Probability & Random Variables

Prof. M. Chakraborty
Department of electronic & Electrical Communication Engineering
Indian Institute of Technology, Kharagpur

Lecture - 34
Introduction to Spectral Analysis

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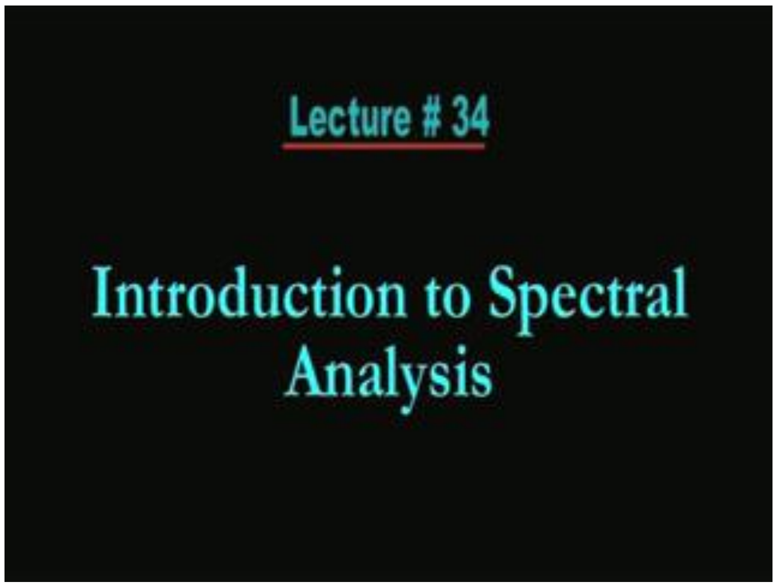

Today's topic is one of the most important topics in this entire course. It is called spectral analysis.

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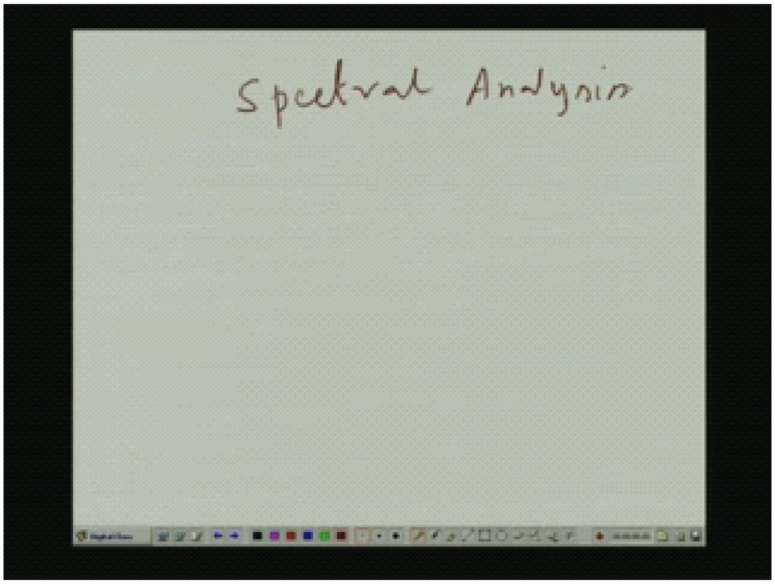

It is also call in different names, sometimes we call it estimation of power spectral density. Or spectrum analysis power spectral analysis you know there are various names which are available actually, but they all mean the same its spectral analysis. Now, before I going to it let us first mix a background and then we will define what is call power spectral density and all those things. And it is physically integration and properties and all those things. First we start with what is called and again this at that thing will be you know mean dealt with total in that, now context of analog signals.

There is a continuous time random position, but it also for discrete signals in fact, towards the later part. We will be considering more on the discrete random positions or discrete random signals and spectral analysis for them, because many sophisticate algorithms are available today. To estimate the power spectrum of discrete random sequences, which are of course, where which what kind of course, relate to the original spectrum of that corresponding continuous time signals but anyway we will not go right now. We will not go that far, because most of the things will not be understood by you now. Unless I give a detail description of what is spectrum and all that. So, before you going to it just of couple results 1 of them possibly I have covered earlier, but I am not sure.

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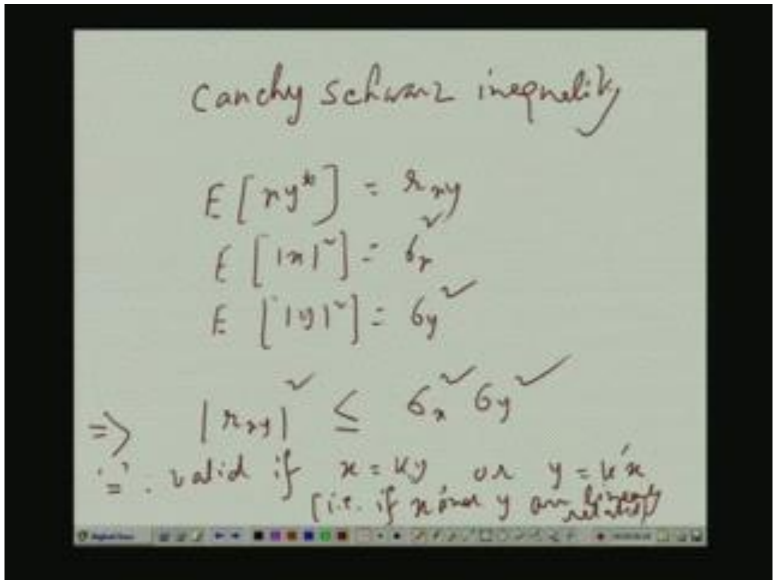

There is nothing in covering again it is called Cauchy Schwarz inequality. Actually this is a very famous inequality and it applies not only to random variables and all that it applies to. In fact, it is an inequality which is available in the study of vector spaces. It relates to it was a normal inner product of vectors and all that. But we will not going to that. We will confine our self to the context of random variables only that is we will be studying this inequality in the context of random variables only. But remember that is only a limited way of looking at Cauchy Schwarz inequality scope is much more.

Nevertheless, suppose x and y are 2 random variables given to you. Then inequality says that if you take that correlation Exy star and call it say rxy and E of mod x square equal to it sigma x square; if you define this term sigma y square. Then this says that if you take the correlation which in general could be complex take the mod that and square that is less than equal sigma x square sigma y square. With the inequality there can be either less than or equal to the equal to valid if x and y are linearly related, if x equal to some ky or y equal to sum k prime x.

We say that if that is if x and y are linearly related. Now, you can ask 1 question here that if I write x equal to ky. It covalently means where k is any number that is. Firstly, what is the meaning of x and y linearly related. That is x is some multiple of y or y is some multiple of x in that case, we say x and y are linearly related. And we are making the claim that if this is. So, if x and y are indeed linearly related there is inequality actually becomes an equality. Otherwise it is treaty less than now you ask me a question here that see once I write x equal to ky does not it automatically imply that y equal to 1 by kx.

