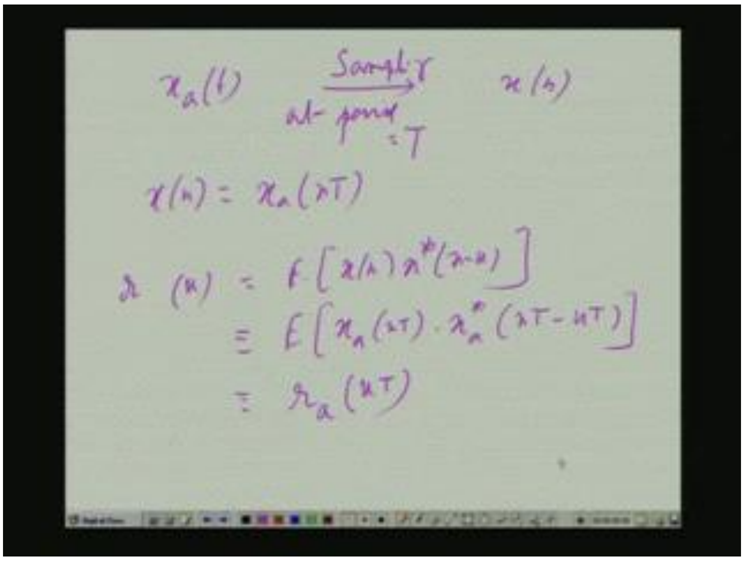

Now suppose, why I mean actually now it will be clear as to why I suddenly went for discrete sequences. First suppose that we have got an analogue function xa t analogue random sequence xat and you are sampling it sampling at period sampling period equal to say T to get a sequence xn. So, if xa t is random xn also is random. From a random continuous value or analogue signal I by sampling I generate a random sequence xn.

So, what is xn? That is n T n th sample sampling period is T. So, 1 sample at is taken at T equal to 0 another is T then 2 T three T four T. So, n th sample that is xn is n th sample of there is a value of this function xa t at what time point at n in to T fine. So; that means, what is rxx k whenever I write rxx k. You look at the you look at what I write in side if it is k k stands for integer. Then; that means, I am talking of the random sequence its correlation function.

If i write rxx tau then i am not talking of the autocorrelation function for the random any random sequence, but autocorrelation function of a continuous value that is analogue signal function because tau is random. Now, what is rxx k it is e of xn x star n minus k which is equivalent to e of x a nT in to x star a nT minus kT, which is equivalent to r. May be if you permit me to avoid confusion here I remove this subscript r this x x just r k and this ra ra stands from the correlation functions for the analogue function. When there is no subscript then it is the correlation function for the random sequence.

So, rk and since i my dealing with no other person no other sequence no other function other then x let us drops the subscript related to x. This rk which is this e of xa nT into x star n T minus kT that is if you call kT to be tau. So, this is nothing but ra at a lag of tau that is kT.

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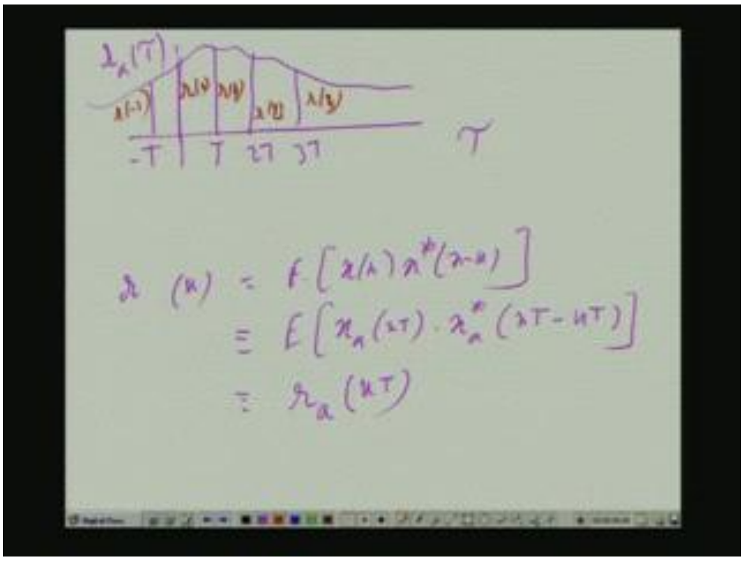

What does it mean it means that I have given a function ra tau as function of tau autocorrelation function some autocorrelation function ra tau. I am taking the samples at 0 at T at 2 T at 3 T at minus T like that. So, I am getting these things this is r 0 this is r 1 r 2 this is r 1 r 2 r 3 r 0 r minus 1; so on and so forth. So, this autocorrelation sequence rk is actually obtained by sampling the corresponding analogue autocorrelation function at a sampling period of sampling period of T.

