Probability and Random Variables
Prof. Dr. M. Chakraborty
Department of Electronics and Electrical Communication Engineering
Indian Institute of Technology, Kharagpur

Lecture - 33
Ergodic Processes

So, today we begin an important topic called ergodic processes or let me put it this way ergodicity

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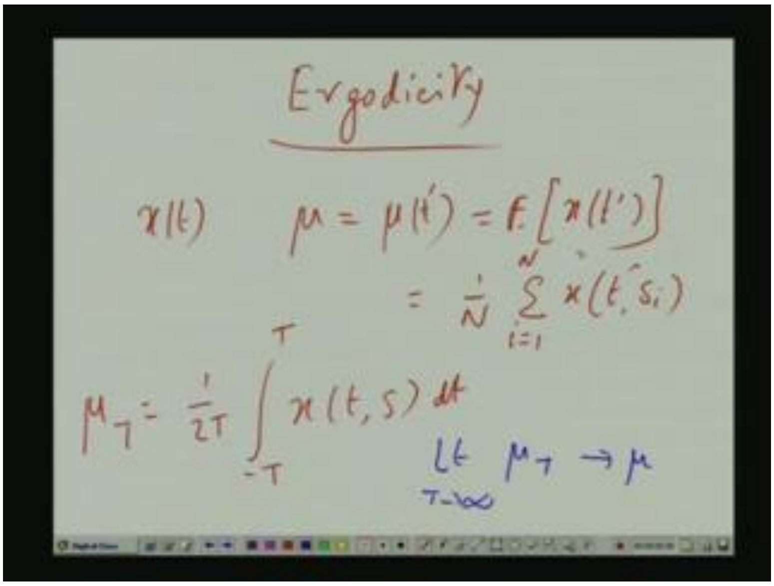

Now, you see suppose we are given a random process xt and we want to find out it is we want to determine it is its statistics say variance moments for that matter say mean. Now, how to estimate, the mean that is suppose you want to find out the mean at some point of time of t prime what do you do. So, this is what is this by definition this is this; now, how will you estimate this. So, naturally you have conduct several experiment with every experiment you will get a waveform xt. So, we get the sample xt prime again conduct the experiment again gets away another waveform again we get the sample t prime.

So, and if you conduct this experiment several times say N number of times then add all the sample values and then divide by F. So, that will be a sample average and if N is very large it tends to infinity that will become a better and better, and finally the correct estimate for this mu. That is we do what is called ensemble average, that is may be xt si where xt prime si per x t prime si, si stands for the particular experimental outcome si. For that outcome the corresponding function is xt comma si that is for the experimental outcome si you have got a waveform which is x within bracket t comma si. So, there I am measuring the sample value at a point of time t prime. And next again I conduct the experiment. So, I will get another experimental outcome for that also I have another I waveform.

So, I measure the same sample value measure the sample value at the same point of time t prime, so and so forth. So, I add all the sample value. So, what the previous experimental outcomes that is over the ensemble and then divides by N. So, that is an ensemble average. And if N is very large if again N tends to infinity this right hand side tends to the actual statistical mean this we know. Problem is in real life we may not have the time to have, so many experimental observations just to calculate 1 mean. Because everything you know most of the operations that we do in our engineering they are done in real time. So, you would not get time to conduct the experiment or to measure the waveform time and again.

So, it may 100 times or 200 times then estimate mean and then proceed for your next job. So, ergodicity is a concept which helps us a lot in this. Now, suppose you got only 1 waveform, suppose you got only 1 waveform for a particular outcome s to this part for a particular outcome s or for any outcome s you got 1 waveform. Suppose, I do some kind of a time average of that and you call it give in a mu t. Question is can this mu t for a very large value of this T. You see what is happening is I am integrating the function and then dividing by the length of the I mean this duration that is 1 by 2T that is 2T.

So, it is an average time average I am integrating it and then dividing by the total time period that we have here minus T 2T that is 2T. Is this time average will this time average give me some idea or may be exactly this statistical average that is the actual mean. Now, before we consider such a possibility we have to see 1 thing that if I really carry out this time average I get a constant which is independent of time it is a constant. That means, for this to work for this to be valid this process. You know the mean, of process should not depend on the particular time instant t prime t prime while you are measuring; that means, it should be constant at all point of time which means it should be stationary in mean.

That is this mu t prime should be just mu I mean it does not it should not matter whether what t prime you have chosen, because if I have to have mu is equal to mu T for very large T or may be as T tends to infinity. That is the question that I was to pose is can we have a situation where mu T tends to mu as limit T tends to infinity. That is if I fI divided by a very large period do I get mu T closer and closer to mu. Now, for this to work first you see this integral as I said this is independent of the time you get a net constant quantity. So, for this to be equally here; then this the process should have mean constant that is it should not depend on the time period instant T prime where it is measured, which covalently means the process should be stationary in mean. Suppose it is, so suppose the given processes give the process xt is given to be stationary in mean, even then this question lies then this question lies that.

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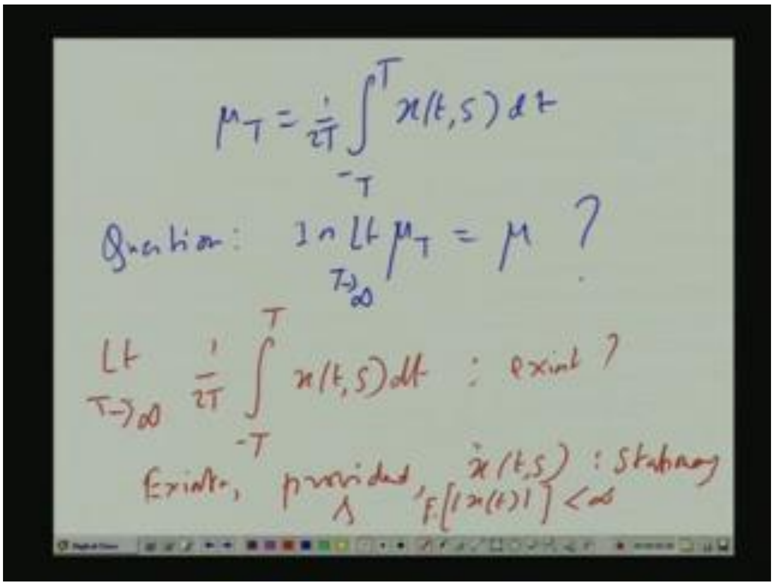

If mu T is equal to give to be question limit T tends to infinity mu T is it equal to mu. This is the question is limit mu T where T tends to infinity is equal to mu this is the question. Now, if that happens then I will say that the process is ergodic in mean, if that happens , but before that what we are doing here is we are doing some kind of time average. So, a mathematical question arises whether this limit exists at all not that is this limit after all limit mu T, T tends to infinity means what. Limit T tends to infinity does it exist?

