Probability and Random Variables

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Lecture - 24
Correlation Matrices and their Properties (Cont.)

So, in the last class we were discussing these correlation matrices, we discussed some properties of hermitian matrices or hermitian matrices which are also I mean positive definite matrices. We found out that the high-end values of any hermitian matrices are such not even positive definite. They were real and they again vectors are orthogonal to each other. You found out then if in addition the matrix is given to be positive definite, then again values are also positive. If it is non-negative definite, they are non negative. This we have seen. Then we were discussing the linear dependents and independents random variables and all that.

So, we just start from there. I continue with the previous discussion on the correlation matrices first. Do you remember our previous treatment on what is called vector space of random variables, where each random variable was treated as a vector in an abstract vector space of all possible random variables in the world, and there we defined addition of two random variables as addition of two vectors. What is meant by zero vectors? That is random variable which will take zero values. Then, negative of a random variable means again a random variable which takes negative of the original random variable and likewise is just in that framework, we can discuss what is called linear independents and dependents.

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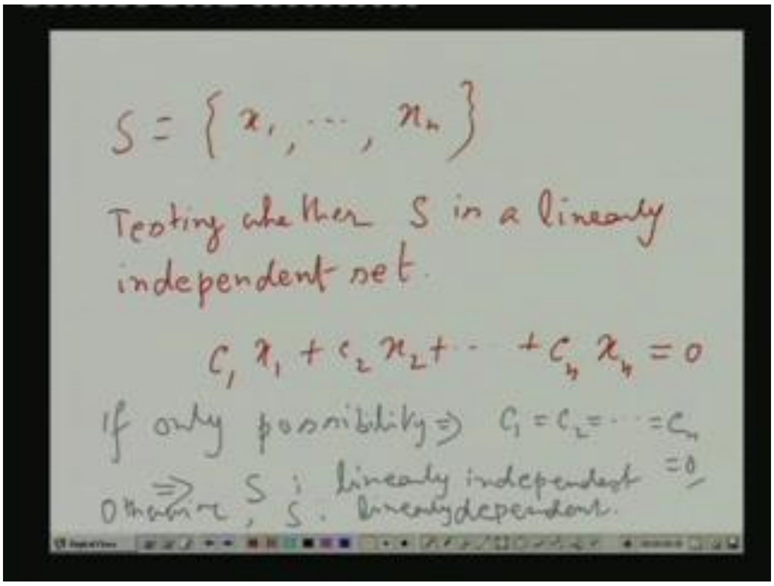

Suppose we have got A set s of random variables x 1 dot dot x n. If these random variables are such that none of them can be expressed as a linear combination of the raised or part of the raised, then I can say that this set has no redundancy. Each random variable is independent of the raised and I probably call it linearly independent of the raised. Therefore on the other hand, if it is the set is such that at least one vector is expressible in terms of the rest as a linear combination, then it is called a linear dependent set. So, that means a set has some redundancy because at least one value which can be expressed by the others as a linear combination is a redundant one, right. This is called linear dependent set. We all just look to linear independents.

Anyway, you can easily see that if I say that this set is linearly independent, then any subset of that set say s is also linearly independent. That is because if it is not, so that is if the subset is still linearly dependent. That means, within the subset there is one vector at least which can be expressed as a linear combination of the remaining ones which means a vector or a random variable of this set is expressible as in a linear combination of some other members of this set which makes it linearly dependent. So, I repeat again given the linearly independent set s is of n entries n random variables x 1 x 2 dot dot dot x n, any subset of that also is linearly independent.

However, this cannot be said about sets which are linearly dependent. For instance, the set s can be linearly dependent set for I can find out and subset which is linearly independent. For example, may be x 1 can be written as linear combination of the rest, but no other members from x 2 to xn can be written as a linear combination of the other ones. In such a case, if I take a subset of a, is by excluding x 1 from s, then that set, that subset you can call it s prime is obviously linearly independent. So, I repeat. Any subset of a linearly independent set is still a linear independent set, but any sub set of a linearly dependent set may be dependent, may be independent.

Why you are so bothered about linear independents and all? Before that, we just say how to test mathematically whether the given variables from a linearly dependent set or not. So, testing that is, we are testing whether s is linearly independent set or not. How to test a formal linear combination like this c 1 x 1 plus c 2 x 2 plus dot dot dot cn xn equated to 0. What is zero? Here it is not scalar at 0. It is not arithmetic 0, it is a vector 0. That means it is a zero random variable, because you see on the left hand side I am combining random variables. So, we are given x 1.

What is c 1 x 1? C 1 x 1 is also random variable which takes values randomly like x 1, but in each experiment on x 1, whatever value I have for x 1 that I have to multiply by a scalar as c 1. That will be the corresponding value for the random variables c 1 x 1, same for c 2 x 2 dot dot dot and then, if we add several random variables like this, we still get a random variable, right. If I equate it to something else on the right hand side that cannot be scalar, that can be real, that can be just a number. It is a random quantity again. So, it is though I write 0 here, actually mean a zero random variable which means a random variable which in each experiment takes a value 0 only and when we say if using probabilistic terms, that it is a random variable which always takes the value 0 with probability 1.

In this case, suppose you form a combination like this and then, will try to find out what are the possible solutions for c 1, c 2 up to c n, so that the equality is satisfied. Now, obviously, one solution is obvious you know c 1 equal to 0, c 2 equal to 0 dot dot dot c n equal to 0. Then, again this equality is satisfied obviously, because 0 x 1 scalar at 0 times random variable x 1 means a 0 random variable, because the product will always return zero value. In this same experiment, same here for c 2 x 2 dot dot dot same power cn xn. When similar zero random variables are added, you still get a zero random variable which means the equality is satisfied, but is there any other solution where at least one or more of the coefficient c 1 to cn are non-zero.