You know it is very much like sampling a continuous time signal you have already we have studied Nyquist sampling theorem. And all that, time continues time signal is sampled with sampling period T you get a sequence. You find out the dtft expression for the sequence and relate it to the analogue Fourier transform of the analogue function. And then you get an expression and then find out when aliasing will occur, when aliasing will not occur and all that.

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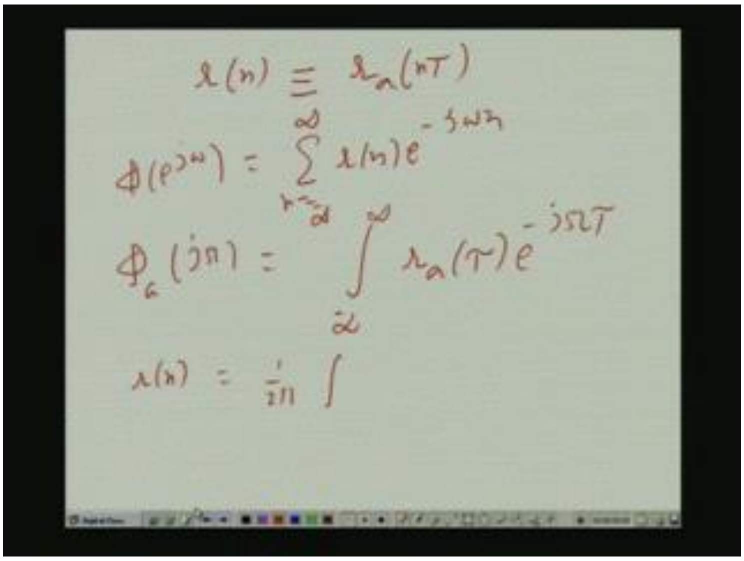

Same thing we can do here. So, what we were given here is this. That this sequence r n is a sampled version of an analogue autocorrelation function of ra tau, it is just sampled at a sampling rate of 1 by T or sampling period of T and you get this sequence. So, if you have phi e to the power j omega is and phi a. Now, I am writing because i am again dropping the subscript related to x. Phi a j Omega this is the analogue Fourier transform of. Then what is the relation between phi e to the power j omega and phi a j omega. You can see one thing that r n is given by the inverse dtft of phi e to the power j omega.

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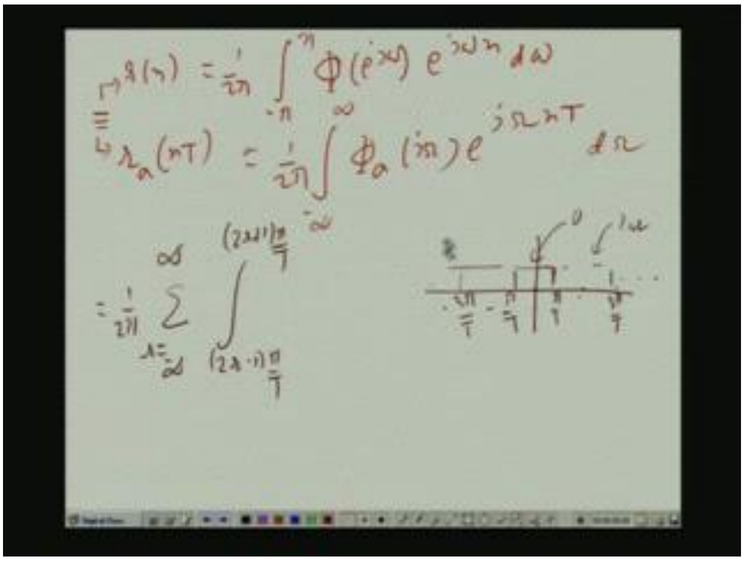

That is in fact, r n the inverse dtft relation and ra nT and this 2 are remember these 2 are equivalent. This is given by in verse the dtft of sorry not in verse dtft in verse analogue Fourier transform of the corresponding analogue power spectral density, but evaluated at a time not any tau, but specifically n in to T. Now, the 2 left hand sides are equal that you have seen.

So; that means, by equating the 2 right hand sides you can try to derive a relation between the 2 power spectral densities. 1 is 1 of them is discrete power spectral 1 of them is phd for the discrete sequence. Another is the phd phd for the analog loading the analog random process. Now, this second integral you can break down like this, this whole interval from minus infinity to infinity you break like this you take the gap. You know 1 1 thing from pi by T 2 minus pi by T. Then take another range from pi by T to three pi by T another range from minus pi to minus 3 by T like that dot dot dot.