That is y also a multiple of x yes, but still I am doing it for some mathematical correctness and you can see that why Suppose, k is non 0 then x equal ky or y equal to 1 by kx 1 by K is that define 1 by kx I mean both are equivalent. So, you can say x is a multiple of y or y is a multiple of x, but suppose x is 0 and y nonzero. You can see 1 thing that even if x equal to 0 and I mean in this inequality. Suppose x is 0 that is x is 0 random variable and y is nonzero. In that case the correlation is 0 and sigma x square is 0. So, in that case; obviously, it becomes inequality, because of the left hand side the correlation between x and y is 0 total x is random variable which always take 0 values.

So, 0 and on the right hand side sigma x square there is the variance of at the power of this 0 random variable that is 0. So, right hand side and left hand side are equal it is equal. So, did it equal so; that means, when x equal to 0 or further matter y equal to 0. The equality is valid this we can see by instruction. Now, does it fall under this category that if x equal to 0 or y equal to 0 x equal to ky. And suppose, x equal to 0 and y non zero we can unless that look x equal to 0 or nonzero, why not take why not take k to be 0 a scalar 0 number?

Then 0 times any nonzero random variable will still give a 0. So, x equal to 0 y non 0 that is satisfied here thus k becomes equal to 0. But in that case 1 cannot write that if x equal to ky y is also 1 by kx, because y is nonzero and x is 0. So, no finite number times 0 gives you another nonzero number. That is why I write separately that y equal to k prime x that x equal to 0 case fine 0 equal to some this scalar 0 times y this is satisfied. But the y equal to 0 case that why suppose y equal to 0 for that y x may be non 0 k prime is 0 why. So, you get again y equal to 0. So anyway, you can easily see 1 thing that if you put x equal to ky or y equal to k prime x 1 the left hand side, then this is becomes inequality. You can see this.

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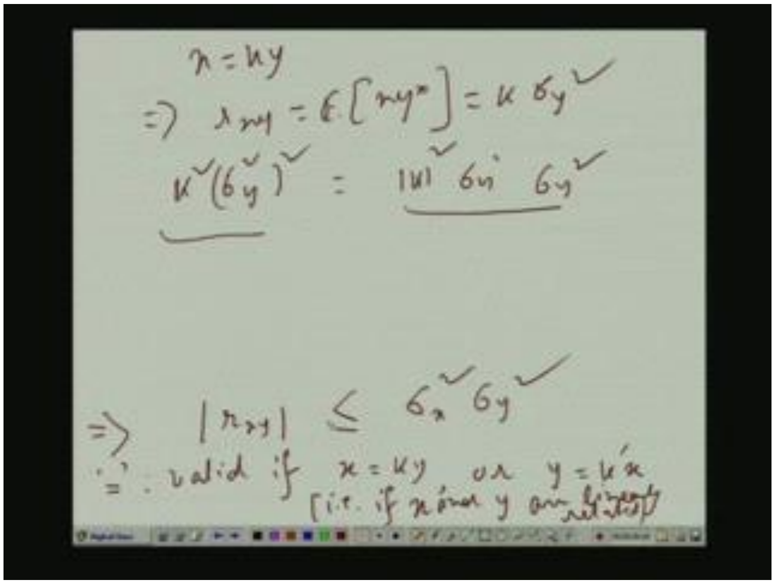

Suppose, x equal to ky then rxy which is Exy star it is simply k sigma y square. You put ky here k you take out y into y star that is mod y square and this is k sigma y square. So, the left hand side if you square it up you will get k square sigma y square whole square. And on the right hand side these are the left hand side around right hand side what you get sigma x square sigma y square sigma x square is already present. And what is sigma x? square sigma x square E of mod x square E of mod x square is E of mod of ky whole square. So, mod k square goes out and again sigma y square.

So, again you can see left hand side and right hand side they are indeed same. Similarly, you can verify that y equal to while y equal to k prime x even then the equality satisfied and this y x equal to ky or y equal to k prime x takes care of the 0 cases. That is x equal to 0 y nonzero or y equal to 0 x nonzero both cases are satisfied here. For instance when x equal to 0 x is a 0 random variable and y is not. Then I can always express that as x equal to ky where k is 0. Similarly, when y is the 0 random variable and x is not, I can write that as y equal to k prime x or k prime is 0.

So, these 2 cases x equal to ky or y equal to k prime x takes care of all types of linear relations between x and y. And whenever this is satisfied we can see that the inequality becomes equal. Now is go for the case where x is neither equal to ky or y nor y is equal to k prime x for any scale appear or k prime. That is x is and therefore, neither x is 0 nor y is 0, because x equal to 0 case or y equal to 0 case come directly under this. Suppose, then we are I mean repeat again; we are taking we are not talking of the case where x is not equal to ky or y is not equal to k prime x. So, there is not linear relation between x and y which also means that neither x not y is 0 random variable. In that case my claim is that inequality becomes a strict inequality that is strictly less than no equal to that this is the major part of the proof. That you consider.

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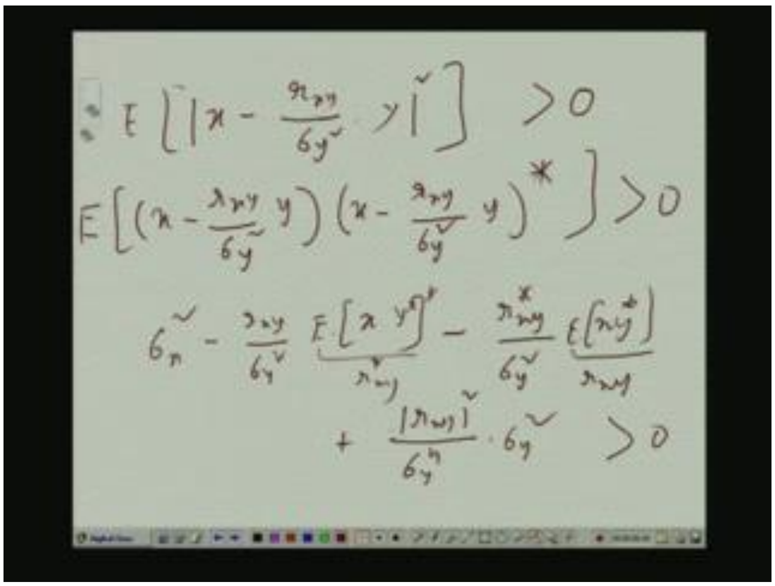

In fact you can go in the next page, in this case, suppose, I consider this x minus rxy by sigma y square into y. Now, rxy by sigma y square this is a constant this not random this is a constant number some number; you can say k or l or whatever, is actually in our case rxy cross correlation between xy divided by sigma y square. This times y if I now take expected value of mod square of this. Firstly, you see 1 thing that we are now considering the case where neither x equal to ky nor y equal to kx k prime x, which also means that neither x nor y is a 0 random variable which also means, that sigma y square is nonzero, therefore it is strictly positive. Therefore, there is no deviation by 0 here. So, I can happily write rxy by sigma y square without any problem. Further x minus this constant time y is nonzero, because that is what I said that we are not talking of the case where x and y are not linearly related. That is there is no scalar by which if you multiply y you can get x x minus any scalar types y is always non 0 random variable. For any choice whatever scalar you choose, because x and y are not linear related.