Suppose, it is not I first make a claim and then, I just verify it. Suppose, if only possibility is then s is linearly independent as my claim, otherwise s is linearly dependent. I consider this that if there is some other combinations of c 1 c 2 up to cn, where not all the coefficients are 0, then I will say that the set becomes linearly dependent, and if no such thing exists there, obviously I will show that the set cannot be linearly dependent. If I set cannot be linearly dependent, it has to be independent. So, there is no other possibility which means if you cannot express any of the elements as a linear combination of the rest. That means, effectively that the set is linearly independent. So, suppose just to verify this claim what the statement is…

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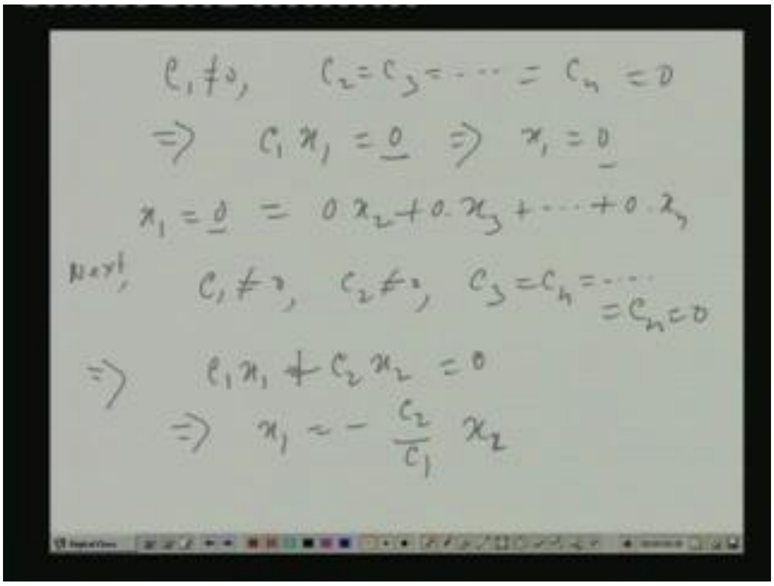

Suppose only c 1 equal to not equal to 0 and c 2 equal to c 3 equal to dot dot dot c n equal to 0. What does it mean? This means c 1 x 1 is 0 vector, and since c 1 is non-zero, x 1 has to be the 0 vector is a zero vector. That is zero random variable. So, that means, set s contains one variable which is a zero random variable which is x 1, but then that means I can always write x 1 which is zero random variable as nothing but zero times x 2 plus scalar 0 as zero times x 3 plus dot dot dot zero times xn. These zeroes are scalar zeroes, not random. They are not zero variable, not zero random variable, but even the less zero scalar as zero times variable x 2 produces zero random variable.

It is same to other terms and when they are added, when the zero random variables are added, you get zero random variable. Just to differentiate between scalar zero and random variable zero, I put underline between this, so that you understand which one is the random variable zero and which one is the scalar at zero. So, that means, if we have got a zero random variable, it is always expressible as a linear combination of any other set of random variable. So, if a set s contains zero random variable x 1 or any element being zero number variable, it is a linearly dependent set and not independent because that particular element can be written in terms of the rest. So, it is a redundancy.

So, I first started with only one coefficient non-zero, others zero. Now, you go for set two coefficients non-zero and others zero. Next suppose c 1 is not equal to 0, c 2 not equal to 0, but c 3 c 4 dot dot dot up to cn is equal to 0. This means, c 1 x 1 is c 2 x 2 which means x 1 is minus, sorry c 1 x 1 plus c 2 x 2 equal to 0. So, it means x 1 is nothing but minus c 2 by c 1 x 1, so x 1 x 2. So, it means x 1 can be written as a linear in terms of the other random variable x 2. It is also a linear combination, but having only one element, x 2 multiplied by some scalar number.

So, again the set becomes linearly dependent, because at least one member is expressible as a linear combination of another member. It is a linear combination, but one entry is there. No other entry to combined, but nevertheless redundancy is still there and now you can see that if more and more coefficients at non zero, you get linear relations involving more and more random variables. That means that you know there is redundancy that is one random variable can be expressed in terms of the rest and so and so forth. So, it has become dependent which means if the set is linearly dependent, then you can get non-zero solutions for c 1 to cn. At least one coefficient is non-zero; more than one coefficient is non-zero. In that case, the set is linearly dependent or conversely if the set is given to be dependent.

Then you must be able to get some non-zero solutions for c 1 to cn, at least for one or more than one of them which means if you cannot get it, if you cannot get non-zero solutions for c 1 to cn or at least for one of them or more, then one of them, that means the set cannot be dependent which means the set is linearly independent. The only solution is c 1 equal to 0, c 2 equal to 0 dot dot dot cn equal to 0. The moment that is evaluated a linear relation will emerge. So, these two terms actually mean the same thing. Dependent means existence of linear relations which means you get non-zero solutions for c 1 to cn or part of them. Absence of linear dependents means you cannot have such relation which means in that equation, all the values for c 1 to cn have to be zero. Any deviation from that would result in linear; some linear relation between the members of it is random variables which is a contradiction. Now, why you are so bothered about these dependents and independents? As I told you linear independents means absence of redundancy and dependents means presence of a redundancy. Whenever there is a redundancy, you should I mean not only we try to avoid, but we try to eliminate redundancies.