Answer is yes it was proved by 1 famous I mean statistician called named Barcoff he said that this exists provided exists. Provided xt s is stationary, this is stationary and expected value of mod xt if it is stationary it does not depend on s E of mod xt that should be finite. If that happens this limit exists it is just for a mathematical this thing you know correct I mean a to be mathematically correct or just for mathematical interest we are writing this. We will always assume that this exists, these conditions are satisfied. So, we can happily take the limit, but when you write a mathematical statement like this you should provide the condition under which it is true that is why I gave this condition which was obtained by Barcoff. The process should be stationary and E of mod xt should be finite and; obviously, there is nothing big deal about it, it means that mean should be finite.

If that means, so then this limit exists, so we can happily take the limit this was proved by somebody. But then that brings us to that initial question that now, the limit exists does. The limit give us actual mu that is the actual statistical average the expected value of xt that leads to this concept of ergodicity and again ergodic it is more general it is not only restricted to mean. In fact it is restricted to any kind of time average.

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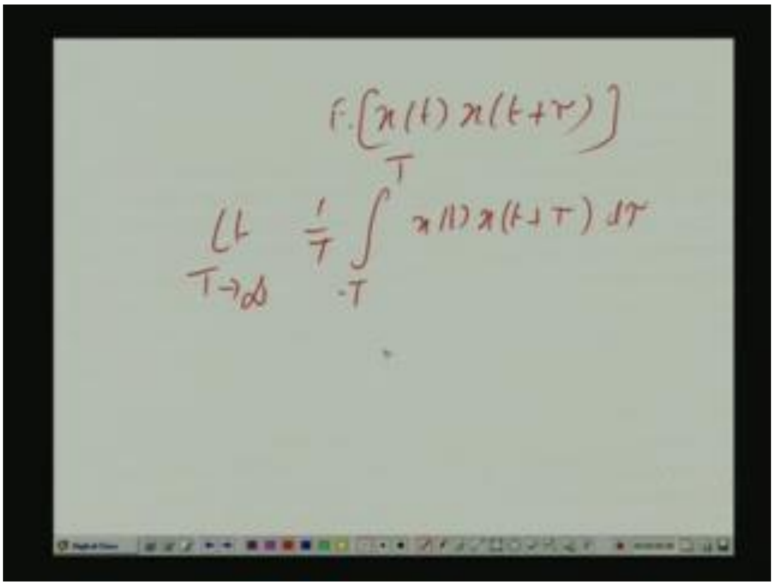

For that matter say we will be doing with this also some after sometime. Suppose, you are doing you are finding out correlation. What do you do to estimate this; you carry out this product over ensemble I mean for each sample function you carry out this product and average over the ensemble that will give an idea of this and estimate of this and if total number of observations over which this average its large and large this will be a very good estimate of this correlation. But again that question comes that does it become equal to this. That is if I carry out instead of taking the expectation over the ensemble. If I do the expectation on this averaging expectation is same averaging overtime this quantity and limit tend to infinity. This does this give you does this give us this and here it is second moment I can take third moment also fourth moment also and all that.

So, essentially if this time average gives rise to ensemble average then we say the process is ergodic. And when I say ergodic, it applies to all the moments not just for mean or correlation it applies to all the moments, that is to evaluate all the moments. Instead of carrying out ensemble average you can carry out a time average like this; if that is true than the process is called ergotic. Now, if the process is ergotic; that means, that just by time average, I can determine all the moments, just from given 1 given 1 particular sample function I can carry out this time average and get all the moments. And if you know all the moments I know the entirely probability density and distribution.

So, I know the full statistical characterization of the process. So; that means, 1 sample function should be enough to obtain the statistical characterization of the process. That is obtained all the moments, so equivalently probability density or distribution. Now, I consider first mean ergodic process that is processes which are ergodic.

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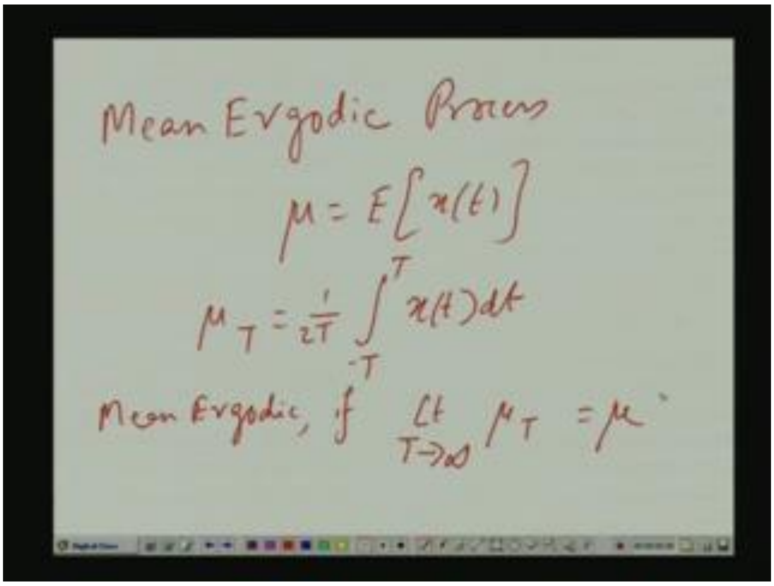

So, only in mean to start with here actual statistical average and expectation which you obtained by taking ensemble average and the number of times you average if number of times you take you observe it before averaging if that is large and large, then that average becomes actually same as the actual statistical average or expected value of the xt. And I am assuming the process to be stationary in mean; obviously, because the region has been told already that if I talk of ergodicity it has to be stationary. Now, suppose stationary process at this process which is stationary at least in the mean; that is, given to me and the actual mean of actual expectation is this which we can obtain by ensemble average provided you take very large number of observations. Question is when I found this quantity mu T then it is mean ergodic if is equal to mean. Now, what will be the condition; what should be the condition on the process what kind of processes should be what should be its characteristic.