Now, linear relation we cover already; that means, x minus this constant times it is not a 0 random variable. So, if you take mod square of that an expected value that will be strictly greater than 0 there is no question of is becoming equal to 0. Now, E a E of mod square of this means E of thus this is quantity multiplied by its conjugate that is x minus this constant times y times within bracket x minus again the constant time y, but conjugate.

Then if you really do that exercise it is very simple to see that first terms will be E mod x square. Then it will be 1 term; involving the cross term that is this quantity rxy or may be if you do not feel comfortable we are not able to erase that. So, let them be there I where may be I draw from here; that means, x minus star this will be greater than 0. And if you can if you can multiply expand what you get is E of mod x square, because x x star which is sigma x square. And another term now you cross term this rxy y by sigma y square times x star expected value of that.

So, that will give you rxy by sigma y square that will go out into E of y start y x star that is x star y. Now, I defined E of x into y star as a correlation rxy here it is x star y. So, this is conjugate you can also write this as you can take this conjugate this conjugate out and put it here in these 2 places here and here, which is nothing but, rxy star this is nothing but rxy star here is 1 cross term. What is the next cross term? x into conjugate of rxy by sigma y square y.

So; that means, rxy which can be complex. So, I am putting a star sigma y square is real no star and expected value of xy star; which is again rxy. And last term is rxy rxy star divided by sigma y to the power 4. So, mod rxy whole square divide by sigma y to the power four into expected value yy star that is expected value of mod y square which is again sigma y square. This entire quantity is greater than 0 now, you can see 1 thing here you get rxy into first r r star xy.

So, mod rxy square divided by sigma y square second term third term also rxy is star into rxy again mod rxy square divided by sigma y square. And last term is mod rxy square sigma y square. So, 3 terms are same 2 are with minus on with plus. So, after simplification then it means.

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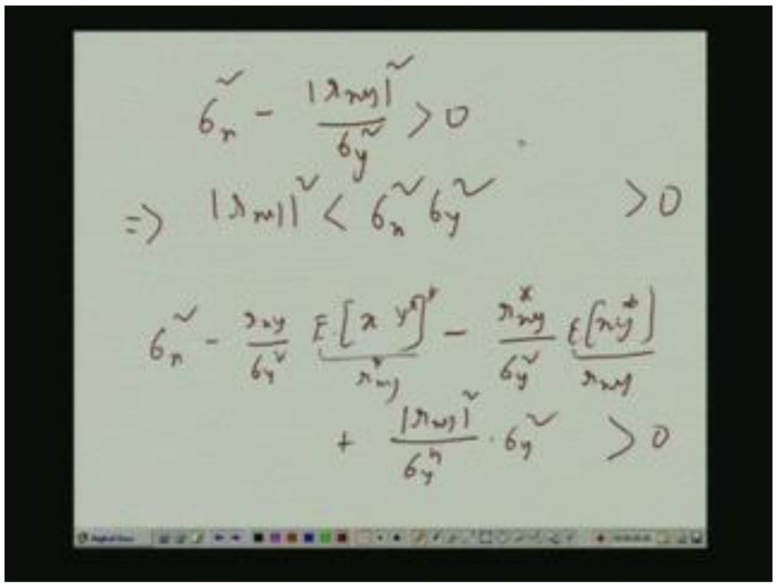

It means sigma x square minus mod rxy is square divide by sigma y square this is greater than 0 which proofs that inequality. If you take this 1 term to the right basically its very simple you get this. This is called Cauchy Schwarz inequality if the we have probability in the context random variables but actually, its scope much this beyond these random variables on it and basically, an inequality which is study in the context of vector spaces.

So, you remember you also said that random variables can be treated as vectors. And abstract vectors space while correlation between 2 random variable is actually the inner product that is. So, hold dot product between vectors. A rule of vector addition that is just addition normal random very usual in motion of addition random variables scalar multiple some random variable all those are discussed earlier. And this Cauchy Schwarz inequality actually is a topic of vector space theory and that is it can be proof in the general vector space context.

Here for the sake of simplicity I just did not bring in the vectors space concept. I apply that say all the usual steps of Cauchy Schwarz inequality purely in the context of random variables. Now, why this what follows from this two things follow from this, which are important in the context of spectral analysis.

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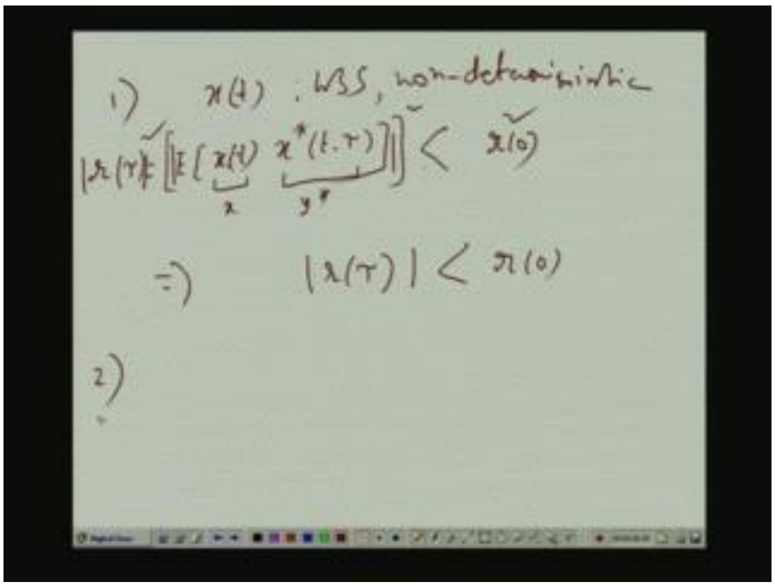

So, we just walked they here number 1 suppose you are giving the random process for the time being I will put xa t or may be a is not required. Whenever you use the low variable t which stands for time it will be automatically in place it into the imply that I am dealing with. The continuous time random process xt is given its it may be complex valued, but continuous time random process suppose it is given. In that case in the previous inequality we had this right rxy that is Exy star.

Now, suppose I consider this that is xt this is suppose your x in the previous case and this is your y star there is y equal to x of t minus tau. Though x is a random variable that is the sample at a particular time index t time at the point of time that are be the sample. That is random variable say x of t is random variable equal to say x. And x star t minus tau I call it y star there is x of t minus tau that random variable is say y, so the star of that.

If I consider this you know correlation which. In fact, is nothing but rtau, because I am assuming WSS process yxx are equals. So, these are tau I should have write in or xx tau, but for the time being I am dropping subscript xx, because I am dealing with any other random variable here random process here. So, at tau by Cauchy Schwarz inequality I can then write it's less than equal to sigma x square sigma y square.