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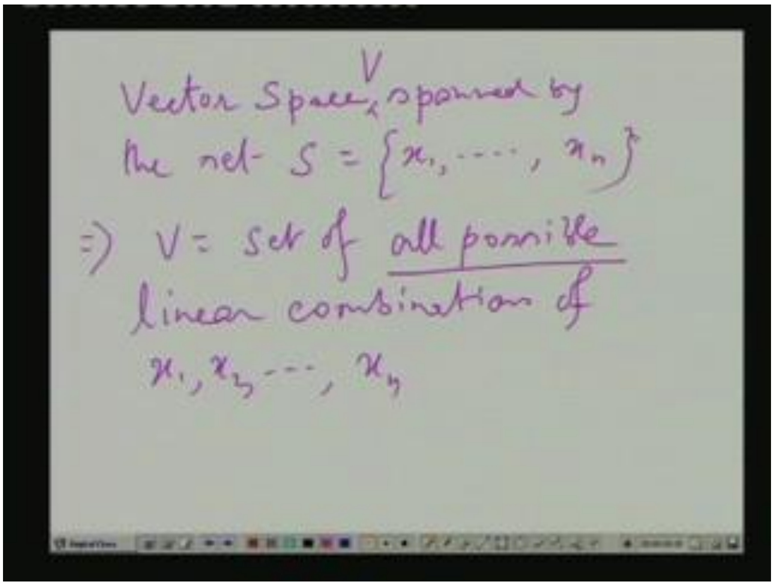

Suppose we consider what is called that is vector space spanned by the set s. That means I consider the set. So, equal vector space v spanned by the set s. What is v? That means, v is set of all possible linear combinations of, that is if you take the elements x 1, x 2 dot dot dot xn, any typical linear combination is c 1 x 1 plus c 2 x 2 plus dot dot dot cn xn. Now, you have infinite choice for c 1 infinite choice for c 2 dot dot dot infinite choice for cn. So, basically you can look for infinite ways of linearly combining them, but if you collect all such infinite varieties of linear combinations, what we said that is called a vector space v. They are actually if you take any vector, if you take any two vectors, one I mean each of them is linear combination of these entries.

So, if you add those two vectors, again you get another linear combination of these vectors. So, that vector is said to be closed under vector addition. There you take any two vectors of the vectors space v and then, you still remaining v, because v consists of all possible linear combinations of x 1 to xn. So, any combination an naught x 1 exchange an again another combination of x 1 to xn. If there added, you still get another possible combination of x 1 to xn. So, remaining v is called we say that v is closed under vector addition.

Similarly, v is closed under scalar multiplication that is if you take any particular vector of v which is some linear combination of x 1 to xn, so call it c 1 x 1 plus c 2 x 2 plus dot dot dot cn xn. Take that multiply by any scalar constant k, you get a term like k c 1 times x 1 plus k c 2 times x 2 plus dot dot dot plus k cn times xn. Now, this linear combination is this is again another linear combination you know I mean going to v. That means another linear combination and since, v consists of all possible linear combinations, we say that v will be a belonging to v only, and you say then v is closed under scalar multiplication.

Likewise you can have the zero vectors. Just multiply the elements x 1 to xn by scalar zeros at them, you will get the zero random variable which is a zero element or zero vector for the vector space v. So, v consists of the zero vectors also. Then, given any particular vector, you can find this negative what is a typical vector of v c 1 x 1 plus c 2 x 2 plus dot dot dot plus cn xn. What is the negative of that? Wherever you have c1, take it minus c 1, wherever you have c 2 take it minus c 2 and like that. So, see again another linear combination coefficient at just reversing the sign, but since v consists of all possible linear combinations, this one also is a member of v.

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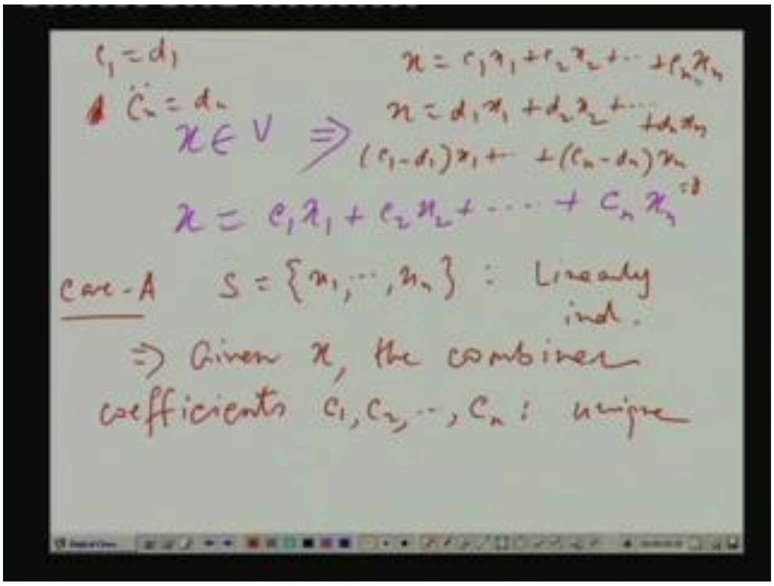

So, negative of each vector remains in v and likewise now that means any element x element of v can be written as c 1 case A set S which is nothing but x 1 to xn given to be linearly independent. Then, my claim is given a vector x, you express it first in terms of x 1 to xn as a linear combination, but this coefficient c 1 c 2 to cn, they are unique. That is given x, the combiner coefficients unique how do you say so. Suppose, it is not, so I should get x equal to first c 1 x 1 plus c 2 x 2 plus dot dot dot plus cn xn, and also suppose same x can be expressed as d 1 x 1 plus d 2 x 2 plus dot dot dot plus dn xn, where d 1 to dn, they are not all same as c 1 to cn. At least in one coefficient, there should be default and left hand side is still x.