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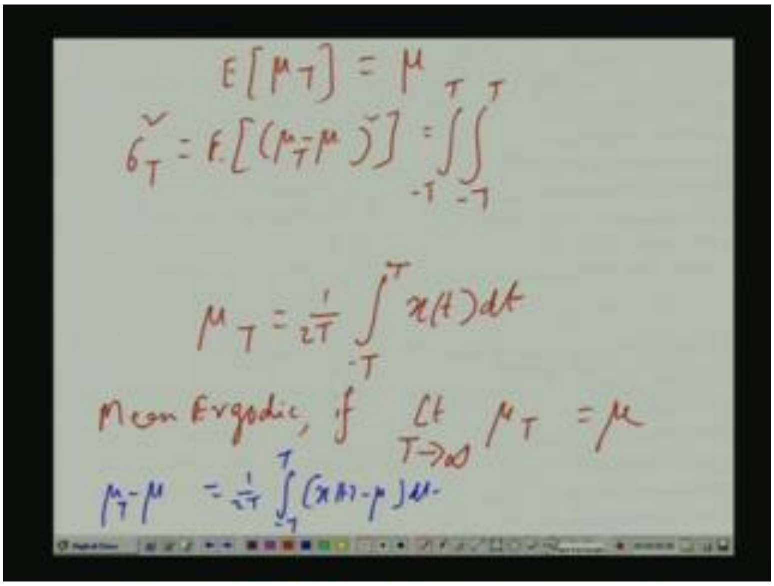

So, that this is satisfied we now investigate that you see, given any xt if you carry out just this integral from minus T to T and this divided by 1 by 2T and you will get mu T naturally mu T is a random variable. Because next time you have another observation for the process say another xt you carry out the integration you get another value of mu T so on and so forth. So, mu T is a random variable what is the mean of mu T, E of mu T clearly you put the expected expectation operator inside the integral, because expectation is a linear operator E working on xt and E xt, because of of xt is constant and is mu again mu comes out of the integral.

So, integral give rise to 2T, so mu in to 2T divided by 2T which is equal to mu; that means, this mu T is a random variable whose mean is mu , but that is not enough. It is true that whenever you observe this mu T for any choice of T not necessary infinity its expected value would be mu, but I am not interested only in that, I want that mu T should become equal to mu. Now, here 1 thing I mention that since mu T is a random variable, and I am taking a limit I must sense, I must tell you in want sense is the limit carried out now, here it is mean square sense. That is, the difference between mu T and actually mu, mu is constant mu T is a random variable mu T minus mu that difference that is an error it will take the variance of that that is expected value of mu T minus mu square, that should become less as T becomes high. So, that should tend to 0; in that sense, I am writing this limit expression then is that mu T of which is mu in the mean square sense as T tends to infinity.

So, far I have only got that for any choice of T mean of this random variable mu T is mu, but for this ergodicity mean, I must have this satisfied that if I call it sigma T square I am taking all real case. So, there is no point in putting a mod, because I am taking a square which is same as what is mu minus mu T by the way? Let us do this thing separately here, mu minus mu T; we can always write as or may be its better that I put you know mu T minus mu its does not matter though, because there is a square,mu T minus mu what is mu T minus mu, mu T I know this; minus T to T xt and I can also write xt minus mu dt; obviously, because if you take the mu part mu comes out integral gives you 2T 2T by 2T cancels you will get back again mu.

Now, this variance this is what mu T minus mu is equal to this. So, I have to take the square, so there will be a double integral and expectation here. So, if you take a double integral; 1 integral with will be will be with respect to say T1, because this side T. T is the local parameter local variable of integration. So, you take T2 T1 you know now, be integral next when I have, because this is a square. So, mu T minus mu times mu T minus mu; in the first mu T minus mu you will replace that mu T minus mu by this expression, but take T to be T1 as the variable of integration. In the next case you take T to be T2. So, double integral with respect to T1 and T2 and then push the expectation operator inside. The double integral that is what you will get here. That is and there will be a1, because 1 by 2T. So, that will give rise to a 1 by 4T square that is an important term.

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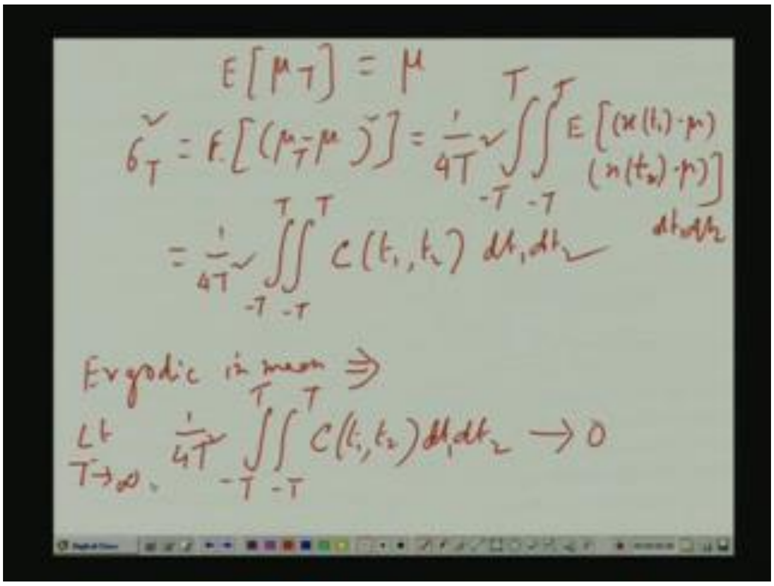

So, I should not ignore that 1 by 4T square double integral and now, instead of working out step by step; I think we are now, sufficiently mature to see what result we will get. If you have xt1 minus mu and again xt2 minus mu and you carry out this and then apply the expectation operator on this product of these terms. What you will get is simply the covariance of the process xt when measured between the 2 Time indices T1 and T2. I have assumed the process to be stationary in mean, but not necessarily stationary in correlation also that is not necessarily secondary or stationary and so. That is why I will not say that the covariance is not a function of T1 or T2, but is a function of T2 minus T1; no, not still not yet it is stationary in mean that much I am using.

So, I repeat again or maybe I write down in case, you have got terms like this xt1 minus mu xt2 minus mu and dt 1 dt2. Now, this clearly gives you the covariance of the process x when measured between the 2 Time indices t1 and t2. And is not necessary function of the the lag T2 minus T1 or T1 minus T2, because I am not assuming the process to be stationary in covariance or correlation yet. In that case, this simply becomes equal to 1 by 4 T square; c is the covariance t1 t2 dt 1 dt2. Now, if this quantity tends to 0 as T tends to infinity. That means, this error variance sigma T square is becoming 0 in the mean square sense is up which means mu T is approaching mu in the mean square sense as T tends to infinity.

If that happens, then I will say this process is ergodic in mean, that means the condition for ergodicity in mean is simply this. This fact thus limit T tends to infinity this integral 1 by 4T square c t1, t2 dt1, dt2 This should what this should tend to0. If that happens, then the process is ergodic in mean. This process is yeah ergodic in mean you can take an example.