For the time being you assume I am assuming that this 2 variables x. And y there are not linearly related that is the random process is such that no sample is a linear combination of some other samples. It is a purely non deterministic process those processes are called deterministic where 1 sample can be written as a linear combination of some other samples, but we are considering in non deterministic process. So obviously, no 2 samples are linearly related non deterministic.

So, it will be strictly this will be strictly less than what less than what sigma x square sigma y square. What is sigma x square that is variants of the process xt that is E of mod xt square. Because, of stationarity that will not depend on t so that, you can say r 0 which is real and positive in our case. And again E of and then sigma y square nothing but E of mod xt minus tau square t minus tau whole square now as I told, because of stationarity E of mod xt minus tau whole square. We will also be same as r 0, because the time in this does not matter, because of the stationarity.

So, just a minute the inequality had square of that. So, r square tau remember that you will get square of that rxy square mod rxy square. So, it will be actually mod mod of this in the Cauchy Schwarz inequality mod of rxy whole square was less than equal to sigma x square sigma y square. So, if I apply mod on this coefficient x square that is less than equal to r less than r 0 into r 0. So, r square 0. So, mod of this that means; you take the positive square root from both site.

So that means, for any non deterministic wide say stationary random process where you have the maximum value of the correlation what are correlation that is at lag 0. For any nonzero lag tau that correlation value has a magnitude that is less than r not that follows from here. Secondly, so I erase this thing.

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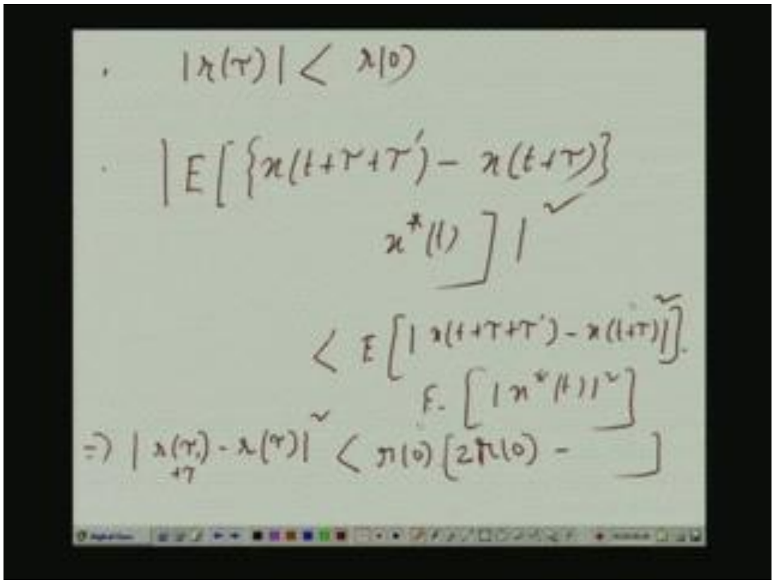

So, what followed from previous discussion only that we write rtau less than r not, this is we have already obtained. The other thing is if you now consider again Cauchy Schwarz inequality, but in this form expected value of say xt plus tau plus say tau prime minus. This quantity time x star t, so again by Cauchy Schwarz inequality that will be less than here less than always. Because, I told there is not linear relation between the samples of the process. But it strictly less than that will be less than expected value of that is the variants of the first term into the variants of the second term.

There is expected value of mod of t plus tau plus tau prime minus xt plus tau this square into expected value of mod x star t square. Now, what does it imply this this is the from the Cauchy Schwarz inequality. Now, what does it imply in the left hand side you consider the first product. This is what x the, this is a product that is why it will minimize to this auto correlation, but what the time points 1 is t plus tau plus tau prime another is just t.

So, lag is tau plus tau prime. So, it will give there is nothing but r you can rx x, but I am dropping that is subscript xx or of what tau plus tau prime. In the other case what it will give it is just r tau and this is you take mod of other post mod square. So; that means, left hand side gives you tau plus tau prime minus rtau square that is less than. Now, here the second term mod of x star t is same as mod of xt at E of the square of that mod xt square E of that is simply r 0.

So, 1 term is r 0 I see how about the first term. So, first term let us see what it is mod square mod square. Means this expression times its conjugate I will not work out those, because of as simple as steps or workout minutes few minutes back. So, this quantity times its conjugate. So, there will be four terms four product terms. First term we minimize to what x of t tau plus tau prime times its conjugate that there is minimize nothing but r 0. Similarly, another term xt plus tau into xt plus tau star that will again minimize r 0, so there will be 2 r 0 2 r 0.

That there are 2 cross terms 1 of them will be what x of t plus tau plus tau prime into x conjugate of t plus tau that we minimize to auto correlation until act tau prime. Other time other cross term will be minimize to auto correlation untill act again tau prime, but conjugate of that. So; that means, you will get twice this you can verify a real part remember z plus z star is twice real part of z. I repeat again if you consider this, because what are cross terms.

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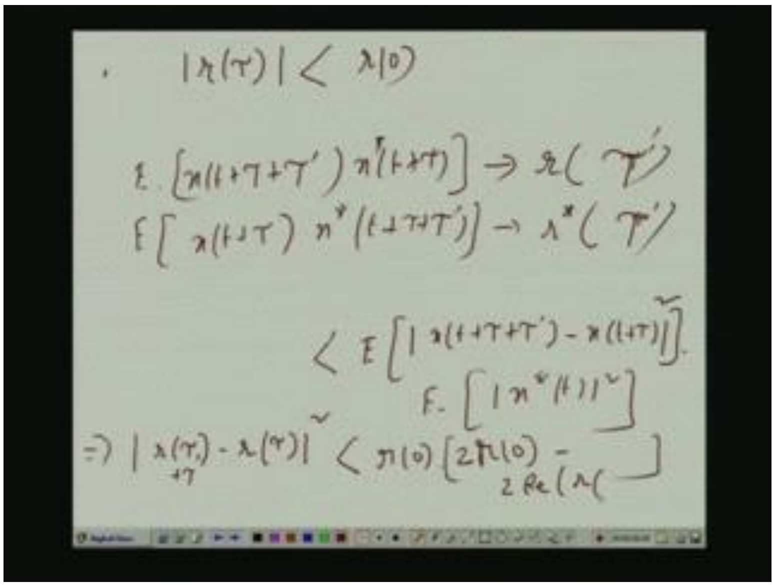

One cross term is we I work out here One cross term was between these 2 times xt plus tau and this there is conjugated it expected value of that. That will minimize to what rtau plus tau prime other cross term will give as to conjugate of the first xt plus tau not conjugate but x star. So, it is nothing but previous thing, but its conjugate value. So, complex number it is conjugate nothing but twice real part of the complex numbers there is what I write.

So, this will not be tau plus tau prime if you this only tau prime, because the lag here you can see in one case t plus tau plus tau prime another case t plus tau. So, gap or lag is only tau prime. So, these are these are purely tau primes tau primes. So, to real part of rtau prime. What is the implication of this, we will see?