Suppose, there is a situation or question it occur by or n side will be there. It cannot occur. So, the coefficient c 1 c 2 up to cn is unique now. You subtract left hand side from left hand side and for right hand side from right side, what you get is c 1 d minus d 1 x 1 plus dot dot dot plus cn minus dn xn is equal to 0. So, you got another linear combination with new coefficient c 1 minus d 1 c 2 minus d 2 up to cn minus dn, and not all coefficients are 0, because I mean at least in some coefficients, the c and d will differ. So, on the left hand side, you have got this and this is equal to 0, but if x 1 up xn, this set is linearly independent, this is the only way such equally to will be satisfied this by taking c 1 minus d 1 equal to 0 which is c 1 equal to d 1 c 2 minus d 2 equal to 0 which c 2 equal to d 2 cn minus dn equal to 0 which means cn equal to dn. This is because a set is given to be linearly independent.

So, we get c 1 equal to d 1 dot dot dot cn, sorry cn equal to dn. So, the coefficients are unique. You cannot have two sets of coefficients which can be used to linearly combine x 1 to xn producing the same vector x, so the uniqueness of the combiner coefficients. So, vector x can be just described in terms of the coefficient c 1 to cn provided x 1 to xn, they are fixed. So, each vector x has its own unique representation in terms of the coefficients c 1 c 2 up to cn.

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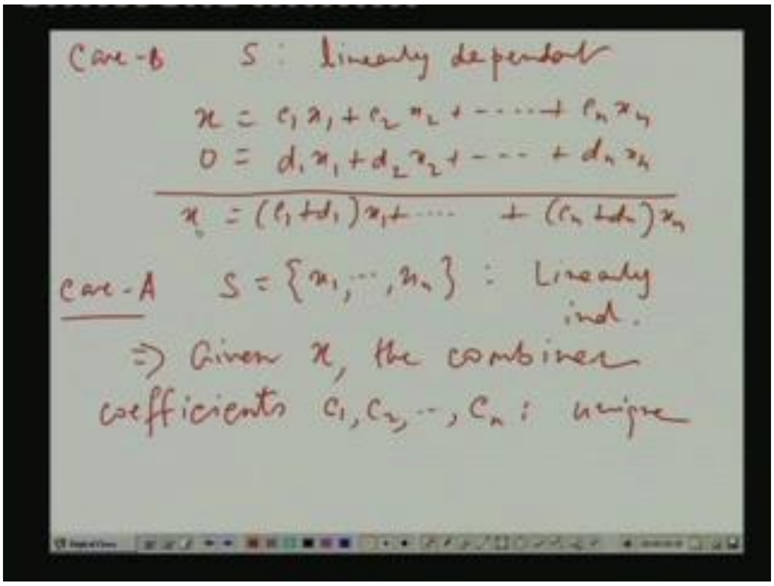

On the other hand, Case B, S linearly dependent. Now, we are already given this suppose c 1 x 1 plus c 2 x 2 plus dot dot dot cn xn, that is given any x you already have such combination cn xn, but now additionally you have given the fact that s is a linearly dependent set which means you can find if you have some coefficients d 1 d 2 like that dn xn, then equate to 0 for all the coefficients d 1 to dn that not 0 because it is a linearly independent set. So, some coefficients at least one or more than one will be non zero. So, I write that equation here as a second equation. Now, add left hand side with left hand side you get x, and here you get c 1 plus d 1 x 1 plus dot dot dot cn plus dn xn and since, at least one coefficient or more than one coefficient is non zero for this.

You get another linear combination expression for x for the coefficients are neither c 1 nor d 1, neither c 2 nor d 2, but something else which means there are many possible linear combination type expression for x. So, the coefficients are not unique that shows the redundancy. So, that is why you will always look for linearly independent set. Now, why this is important for us?

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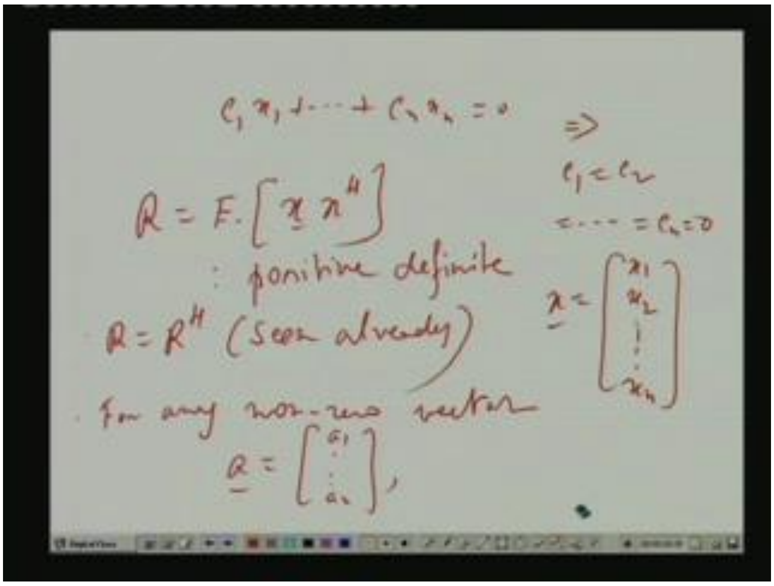

If the set is linearly independent, you remember we have got such things c 1 x 1 plus dot dot dot cn xn equal to 0 implies c 1 equal to c 2 equal to dot dot dot cn equal to 0. Under such case, the correlation matrix of the vector x that is if R is E x x hermitian for x is, this will be positive definite. Now, positive definite means hermitian matrix that is easily seen last time if you take the hermitian transpose of R, you have to take the hermitian transpose of this vector x x H.