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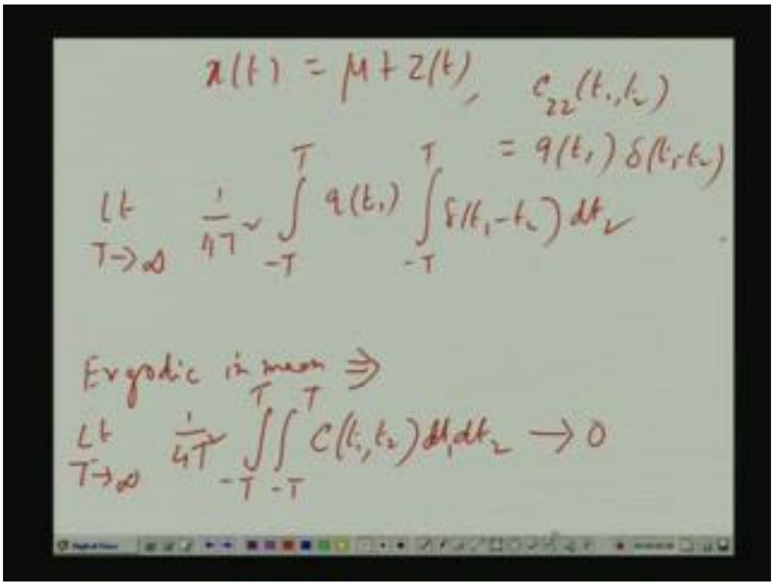

Suppose xt is given to be some mu plus zt where zt is a 0 mean, random process mu is constant zt is a 0 mean random process. What is expected value of xt then mu, because of E of zt is 0 and therefore, xt is stationary in mean, because the mean that is expected value of xt is constant that is mu independent of t. So, that first condition is satisfied then is stationary in mean. And it is given also that zt has a correlation that is the correlation of zt is given to be correlation or covariance, because zt is 0 mean. So, the correlation covariance means the same.

So, c zz if you call it t1 comma t2 is you know it is a wide sequence. So, it is given to be some function q t1 in to delta t1 t minus t2. So; that means, if t1 and t2 are same then only it has got a value, but that value is not independent of t at t1 it has a value q t1. So, if t1 changes this value changes. So, variance changes from point to point. But the process is uncorrelated that is between t1 and t2; There is no correlation or no covariance its 0 mean. So, no correlation covariance means the same, so that kind of thing here, if the process xt is to be ergodic in mean what is the condition I simply apply this condition here, that is 1 by 4T square if we apply minus T to T.

Now, c t1 t2 if you put here, qt 1 is a integral outside, because t1 I will take to be the variable of integration for the outer integral and inner integral will have delta t1 minus t2 with respect to t2. So, that integral will have value only 1 you know, it is a delta function t1 is constant within that that is it will have things like this q t1 dt2. And this integral is 1 this is a property of delta function this is 1 and this we basically get this fact.

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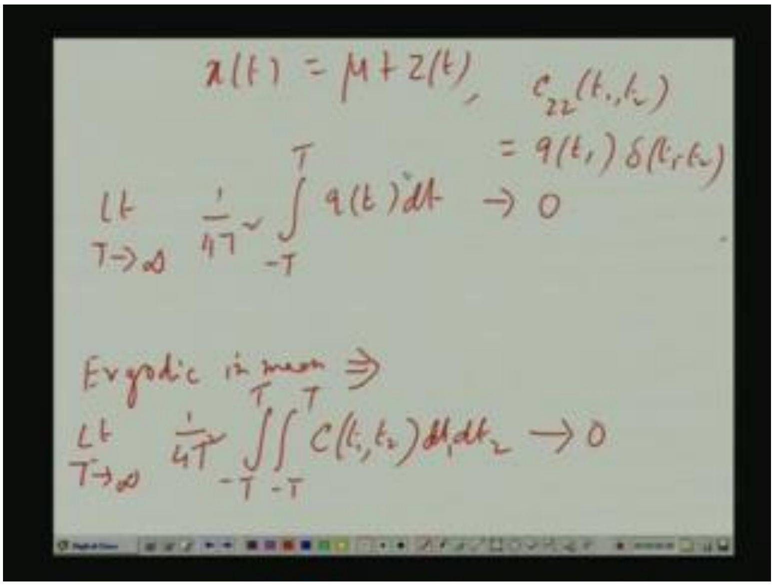

So, I am not I can eliminate this what do we get. In fact, you can even ignore this t1 now, call it t this. This should be as this should tend to 0; this qt variance integrated divided by 1 by 4T square as T tends to infinity. So, it should be 0, we now take another important case.

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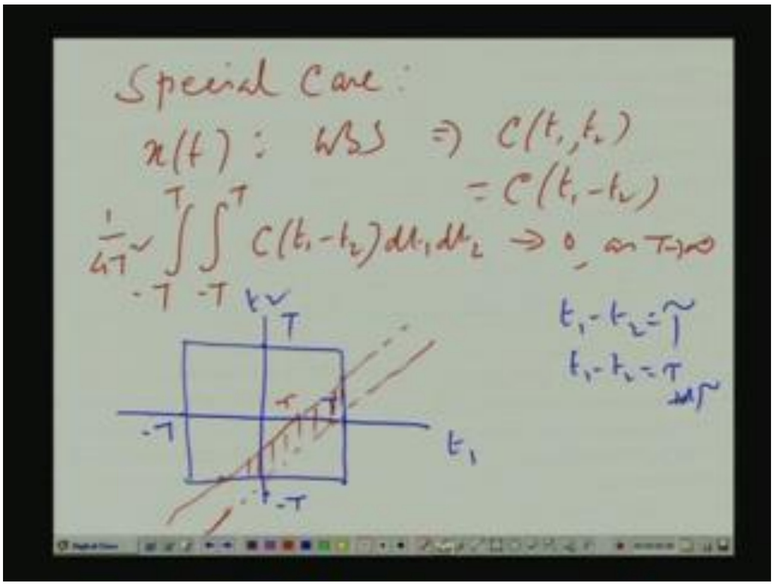

Special case xt is given to be wss if xt is wss then; that means, instead of c t1 t2 it should be a function of c t1 minus t2 and I should have this d 2. So, tends to 0 as T tends to infinity this is the thing. What does it imply let us see, let us see what is this how to carry out the integral may be, I am not sure we have done this exercise earlier. This is double integral within this box this function is to be integrated within this box. Now, what function c t1 minus t2. So, this such a function, which neither depends on t1 or t2, but depends on t minus t2. So, if t1 minus t2 is constant in a region or a in this in a segment of this only a part of this square then c t1 minus t2 is a constant over that segment.

So, therefore, if I consider the straight lines like this you know t1 minus t2 equal to say tau this axis is t1 this is say t2, t1 minus t2 Tau. To start with suppose I take tau to be say here tau, what is the slope of this; slope is 45 degree 1 this curve if it is tau it would go like this if it is tau. So, what this line this function has same value, because what this line t1 minus t2 is constant tau. So, I have a constant valve c tau over all the point from here to here from here to here for all the point within the square have got constant value.

I when I take a line close to it t1 minus t2 is equal to tau plus d tau; that means, I will have 1 more line if it is tau plus d tau. So, I will take this area, this area over this area. So, this d tau infinitesimally small over this area the function c has a constant value that is c tau. So, what is the integral over this infinite small segment c tau times this area and then I will sum up by this integral to include the entire square. What is this area?