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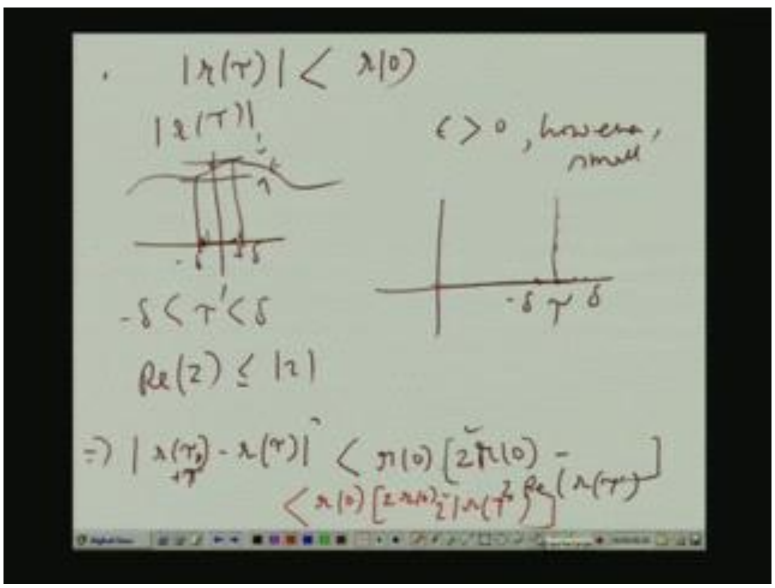

It means shown that rtau is a continuous function, because have to very soon do some integration involving rtau. So, it establishes the condition . Firstly, suppose rtau whatever it is a tau equal to 0. Suppose, in this continuous a tau equal to 0; that means, given any number real number epsilon greater than 0; however, small. Around 0 you can find a range may be delta minus delta so that, when you have when you are within this zone.

Then the gap between r 0 and rtau this range this is within a epsilon within a epsilon; epsilon meaning very small. What is the, what is the meaning of continuity? However, small number we choose; however, small very close to 0 infinite may be small. Still around the point what I am looking for continuity if it is continuous they are around the point. You can find neighborhood see plus delta minus delta; however small again. So, that if you are anywhere within the neighborhood.

Then the difference between the function at that point and at the or say the center point 0 that will be less than its magnitude will be less than epsilon. That is generally speaking around the point of interest that is 0. In this case, you can find this small neighborhood; however, small. Where all values of the function are very close to the where function that say center point 0 very close means within some limit of epsilon, there is a definition of continuity; then we said is continuity continuous.

Now, this means if you see this left side that if it is continuous at tau equal to 0. Firstly r 0 is positive we know and if it is continuous at tau equal to 0. That is, we have 3 plotting not rtau, because rtau is complex I should rather plot at or mod a tau. Then only I can do 1 plot say we are plotting say mod at tau mod at tau. And r 0 and mod of r 0 they are same, because r r 0 is a real number. So, mod at tau is function which is continuous; suppose, it is given to be continuous at tau equal to 0.

That means at around tau equal to 0 I can find it neighborhood where all the values of this function mod a tau are very close to mod of r 0 that is simply r 0. A very close means within epsilon for a given a epsilon; however, small. If that be the case, then what does inequality say that for any other point say tau not it origin, but tau there also there also if I choose this tau prime. Say tau prime within that range delta and minus delta if I take, because tau prime here was varying within the range delta to minus delta tau prime. And continuity it was moving to be continuous.

So, this right hand side was this r 0 minus real part of 1 thing 1 thing I forgot to mention that a real part of you know any complex number z is less than equal to mod z, because what is the real part of any complex number. So, z if the complex number is z what is the real part mod z cosine of the angle and cosine of the angle less than equal to 1.

So, this this is obviously, satisfied. So, this right hand side inequality. In fact, I should modify to the following noting that real part of r tau prime is less than or equal to mod of rtau prime. I can write this as r 0 twice r 0 minus mod of twice mod of rtau prime may be I write it somewhere else, because it is becoming a bit congested.

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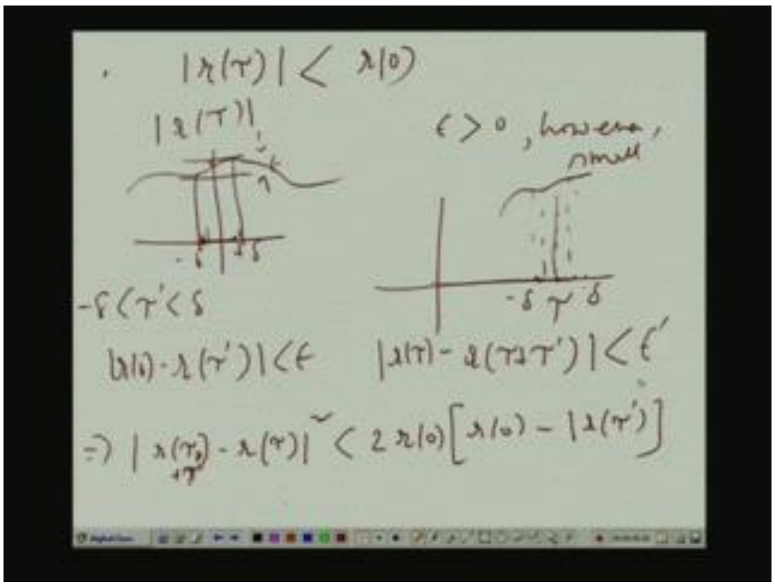

I think the 2 common then r 0 minus mod r tau prime and if either I come back to what we are discussing. That suppose, r tau is a function such that mod for r tau is given to be continuous at origin tau equal to 0. That means given any small; however, small epsilon; however, small it may be epsilon, but real number greater than 0. You can find out a neighborhood of size plus minus delta around the center point; point of interest that is tau equal to 0 here.

So that, for any tau prime within that range the function r tau prime its mod value remains close to very close to r of 0 within a limit of epsilon. That is for tau prime mathematically for tau prime in the range tau prime in this satisfied that r tau prime and r 0 difference that mod of that will be less than epsilon. If you can find out such a delta; however, small that neighborhood be. Now, it is given; that means, from this inequality I can see that right hand side is less than epsilon and epsilon can be very small. So, this product also can get very small so; that means, left hand side also can be arbitrary small.

So, far if I now move to some point tau and goal I mean take the neighborhood from tau plus delta in tau minus delta. Within the neighborhood also that is why if tau prime as I told you tau prime is varying from minus delta to delta with that neighborhood also this function mod r tau. That function will be very close to the center point that is say r tau let me write mathematically here. This means this and center point r tau the difference between them also should be very close some epsilon or epsilon prime. That means it is continuous there also now parallel purpose is I will assume that r tau is such a function that mod r tau is given to be continuous at origin. And therefore, means continuous everywhere.