So, first will be xH and this hermitian has a first entry which is x, and second entry will be x hermitian. So, you get back again x x hermitian expected value which is R. So, R equal to RH seen already and now, we say that R is positive definite. Positive definite means for any non-zero vector say a, when I say non-zero vector, I mean it is such a vector where at least one entry is non-zero. If not more, it is not a vector of all scalar zeros.

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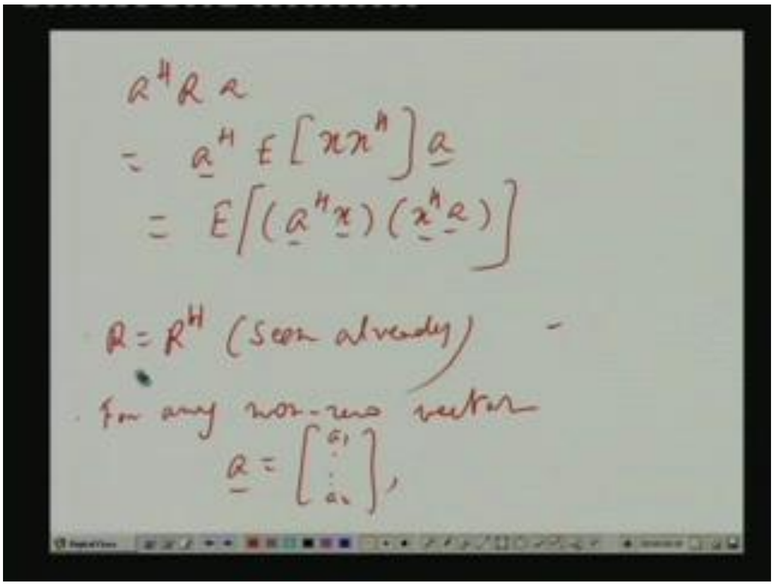

So, if you take like this and this complex numbers in general, we should have HRa and it should be real and positive number. How it is because what is if it is positive definite, then for any non-zero aH Ra is real positive scalar. If suppose a scalar number you can see Ra is vector column vector aH is a row vector row vector tends to column, it is scalar number. It will be a real number also, and it will be positive. If R is positive definite, then that is easily seen.

You can write aHEa which means you can put the a vectors inside the expectation of portion because after all a vectors are just non-random intermediastic vectors which means aHx I think we did it last time, but today I am doing it elaborately because this is very conceptual, and there is nothing wrong in repeating this which is nothing but E of say this quantity is nothing but aHx hermitian and remember aHx is a scalar number is a row vector is aHx is a column vector row times column is a scalar number. It is conjugate, it is hermitian transpose is nothing but just conjugate.

So, I have got a scalar number multiplied by its conjugate. So, it is nothing but expected value of the mod square which is nothing but if you call it a random variable a v, it is nothing but E expected value of mod v square which is nothing but the variants of the random variable v. It is not variants, but it is expected to the mod v square which is always greater than or equal to 0. If its variants have to be separate, so I am not calling it variants, but expected value of mod v square. Now, this always is greater than equal to 0 because after all this mod v square cannot be negative. This is square term mod v is real, then square. It is 0, it is greater than equal to 0, all right, but when it is 0. If v transpose to v a zero random variable that is it is in all experiments, it always takes zero values.

So, expected value also zero value, but question is can it be zero random variable, given that x 1 to xn are linearly independent and a 1 a 2 up to an, they are from a non zero set of coefficients. Answer is no because if it is linearly independent, the only way this linear combination a 1 x 1 plus a 2 x 2 plus dot dot dot an xn which is equal v. Only it can be made equal to 0 is by choosing a 1 equal to 0, a 2 equal to 0 up to an equal to 0, but we are choosing a non-zero form, non-zero values for a 1 up to an at least for some of them which means v cannot be a 0 vector, that is it cannot be a zero random variable which means E of 1 v square is greater than 0. Obviously, real part is easily seen expected value of mod square mod v square.

The mod square makes it real. So, aHRa that is a real number you have seen now. You see that? If the set of random variable to x 1 to xn are linearly independent, then it is a R is a positive definite matrix because a hermitian R a is greater than 0, where a is a non-zero vector. Some more properties of R 1 are the determinant.

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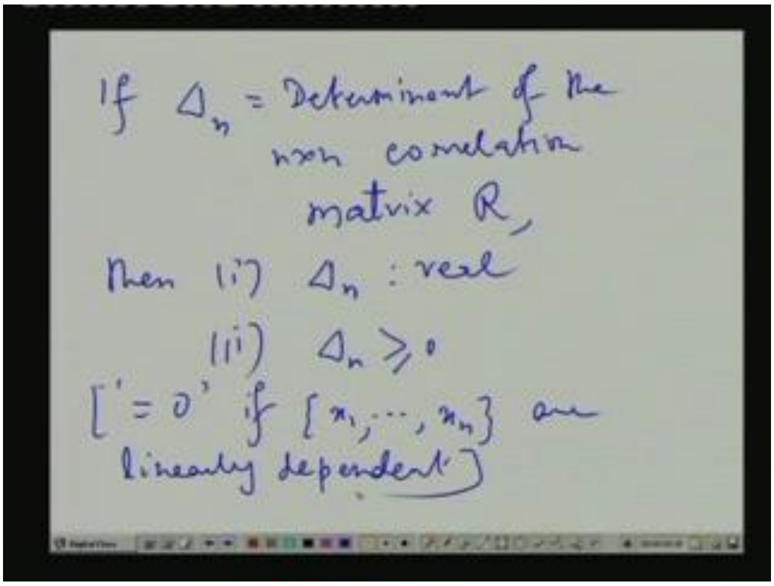

If delta n is a determinant of n cross n correlation matrix R, then number one delta n is real and delta n is greater than equal to 0 if x 1 to xn are linearly dependent. How to show that first?