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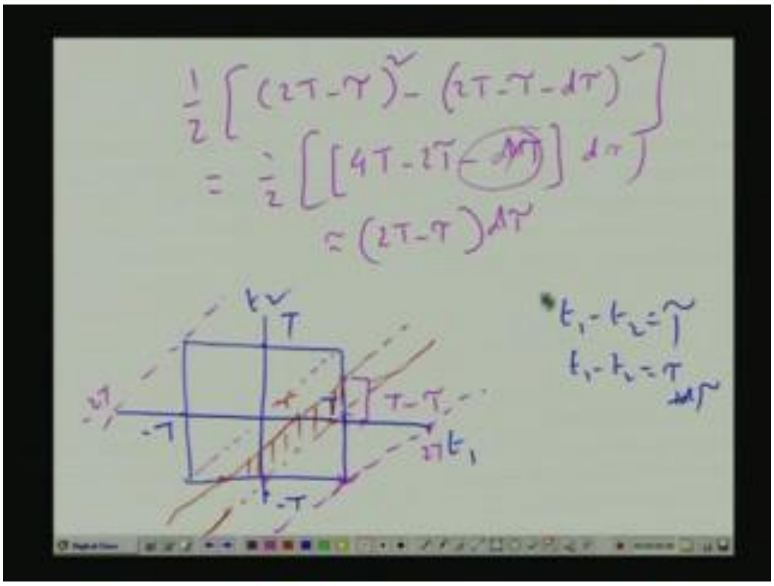

Firstly you can see 1 thing from elementary geometry the slope is 1. if you had taken this line this also has slope 1; that means, all these lines are parallel right, but if you in this triangle this side is equal to this side therefore, in this triangle also this half is equal to this half or in this triangle also this half is equal to this half from similarity of triangles So, I want to find out to find out this second area what will you do I will take the bigger triangle find out its area subtract the area of the inner triangle this small triangle.

Now, find out the area of this triangle it is enough to find out 1 of the side, because it is isosceles it is right angled isosceles is as I told. So, you find out this now, look at this curve straight line t1 minus t2 equal to tau the straight line. When t1 equal to T then this is T minus tau and this is T. So, this length is 2T minus tau and therefore, this length is 2T minus tau .So, I have got half 2T minus tau square this is the area half 2T minus tau square. And how about the inner triangle instead of T minus tau it is T minus dt. So, it will give rise to d tau square so; obviously, you get half 4T minus 2 Tau minus d tau and d tau. And this d tau you can ignore, because d tau into d tau that will be negligible and this 2 cancels here. So, you get 2T minus tau d tau that is the area.

If on the other hand your line was passing on the this side here, tau was taken to be positive if your line and mind you tau will go up to 2T, because this is the limiting case. This is the limiting case of this straight line and this is 2T and from here it will go up to minus 2T. So, tau will go from 2T to minus; T I took positive tau let us take negative tau also.

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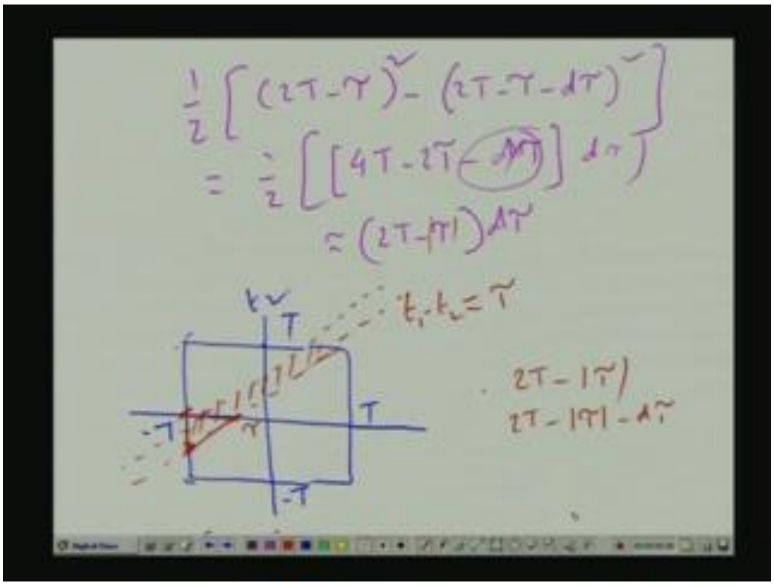

Let us take negative tau, so here tau, so tau is actually negative and if you draw a line and another line through this tau plus d tau. So, I have to find out this area. So, I will find out this bigger triangle and then minus this inner triangle this 2 areas. I will take the area of the bigger outer triangle or bigger 1 and minus the area of the inner 1 that will give you second area. Now, what is this point is you look at the equation t1 minus t2 is tau, tau could be negative here I agree. So, what is this point here; t1 is minus t minus T so; that means, t2 is minus T minus tau. So, what is its length, this part is isosceles that counts, because this slope is 1. So, what is this length; this length is simply let us do this way.

This is length T and this is length mod tau, because tau is negative. So, this length is not tau, but mod of tau. So, total length is like this, length this point is minus T, but actually length is T this is tau, but actually length is mod tau. So, this length is T minus tau. So, this is T minus mod tau. So, this is T minus mod T add to that another T, so 2T minus mod tau here. In the previous case 2T minus tau tau was positive. So, there also I can put a mod does not do any harm and here 2T minus mod tau. And in the another case 2T minus mod tau minus d tau. So, even here if you calculate the same you will get this same figure 2T minus mod tau d tau. So, tau will go from minus T to minus 2 T to 2T minus 2T to plus 2T ; that means, what is the integral now everything is very clear to you.

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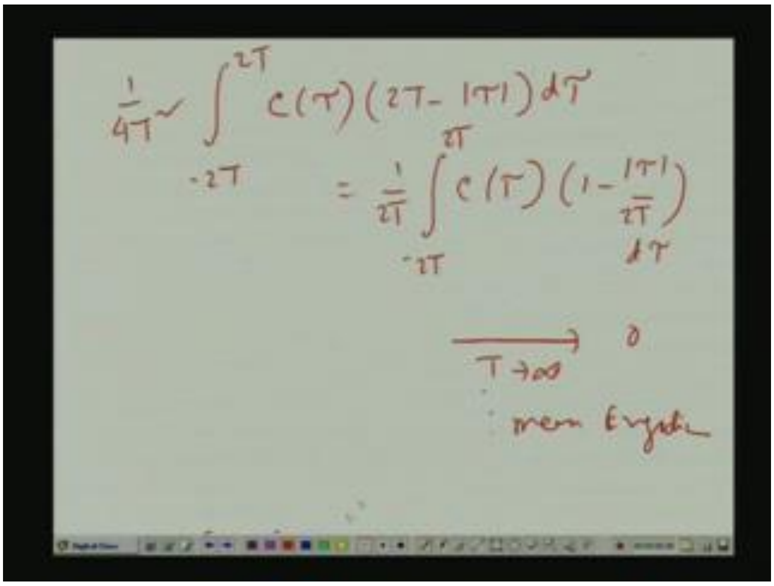