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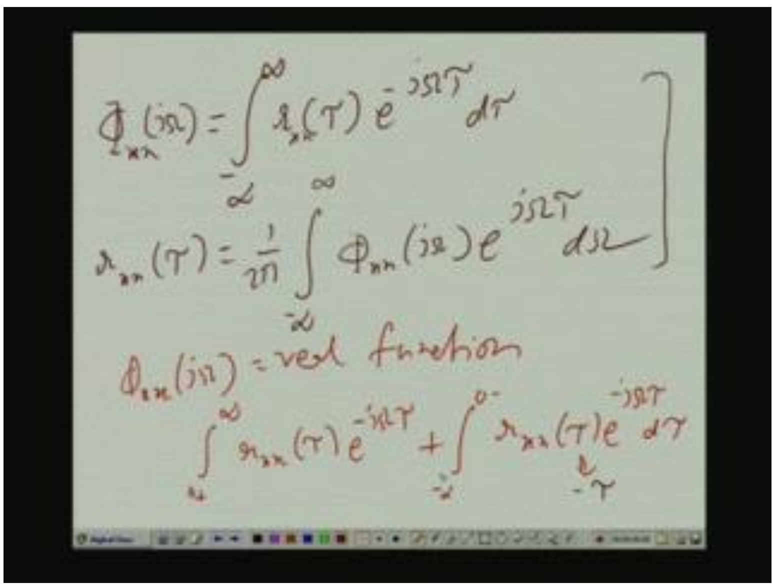

Under such case I now define 1 quantity that is Fourier transform of analog Fourier transform. I should write phi xx here if I put xx yes once I will do so that, you know what it is. Omega is a usual analog frequency radian per second and this is Fourier transform. Fourier transform existence means that you know sufficient condition is that rx tau must be or show this unable that is mod rx is tau integral should be finite. We are assuming that is to be case. So, this integral exists this.

This quantity is called the power spectral density or power spectrum of the random process xt. Why power spectrum how is it related to power all those things will come later. But; obviously, if it is Fourier transform you have got this the other 1 is inverse Fourier transform. That is given the power spectral density. Now, consider this power spectral density function my first add is it is a even though rx x tau could be complex. And it is a complex integral, because there is j factor here.

This phi xx j omega is a real function number 1 phi xx j omega real function real valued function actually. Why, because I can write the integral divide integral are 2 parts 1 is from 0 or may you can say 0 plus to infinity rx x tau e to the power minus j omega tau another will be from minus. Minus infinity to 0 minus same thing now here replace tau by minus tau or. That means integral limits will be from plus infinity to 0.

When tau is minus infinity tau minus tau is plus infinity plus infinity to 0. D tau will be minus d tau now. So, there will be minus sign outside the integral. So, minus and integral from plus infinity to 0 2 can be combined. So, that minus becomes plus and integral instead of from infinity to 0 becomes 0 to infinity. So, both the integrals then have limit from 0 to infinity and 0 plus to infinity. Only problem is you get rx x minus tau e to the power plus j omega tau.

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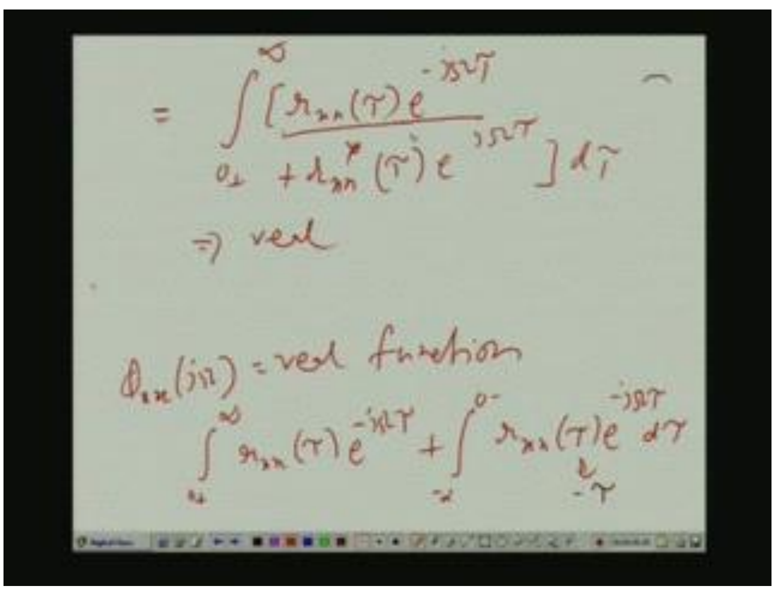

That means, what you have is 0 to infinity rx x tau there is 1 term another 1 rx x minus tau. Now, remember rx x is rx x star of minus tau rx x tau we had this relation rx x tau as rx x minus tau star. So; that means, rx x minus tau is rx x tau star remember that. So, I can write r and e to the power j omega tau d tau d tau. Now, you see if this quantity called z then the other quantity simply conjugate of z rx x tau. So, rx is star tau e to the power minus j omega tau it is plus j omega tau. So, z plus z star is always real is not it. So, this is real even in the processes complex valued correlation is complex valued.

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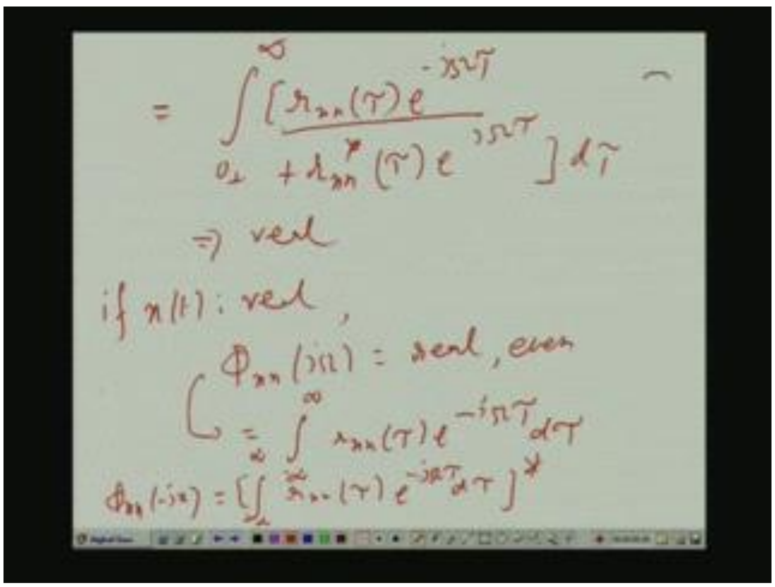

This number on property further, further for the special case if xt real then power spectral density phi xx j omega it is not only real, because there is real always that you have seen, but also even function. Even means, whatever this value at omega same value you have at minus omega just of a difficult to see. In fact, you know I mean, what is phi x is omega after all it is a Fourier transform of the of a function rx x tau.

So, if the function is real valued for any real valued function Fourier transform again we know. Just for difficult to show if you have still insist to I can you can easily see this is equal to what rx xtau e to the power minus j omega tau d tau minus infinity to infinity. So, what is phi xx minus j? Omega minus j omega means everything else remain same it only it becomes e to the power j omega tau. And, then entire thing can be written as minus infinity infinity within bracket rx x, because its real again minus j omega tau d tau and then a conjugate.