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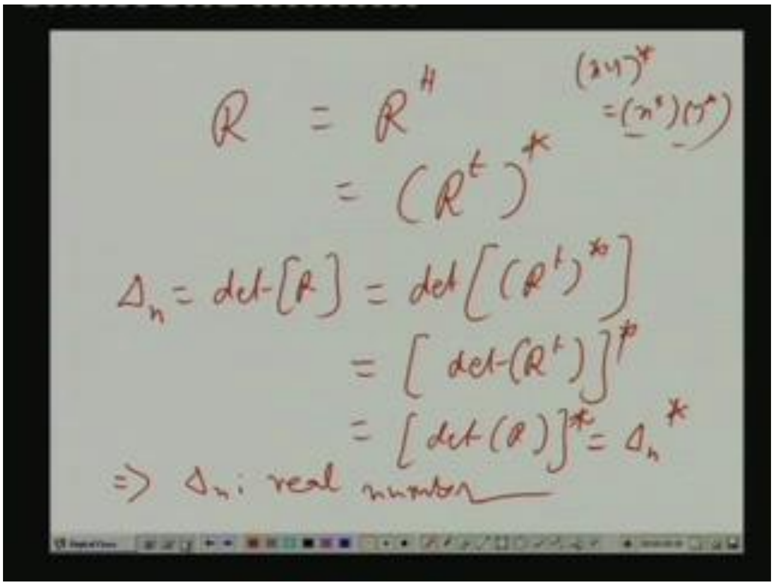

We know that R is same as RH because correlation matrix is hermitian. What is the hermitian transposition? That is R. You take the transpose and then conjugate. Okay which means delta n which is nothing but determinant of R is same as determinant of R transpose conjugate. Now, R transpose is a matrix just ordinary transposition and then, conjugate we can easily visualize. Then, if you take this matrix R transpose and conjugate all the elements and then, take the determinant, you will get the same thing. If you fasten the determinant of R transpose and then, conjugate because after all what is there in determinant, there will be many product terms which are added.

So, if you have first conjugation done on each of the terms, if the each of the terms are present, then compute the determinant that is if you first take you were considering the matrix R transpose, if you first conjugate each elements and then, take determinant. What we will be doing is all the conjugated elements, there will be product terms involving those terms. There will be product forms of those terms which are added. That will give you the determinant, but that should be same as if you first compute these and then conjugate because of the fundamental fact that the conjugate of the product is same as product of the conjugates xy multiplied, and then suppose xy multiplied and then, conjugated is same as x conjugated y conjugated. That is used here.

I think you can see that if you consider R transpose to be a matrix, then if you take this conjugate of all the elements, and then calculate determinant during calculation of determinant, you form products. So, they will be products of this form x star y star that kind of form, it forms products which are added. I am saying instead of first conjugating, you first take the determinant. So, you get the similar product terms will be there only, and then conjugate sign will be seen and then bringing conjugate on each other product terms and use this fact that conjugate our product is nothing but product of individual conjugates.

See if they are conjugated first and then, determinate is computed, you will get the same thing. If determinant is first computed on odd transpose and then conjugated, and here determinant of R transpose is same as determinant of R because transposition does not change which is nothing but determinant of R is delta n. So, delta n star. So, delta n and delta n star are same which means delta n is a real number. Next we show it is not only real, it is greater than equal to 0. In fact, this is greater than 0 if x 1 to x n form a linearly independent set.

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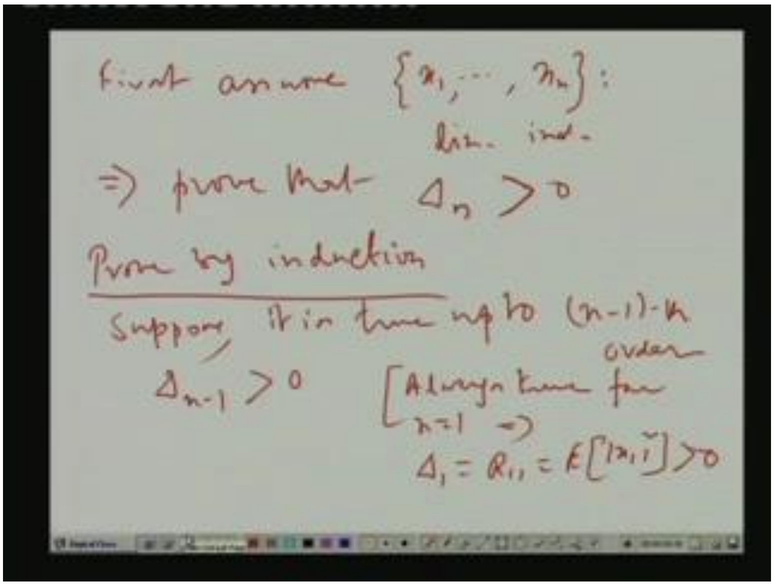

So, first assume linearly independent, then we prove that delta n is strictly greater than 0. Then, we go assume, and then we will go for the fact where for the case this say it is linearly dependent set, and there we will show delta n can be made equal to 0 can be 0. There is not basically strictly greater than 0. It can be 0 also. Sorry, it is delta n. Actually it is 0. When it is linearly dependent delta n is actually zero, so determinant is unique number given matrix. Anyway prove this by induction, we prove it by induction. Suppose it is true up to n minus 1th order, that is delta n minus 1 is not only real but it is greater than 0.