What is the integral, that integral is 1 by 4T square with respect to tau now. So, minus 2T to 2T c tau into 2T minus mod tau d tau.You can take 1 2T common here, because that is the style. So, you write 1 by 2T, so if this quantity, if this tends to 0 as T tends to infinity then means ergodic. So, this is the condition, but we can have some sufficient conditions you know sufficient condition for this. It will go to the next page we rewrite what we wrote that is…

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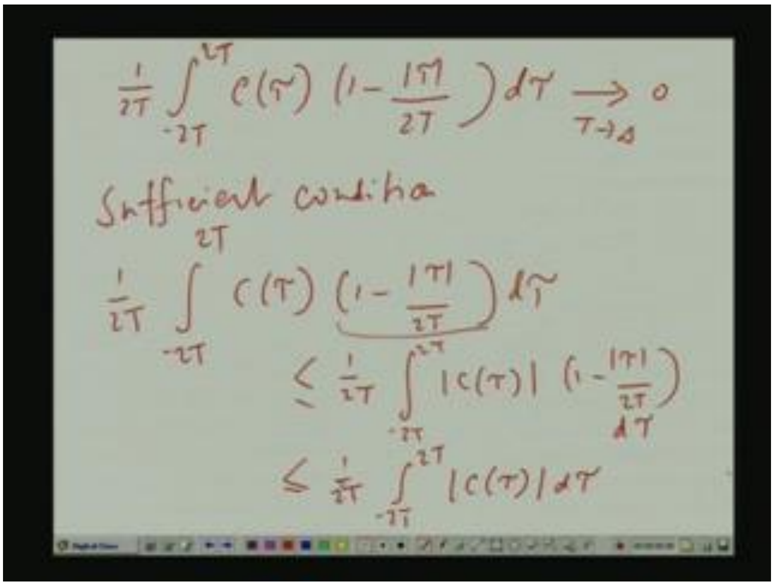

What is the condition was 1 by 2T; I am rewriting only nothing else for my reference I am rewriting here. C tau 1 minus mod tau divided by 2T d tau this should become this should tend to 0 as T takes to infinity. This should tend to 0 this is the condition for ergodicity mean for a wss process. Now, sufficient condition that is if this is satisfied, what is meant by sufficient condition stronger condition that if they are satisfied this will always be this also be all satisfied it is not that it is not necessary. What is sufficient condition some condition which if satisfied would automatically imply to this above condition here also satisfied, but that sufficient condition may not always be necessary should be stronger, but still we go for them, because it makes life easier.

So, here you see one thing look at this integral tau becomes positive and negative both, but mod tau remains positive and mod tau by 2T it is never exceeding 1. Because where tau's range is 2T and minus 2T minus 2T to 2T. So, mod tau by 2 Tau is a 1 factor whose value is always less than equal to 1. So, 1 minus mod of tau by 2T this is the factor this is the term which lies between 0 and 1, but positive always positive, but between 0 and 1.

So, it is a fraction that time c tau. So; obviously, this integral is what you can write that this above integral. C tau can be positive and negative both. So, therefore, if I instead of take I c tau I take mod c tau. So, always I get positive so; that means, the net integral and this factor is positive this is never negative.

For any tau this 1 minus mod tau by 2T it is always this is important it is always non negative lying between 0 to 1, but c tau can be negative; So, instead of c tau I put a mod c tau so; obviously, net integral will be higher, because then there is no negative value. This term, this phrase is positive or non negative the other terms C tau if I take only this mod value so; obviously, product is always positive. So, integral becomes larger than what we have here so; that means, this integral becomes less than and now, as I told you this factor is always 1 minus mod tau by 2T is less than 1. So, if I remove this integral becomes still higher, because earlier I was multiplying this mod c tau by some number which is less that 1 say.

So, if that less than 1 factor is removed that is replaced by 1; obviously, the total product value goes up. So; obviously, the integral becomes higher so; that means, this becomes less than less than equal to say if you want to be very correct. See the right hand side now, in such function that it tends to 0 as capital T tends to infinity then; obviously, left hand side also should tend to 0, because this is always less than equal to this. Remember 1 thing, that left hand side is less than equal to right this right hand side. So, if right hand side tends to 0 as tau tends to infinity what happens to the left hand side. Now, left hand side you remember we started with it was a variance function. So, it is lower bounded by 0 this is important it can be never be negative.

So, left hand side can never be negative if it is lower bounded by 0 on 1 hand, but is again upper bounded by a function which is becoming 0; that means, this left hand side has no other option, but to become 0 as T tends to infinity you understand this. This left hand side functions if started if you look at the early part of this today's lecture. This was not nothing but a variance function and variance function is what that was lower bounded by 0. And this is then upper bounded by this function, but this function if it tends to 0 as T tends to infinity; obviously, this left hand side has to become 0. There is another condition which is according to me mod useful another equivalent condition.

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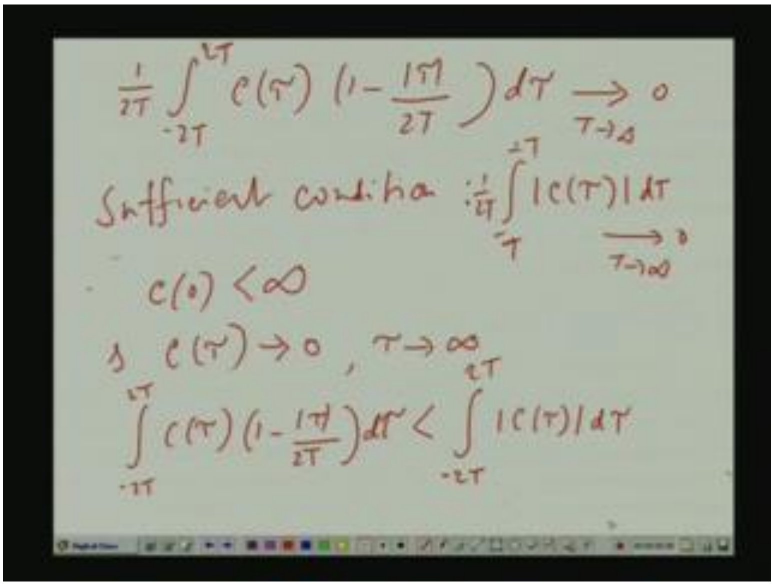

This condition was 1 by 2T, so tend to 0 as T tends to infinity. But I have got another condition if you forget this. If this is given to be that c 0; if this is always real. So, no problem and this is suppose given to be number 1 finite that is variance is finite there is nothing big deal. Variance is that instantaneous AC power you can say th at expected AC power that is you have the mean, around the mean there is an fluctuation that fluctuating part what power has it has this c 0; obviously, this is this I mean for all real processes the it cannot infinity. So, it is finite. Suppose this is true and other thing is c tau if it is given that it is greater than 0 it is tends to 0 as tau tends to infinity.