Conjugate if you bring in rx xt au its conjugate will be same thing, because it is given to be real and e to the power minus j omega tau it becomes it is plus j omega tau that corresponds to minus omega and what is the quantity within the bracket that is phi xx j omega itself and phi xx j omega star, but phi xx j omega is real function. So, its star and itself are same which is phi xx j omega. So, it is an even function. Why it is called power spectral density?

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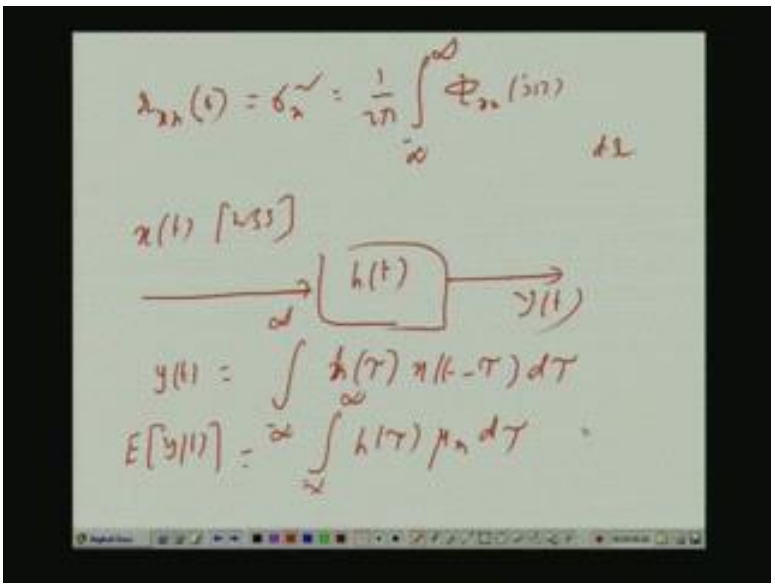

Now, if you see rx x tau that inverse Fourier transform we have seen. So far is special case rx x 0 which is also power sigma x square by that relation 1 by 2 phi minus infinity. Infinity phi xx j omega then the other term was e to the power j omega tau the omega, but tau is 0 now that is our choice. So, e to the power j omega 0 d omega and this is equal to one. So, you can as well remove it. That is 1. So, you see this gives all the information that phi xx j omega such a function real valued function that, when you integrate it over the inter frequency range.

Then you get the averaged power of the signal. So, that mean the phi xx j omega gives you see an idea of how the power is distributed along various omega. That is why it is called power spectral density that is 2 frequencies omega and omega plus the omega net power is phi xx j omega the omega. The total is integral. So, once again I repeat gives an idea of this is what the integral gives you the total power average power.

That means, phi xx j omega is what its gives you an idea about how that average power is distributed over omega. That is why is called power spectral density it is is very important. What will be the unit, unit of power is what and omega is radian here? Radian per second radian per second. So, what will be the need you work it out means actually what power radian per second. If it is capital omega use the frequency F that is the hertz then this becomes watt per hertz to the power frequency. How much power and then you integrate overall frequencies.

This for power spectrum a similar treatment in this also for the discrete time case similar treatment exists also for the discrete time case. May be we carry out the discrete time case later today we consider still 1 this case only. Now, suppose and xt WSS it goes into a system linear time in variant system of impulse of response h t output is a random process Yt, but this Ht WSS answer will be S.

What is Yt if the convulsion between h and x h tau xt minus tou d tau so; obviously, E of Yt is E of this and you can be expected expected say linear operators the expectation can be got inside the integral h is not random. So, E will operate only on x and E of xt minus tau is nothing but the mean of x, because x is WSS. So, the mean of the process does not depend on what time you have what time variable you have here. So, it will simply be h tau expected value that is mew x independent of time d tau integral. So, once you take that tau disappears you get a constant number. So, again mean of the output also is constant does not depend on time. So, its stationary with respect to mean you can call its mu y.

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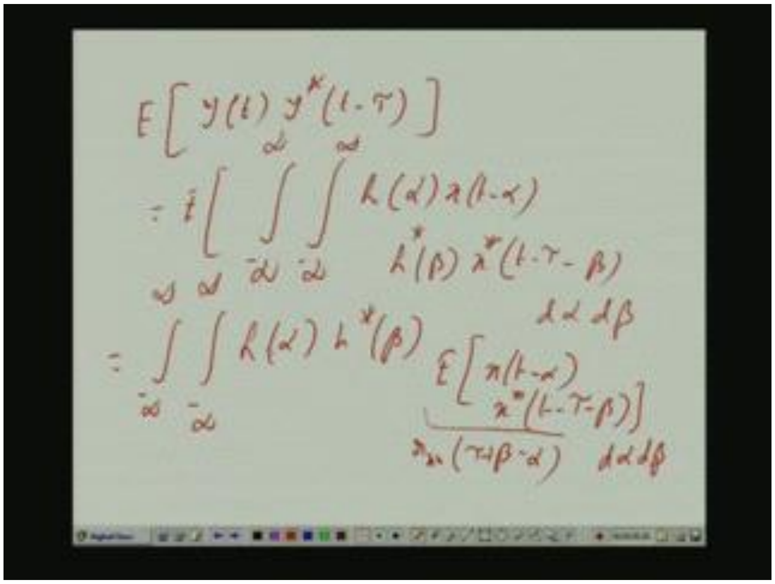

Now, you consider the case with correlation ryy say between 2 points t and t minus tau. Or may be let us find out this first E of Yt say y star is it of a function tau alone then its stationary with respect to correlation also. So, that is find out Yt will be 1 convulsion integral. So, will be y star t minus tau. So, there will be double integral. I am writing integral directly Yt will be say h alpha xt minus alpha integral with respect to alpha. And y star t minus tau will be h beta star x star t minus tau minus beta.

Here the time of interest is t minus tau. So, convolution variable is beta so x star beta x star. Star, because y star here present here and integrals with respect to alpha beta. If push E inside, because of the linearity of this E operator E will work daily on the product between x and x star, because h is non determine h is non random. So, you have got E of d alpha d beta and you see what is this product expected value of product is a correlation. Input is WSS. So, this correlation depends on what only on the lag, lag between t minus alpha 1 end and t minus tau minus beta another end.

So, that difference between them if you take out t goes. So, here resist quantity is nothing but rx x tau plus beta minus alpha and beta and alpha after you integrate disappears. So, everything becomes a function of tau alone, which means, output auto correlation also a function of the lack tau. So, it is stationary with respect to your correlation also, that means, output also is WSS. If that mean, so then what is the output power spectral density. If you remember w I mean power spectral density the get WSS assumption that is a correlation. And, so depend on only tau and then only you take the Fourier transform with respect to the tau.