What is delta n minus 1? It is a correlation. It is a determinant of a correlation matrix involving how many terms? N minus 1 terms and suppose it is true that if you take n minus 1 terms which are linearly dependent, say you take this subset of this linearly independent set say you will take from x 2, then x up to xn form a vector. Find this correlation matrix, call it I mean take the determinant, you can call it delta n minus 1. Since subset of a linearly independent set is also linearly independent, we assume now for induction that delta n minus 1 in such case will be greater than 0 that is it is true up to n minus 1th order case. This is of course always true for n equal to 1 that is if you have only random variable because what is delta 1 is nothing but R 1 which is nothing but expected value of say mod x 1 square which is always greater than 0 x because x 1 cannot be zero random variable. Zero random variable presence of any zero random variable in a set makes a set linearly dependent. So, this is greater than 0. So, this is satisfied up to n equal to 1 that up to first order.

Now, suppose this is satisfied that is determinant is greater than 0. This fact is satisfied up to n minus 1th order, that is if you take n minus 1 number of random variables say x 2 x 3 up to x n, find correlation matrix for the corresponding vector and take the determinant, and call it delta n minus 1 and then, delta n minus 1 is greater than 0. Suppose it is true, then you have to show that delta n also is greater than 0.

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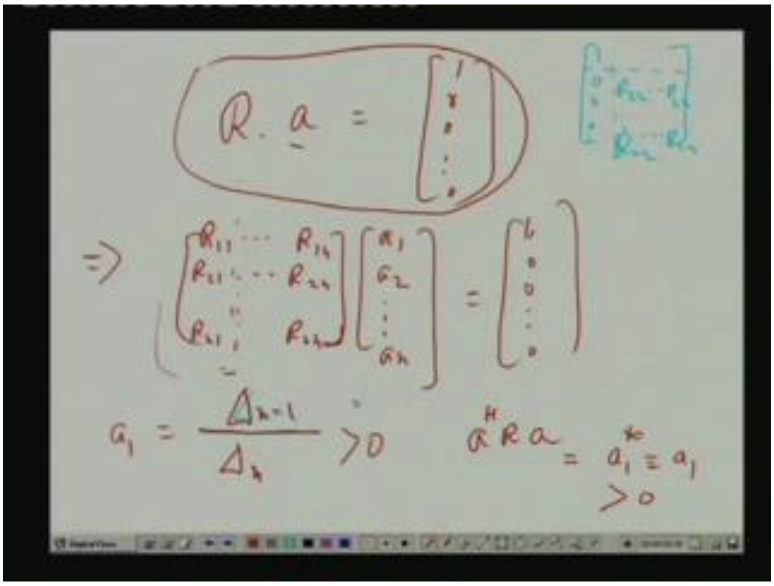

Now, suppose you look for a thing like this R a form an equation like this. That means, equal to say 1 0. We are following suppose equation like these set of equations. Now, we know Kramer's rule. For a Kramer's rule, what is a 1? A 1 will be what you first take out this first column, replace it by 1 0 0 0. This vector takes the determinant of that is by Kramer's rule it will be, sorry ratio of two determinants. What are they that first find out? First form a matrix, where the first column goes that is one first column is replaced by the right hand side vector 1 0 0 and rest of the items are same. So, that means what you have here are 2 2 dot dot R 2 n and then, R Rn 2 dot dot dot Rnn.

What is this matrix? You can easily see this is a correlation matrix of the random variables x 2 x 3 up to xn that is n minus 1 random variable. What is the determinant of that matrix? Determinant of that matrix is just 1 times determinant of the sub matrix because other entries are 0. They could contribute nothing to the overall determinant. So, determinant of such a matrix is nothing but just determinant of the sub matrix and this sub matrix is nothing but the correlation matrix of order n minus 1. So, its determinant is called delta n minus 1. So, it will be say delta n minus 1, sorry and divided by delta n. This is the solution, right.

Now, first you were dividing. So, you do not know whether delta n is 0 or greater than 0, but suppose it is greater than 0, then only it will be divided like this. Now, we make use of the fact that the set x 1 to xn is a linearly independent set which means the correlation matrix R, it is positive definite. So, on this equation, what I do? I multiply left hand side on both sides by a hermitian. So, i here a hermitian R a. What do I get? Just think about it. I give you a minute. Think about it. A hermitian R a. What do you get a hermitian is nothing but a vector transposed and conjugated.

If you now multiply right hand side by a hermitian, what do you get is I mean you get the first entry conjugated a 1 star, but a 1 we have already seen is a real number because determinant of correlation matrix is real. You have already seen. So, a 1 star is equivalent to a 1. So, n is a real number, but we know that R is positive definite matrix. That means, this is greater than 0 and if it is greater than 0, we know delta n minus 1 by your assumption is greater than 0. So, the only way we are saying a 1 is greater than 0 right from here, and delta n minus 1 by your assumption is greater than 0 which means delta n has to be greater than 0.