That is as tau becomes larger and larger covariance between the expected samples situated at a gap of tau becomes less and less that is sample situated far away from each other they become uncorrelated or close to uncorrelated. This is always true I this is true in most of the practical processes, because the samples which are nearby or adjacent they are only correlated if you go far away. If they are far away from each other then the degree of correlation between them is less and less. In that case also this is also then a sufficient condition. Now, how to prove that we have already seen this integral d tau this you have seen this is less than this we have seen just now. Now, you told me so firstly, capital T will become large and large then you told me that as tau becomes more and more c tau tends to 0 that is 1 of the condition.

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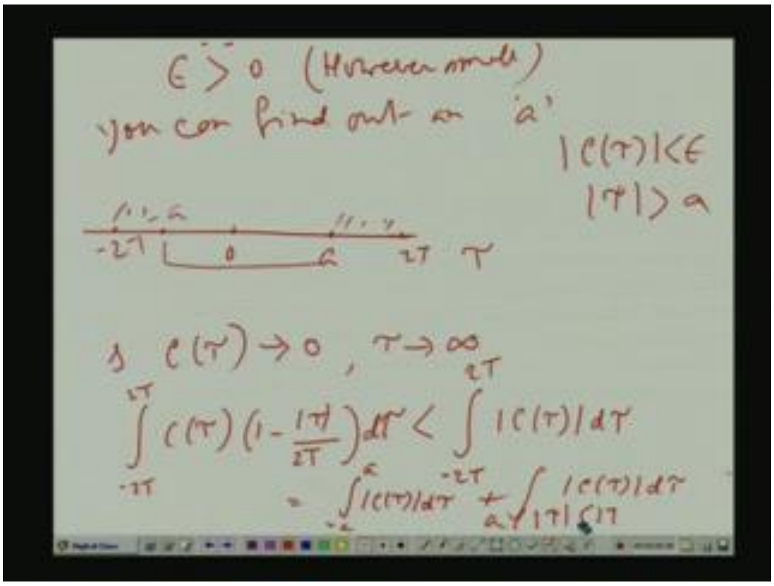

That means given any epsilon greater than 0; however, small that it can be very small, but positive you ca n find out an a. So, that may be this is the time axis 0 you have 2T here minus 2T here a is somewhere here. It could be to the right side, but then T is becoming higher and higher its tending to infinity. So, naturally I can always take 2T to be right of a to in more general and you can find out a. So, that to the right of a the correlation that is if you take tau to be this is the tau axis; if you take tau to be that is the lag tau to be greater than a correlation is almost 0 that is it is less than epsilon.

Give me any epsilon I can also find out some a that is some lag; however, large between which the correlation has come to be very small that is less epsilon, that is that was 1 of the conditions. So; that means, mathematically speaking you can find out a. So, that mod c tau is then epsilon for tau mod tau greater than a. That is, tau can be to right hand side also or this side also. If that be the case and this right hand side integral here I can write as summation of 2 integrals 1 from minus a to a, c tau d tau another that is once you integrate over this period and then integrate over the remaining period this part, and this part do the same stuff, but tau greater than alpha a less than 2T this is the range. If that be now, look at this integral. Now, know we have seen earlier that Cauchy inequality that c tau is always less than equal to c 0.

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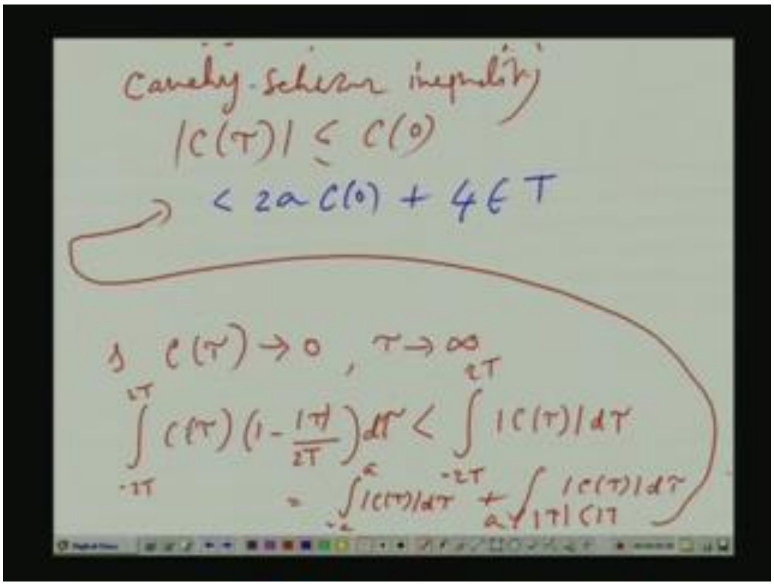

This mod is always less than equal to c 0; c 0 is positive. So, I am not putting any non negative. So, I have not putting any mod, so; that means, if this on the right hand side the first integral is always less than equal to twice a into c 0 and this other integral other integral, since c tau is less than epsilon. This other integral is equal to what how much area from a to 2T and minus a to 2T that much area. This it is very simple, so the first integral here, I am starting from this point just to see the continuity this is less than the first integral is less than equal to this should be less than equal to, but I will make it less than, because the other term for the other term it will be less than.

So, it is 2 integral twice a times c 0, because you know mod c tau is less than c 0 by the Cauchy Schwarz inequality. So, if you put that within this first integral on the other side and c 0 comes out of the integral this is true. And the second 1 within this zone I have said within this zone from a to 2T or minus a to minus 2T the correlation is covariance for some magnitude has already fallen before epsilon. So; obviously, this net integral will be what is less than 2T times if you really take up to 0 to 2T it is and on this side also 0 to 2T. If it is less than 4 epsilon T this integral though integral is taken not from 0 to 2T, but as a fraction of it just a to 2T or minus a to minus 2T. If it were taken from 0 to 2T; it still would have been less than on the right hand side twice epsilon into twice 2T, on the left hand side also epsilon twice 2T. So, four epsilon T, so if we now divide by 2T, because the integral you know if we now divide this by 2T what you get?