If it is not stationary there is no question of power spectral density. So, station WSS nature is must there correlation should depend auto correlation. So, depend only on 1 variable tau and you take Fourier transform with respect that variable. And that is why it is putting on the output. So, output is. So, I can find out the output Fourier output power spectral density what is that be what will that be simply you remember this result.

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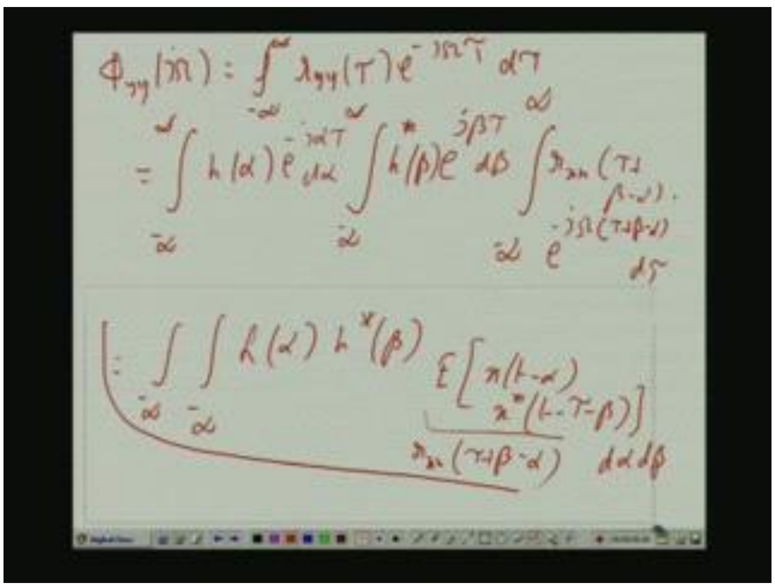

This is the auto correlation of output. So, what is phi yy j omega it is r yy tau e to the power minus j omega tau. D tau Fourier transforms ry tau a tau this is double integral that if you put here it becomes triple integer. And the integral all the integrals exists. So, if you assume all the integrals exists and therefore, convert you can interchange or integration that we know we have done it lengthy of times. If you do that let 1 integral will be with respect to h alpha another with respect to beta. And the inner integral all are from minus infinity, infinity inner view with respect to tau.

So, rx x whatever has tau that will remain inside tau plus beta minus alpha time this e to the power minus j omega tau. Just I know 5 minutes left I will be doing. So, e to the power j omega tau suppose, I write I write that as this. Tau plus beta minus alpha, so to cancel out those extra terms I bring in on this h star here. E to the power j beta tau here d beta e to the power minus j alpha tau here d alpha and this is d tau. First quantity is Fourier transform this is transform function h j omega Fourier transform. The impulse response next quantity is, take the star out.

So, h beta e to the power minus j beta tau that integral is again h it is not tau. There can be tau here this omega, because hard use, so omega alpha omega beta omega beta. So, what is the second integral? Second integral you take star out, so h beta e to the power minus j omega beta that is again h j omega star of that. So, h omega h star j omega and here in the inner integral if you call tau plus beta minus alpha as some variable tau prime.

So, it is simply integral limits remains same everything remain same it just a Fourier transform of rx x tau prime which is nothing but power spectral density of the input. So, phi xx j Omega, so mod this is very important equation this gives rise to the idea of filtering that. You can design a filter system so that, given any arbitrary given any power spectral density phi xx j omega. You remember this is product relation. So, output power spectral density; density can be shaped by your choice of the function.

Then input power spectral density is like this, but I want at certain changes on omega its value to go up. So, my I design a system whose values whose mod value of this function actually its much higher at those ranges of omega. And I want that this is power spectral density have much less value in the certain other changes of omega. So, I design the system such that those send it a omega the value of this function this is transfer function h j omega much less.

So, in this that is actually filtering that happens, because of the product relation you are directly multiplying 1 function by another function. So, by designing the second function that is h j omega appropriately you can give a new shape to output power spectral density that is the idea of filtering. So, I stop here now. Using this relation we will show that this phi xx j omega if you know even for I mean, whether it is a real valued process or complex valued process power spectral density is not only real, but also non negative.

We will show that then we will take up some examples of how to compute power spectral density of certain famous processes. Then I will move to a similar treatment for discrete random sequences what is its power spectral density and all that. And, then we will go for estimation of those spectral. I will relate the power spectrum or power spectral density discrete processes with then analog counterpart. Because, when I put to digital spectral estimation I can only look for digital I mean discrete sequences there is a discrete sequences not continuous valued. So, that is all for today. Thank you very much.

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So, last time we are discussing the spectral analysis and we defined what is called power spectral density of course, we are dealing with these continuous systems. We are dealing with continuous systems you know analog systems. So, you just start from there once again it will help us.

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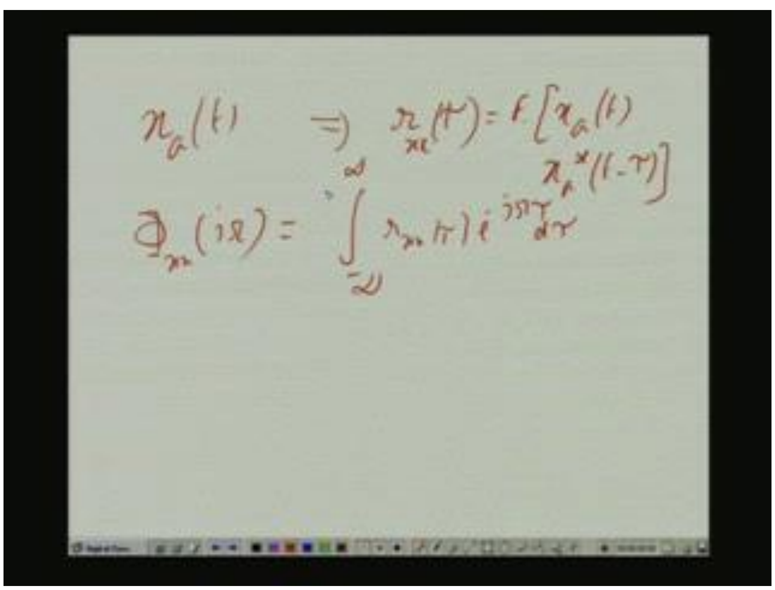

So, suppose there is a analog random process analog or continuous time. So, I just today I put a subscript a just to denote it is an analog signal Xa t let it be random process analog random process and it has a correlation ra say t ra tau which is a. And what is the power spectral density? Here, phi a j omega this equal to maybe I change the notation a little here. So, I put rxx here it is assumed this I will be dealing with only continuous time case or analog cases only.

So, even if I put rxx and do not mention any a here it is assumed that I am I am still considering an analog or there is continuous time random process. So, similarly here phi xx j omega which we know is a Fourier transform of the autocorrelation function rxx tau that is e to the power minus j omega tau d tau minus infinity to infinity. We have start we have seen that even though the random process is complex valued. And therefore, the correlation is a complex valued function the power spectral density is a real valued function.