So, instead of integrating once from all from minus infinity to infinity I want to breakdown the integral over non overlapping sectors. 1 sectors say 0 is sector this is zeroth this is first. So, basically rth sector will be where see zeroth sector centre is at 0 first sector centre is at 2 pi by T second sector centre is at four pi by T. So, rth sector centre will be at 2 pi in to r by T.

So, that range will be 2 pi r by T plus pi by T minus pi by T. And that r will continue from minus infinity to infinity do you understand what I am saying. I am dividing the entire range of the integral minus infinity to infinity in to I mean I am doing the integral as a summation of some integrals and some integrals are finite sub integrals. 1 will be from range minus pi by T to pi by T that I will call zeroth then next 1 will be from pi by 2 to 3 pi by T. That we will be a little first next will be three pi by T to five pi by T that we will call second so on and so forth. What is the rth?

Rth will be what to how to identify quickly what was the zeroth the centre was 0 for the first centre was at 2 pi by T for the second centre is at four pi by T. For the rth it will be 2 pi r by T. So, 2 pi r r by T and then further pi by T to the right and pi by 2 to the left that is rth. And now take r equal to 0 and r equal to 1 r equal to 2 r equal to minus 1 all that and r minus infinity to infinity. So, entire range will be covered. So, that is what I do this entire integral will be written like this sum integrals from 2 r pi by T minus pi by T that is 2 r minus 1 pi by T.

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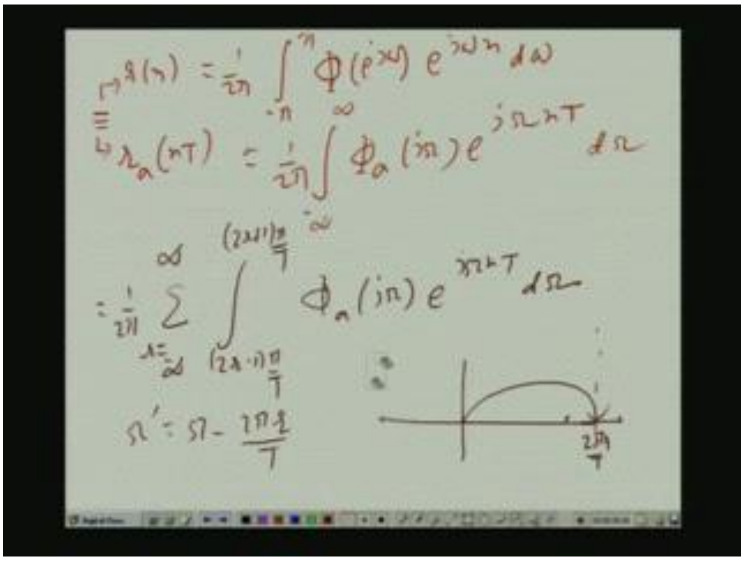

Whenever, you come across double summations and assuming the summations exist they can be interchanged. That is why I will do here, but problem is the inner summation as got r in the limit and outer summation is over r. So, these 2 cannot be interchanged. That is if inner summation cannot be over r. So, r is finished and still you find an outer summation this integral where r comes in the limit. So, for that matter for that matter what I have to do, I do not know whether in 5 minutes I can complete it, but i will carry on in the next class. Anyway let me see as much as I can.

So, what I do what I do indeed about 7 minutes to complete it not 5. Anyway so what I will be doing I will be shifting the origin that is earlier origin was here. And what is the rth? Rth integral was rth zone was here 2 pi r T. So, I will be shifting the origin here. That means, for this particular integral I will replace omega by omega prime where omega prime is omega minus 2 pi r by T. In that if I make the substitution what all I get.

Actually what I am doing is always done in dsp you know while proving Nyquist theorem. So, there is nothing new, but still I am doing it. So, in that case, then what will be the integration limits when omega becomes 2 pi r by T plus pi by T omega prime goes to plus pi by T and minus pi by T here for the lower case. And d omega d omega prime are same. So, I make these changes here.

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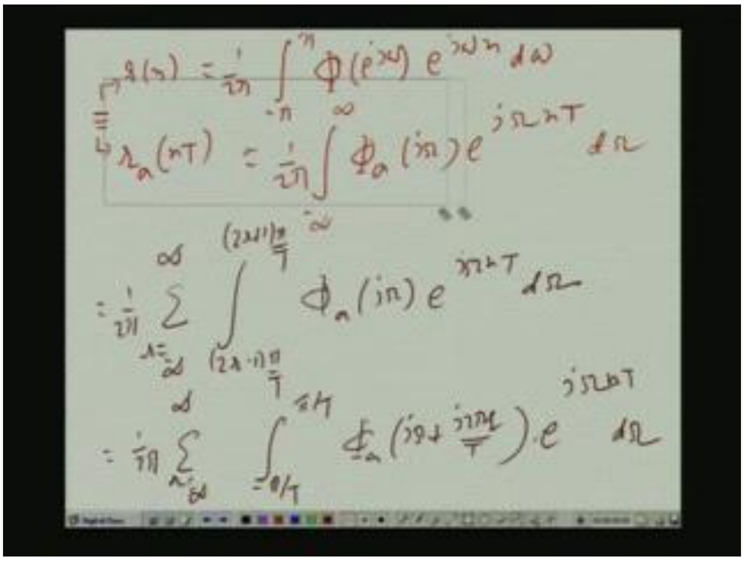