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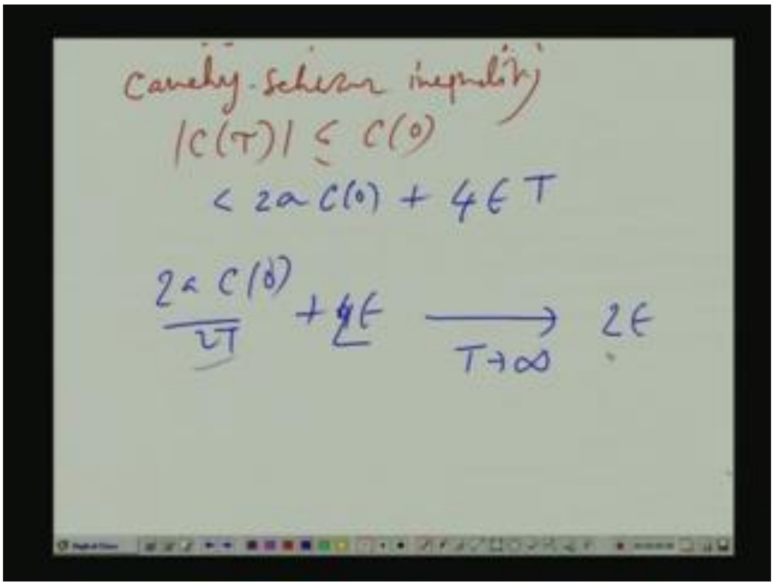

If you divide by 2T, so 2a c 0 by 2T plus 4 epsilon, so as T tends to infinity this part goes to 0 4 and you remain it tends to it is 2 epsilon. So, it goes to 2 epsilon as T tends to infinity and epsilon can be as small as possible. So; that means, that ergodicity mean that condition is satisfied.

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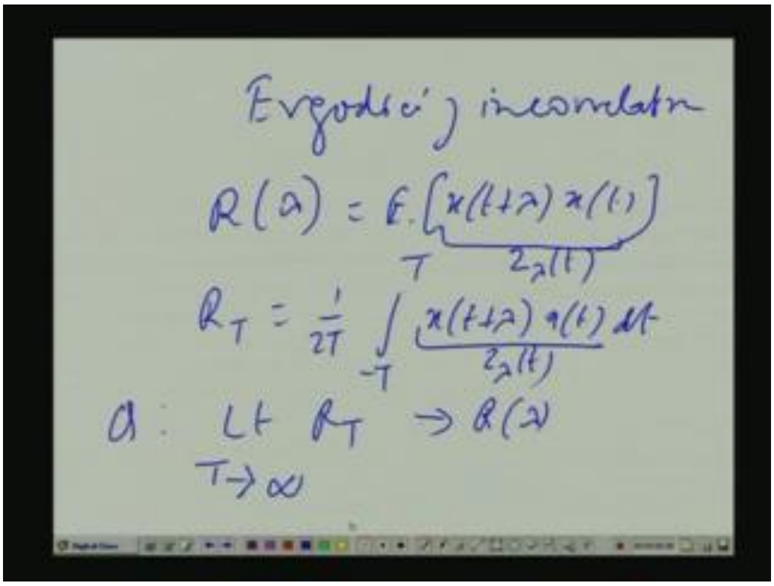

So, we normally prefer this condition and before I conclude this is a the other thing is ergodicity in correlation. Here suppose, you have got R lambda as E xt plus lambda xt that lambda is a lag variable here. So, if you know, if you have any observation xt or from you find out this product again next time you observe E from x find out this product the product between the samples at t plus lambda and t. And carry out this and average over and similar observations which are an ensemble average if your number of observation is higher and higher finally, this right hand side becomes a good estimate. I mean that ensemble average becomes a good estimate of the actual R lambda that is actual statistical average.

Question is if I define RT is equal to 1 by 2T,1 by I will take the 3 more minutes this quantity. Question is does RT question limit RT T tends to infinity is it equal to R lambda. If, so I say it is ergodic in correlation also. You remember 1 thing RT is independent of T, because the integral is a function of lambda only. So, RT here is a function of lambda. So, for this to work at all for this to be valid at all this R lambda if this expected value should not depend on T; that means, the process should be I mean stationary in correlation also, because this expected value of xt plus lambda into xt as written here it should depend only on lambda and not on T, because if we want RT to be equal to R lambda as T tends to infinity.

Now, RT is an integral were T disappears after integral. So, that; that means, R lambda also should not have any T which means this expected value should not have any T it should depend only on lambda which means it is stationary in correlation. So, that is the primary condition the process should be stationary in correlation in that case when should this be have satisfied. Now, you see xt plus lambda into xt you can call it a process said z lambda t and therefore, this is again z lambda t. So; obviously, this is equivalent to saying that the process z lambda t is ergodic in mean, because what is if you consider z lambda t what is R lambda R lambda is nothing but, so lambda is fixed here for a particular lambda you have got a process z lambda t

So, what is R lambda R lambda is the mean of z lambda t and what is this. This is z lambda t plus this is z lambda t time average from minus T to T. And this will become equal to R lambda provided the ergodicity in mean condition on z lambda t are satisfied. Now, we know what are the conditions on the what are the conditions for the ergodicity mean of a process. So, if z lambda lambda, t is a process which is ergodic in mean now; obviously, this will be satisfied.

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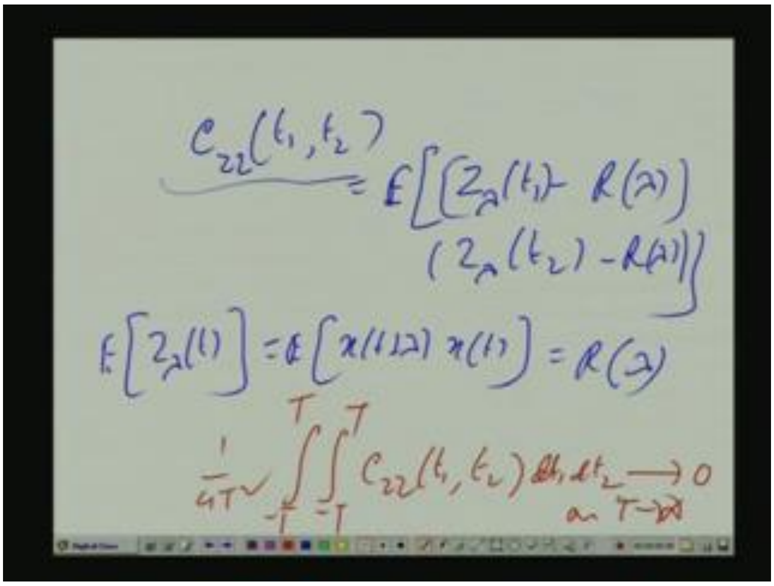

That means if you define Czz t1 comma t2 as expected value of z lambda t1 minus. What is expected vale of just 1 more minute z lambda t that is expected value of x t plus lambda in to xt and this is R lambda. Just 1 more minute this into z lambda t2 just 1 more minute R lambda. You define this czz t1 comma t2 we know what is z lambda t1 what is z lambda t 2 and all that. This expected value you carry out and this is the covariance then this should satisfy that condition. What is that condition that 1 by 4T square czz t1 t2 dt1 dt2 should tend to 0 as T tends to infinity. That is all for today. So, maybe we will just I will just touch about this a little bit in the next class and then we move to the next week.

Thank you very much.