So, what I get Phi a. So, I said omega prime equal to omega minus 2 pi by r T. So; that means, this becomes j omega prime plus j 2 pi by T in to e to the power this is interesting j. What is omega again? Omega prime plus 2 pi r by T in to nT d omega prime. Now, see the second factor 2 pi r by T in to nT T T cancels and e to the power in need 5 minutes. E to the power j 2 pi r n equal to 1 e to the power j 2 pi rn that is equal to 1. So, it makes no difference actually. So, I can always rewrite it is like this. And why again omega prime bring back our good old omega. So, you call it omega sorry and there is a T here of course, omega nT.

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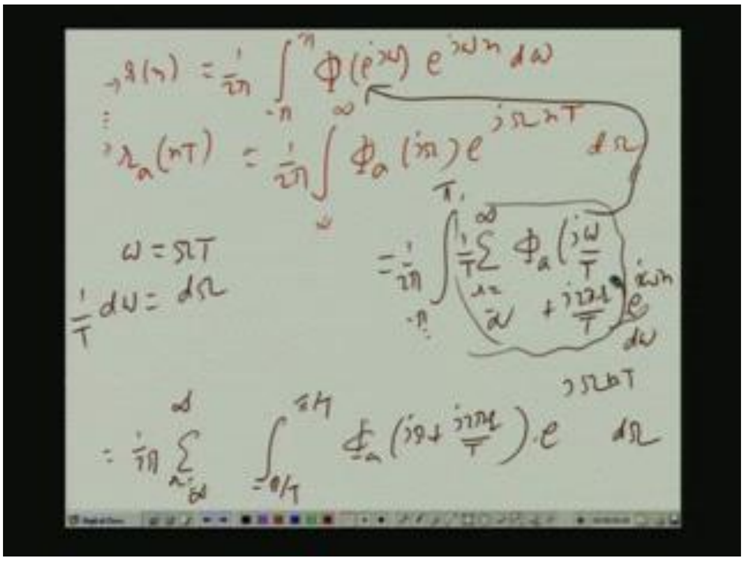

Now, if you have now if you have small omega equal to capital omega T then d omega is 1 by T d omega is d capital omega. And what happens to this integral this becomes and i interchange the 2 summations. Now, I can easily interchange because r is disappeared from the integral limits. So, r and I put here 1 by T here because, 1 by T d omega d omega in to e to the power j omega n d omega.

This quantity I will take up in the next class again this quantity will give you this. You can compare the 2 integrals. And the limits i made a mistake the limits will not be from minus pi by T it will be from minus pi to pi because if capital omega is pi by T product is pi. If Omega is minus pi by T product is minus pi. So, the integral limits are from minus pi to pi.

So, you see the 2 integrals are identical. So, you can easily find out what is phi e to the power j omega. And the relation between analogue and digital frequencies is omega this omega equal to Omega T. Now, this I will give a physical interpretation of this and I will see that by Nyquist condition you can avoid aliasing. And therefore, if you avoid aliasing you can get analogue you can get the analogue. This thing, you know I mean power spectral density directly from digital the power spectral density of the sequence, which you obtained by sampling the analogue random function.

In that case, is if your purpose is estimate the power spectral density of an analogue signal i ts better you sample it at above Nyquist rate, that is no aliasing and find out by some computer algorithm sophistical algorithm the power spectral density of the corresponding sequence r n sorry xn from that, you can easily construct phi a j omega. So, I will start from here in the next class.

Thank you very much.

Preview of the Next Lecture
Probability and Random Variables
Prof. Dr. M. Chakraborty
Department of Electronics and Electrical Communication Engineering
Indian Institute of Technology, Kharagpur

Lecture - 36

Spectrum Estimation-Non-Parametric Methods

So, in the last class we are doing 1 thing in hurry. So, maybe I will just start from that and do it again. That is suppose, there is a random continuous valued that is analogue random process x a t and this is sampled at a time period sampling period of T to generate a sequence xn so; obviously, xn also is random sequence. So, you have xn equal to the nth sample that is if you sample at T equal to 0, then t equal to capital T, then twice T is the TL so on and so forth, Nth sample I call it xn.