By the way I forgot one thing. When I said a hermitian R is greater than 0, I must ensure that a is a non-zero matrix, non zero vector, but obviously if I am solving very equation like this, a cannot be a zero vector, because the moment a is a zero vector R times a zero vector has to be and all zero vector, it cannot be 1 and then 0. See if the solution like this, the solution cannot be a zero vector, but it has to be a non-zero vector. So, this is implicit. The solution of the matrix that I am looking for, the equation that I am looking for is a non-zero solution is a non zero vector, and for such a non-zero vector, this is greater than 0, but this is equal to a 1 and we have already seen that is nothing but delta n minus 1 by delta n. Therefore, if a 1 is greater than 0, then delta n has to be greater than 0, because delta n minus 1 has been greater than 0.

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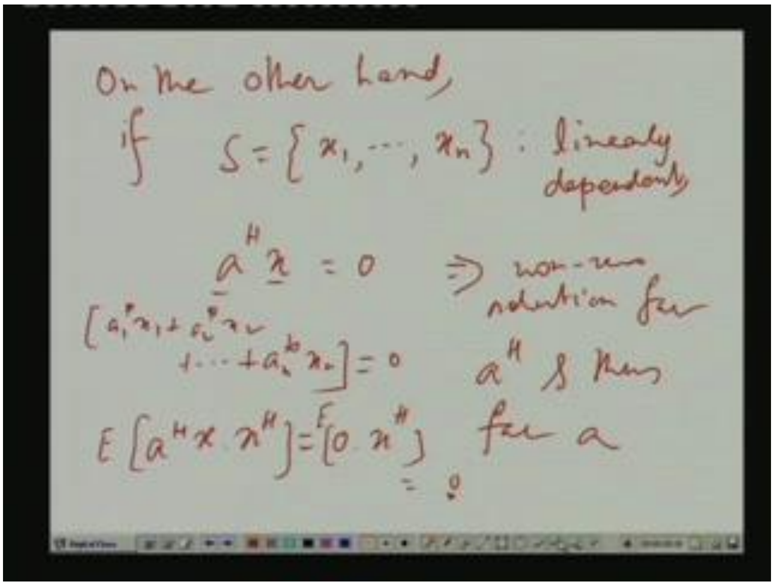

On the other hand, if this set is given to be linearly dependent, then we can say that we can find out something like this, some a hermitian x equal to 0. If you solve, we can get non-zero solution for a hermitian. After all what is a hermitian x? It is nothing but what is a hermitian x is nothing but a 1 star x 1. The first entry, then a 2 star x 2 plus dot dot dot an star xn, and if it is equal to 0, the elements x 1, x 2 up to xn are linearly dependent. I must be able to get some non-zero. The solutions for a 1 star a 2 star up to an star, at least one of them are more than one of them. So, that is why I have got a non-zero solution for this vector a hermitian. Therefore, it is same for a, because if any coefficient is non-zero say a 1 star is non-zero, then a 1 is also non-zero.

So, I can now summarize that since x is given to be I mean x consists of elements which are linearly dependent, I can find non-zero solutions for a for aH and therefore, for a, so that this equation is satisfied. Now, I multiply this by xH which means the zero random variable is multiplied by xH and then, I take expected value same here, but since it is 0, we will get a zero vector, right. I will get a zero vector.

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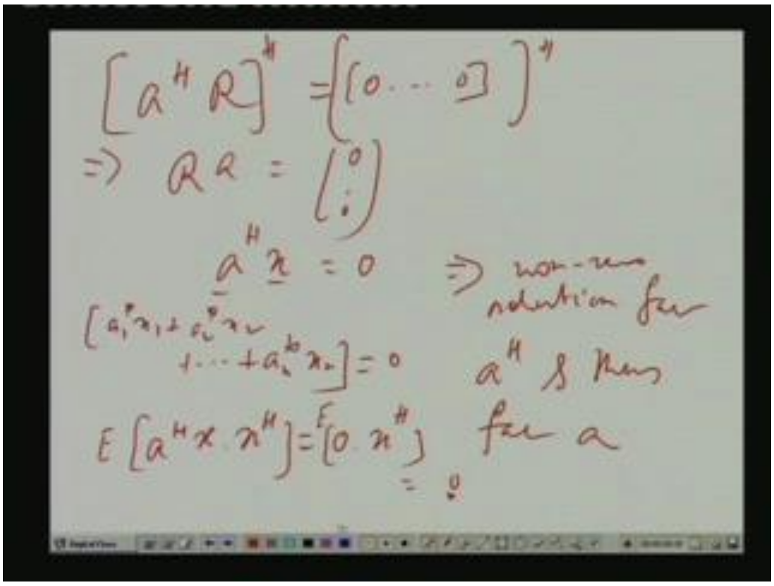

What I get here? It is a hermitian R. A hermitian comes out E of E on xxH. We will give you R correlation matrix and that is equal to zero row vectors. Now, if you take the hermitian transposition of both sides, this actually this vector, right. If you take the hermitian transposition of both sides that is if you now apply hermitian on this side, this side, what you get is R hermitian which is R and a hermitian notice a R a equal to a vector consisting of zeros. So, that means a equation, where R a equal to 0 with equations is of this form R matrix R times of vector a equal to zero vector, and we are guaranteed to have non-zero solutions for a.

Then, it means determinant of R cannot be non-zero because the determinant is non-zero solution is only solution that is possible is all zero solution that is consist a is nothing but a zero vector, but you have been told earlier that can be non-zero solutions for a because of the fact that the set x 1 to xn consists of linearly dependent variables. That means determinant of R is actually 0. This can be argued differently also.

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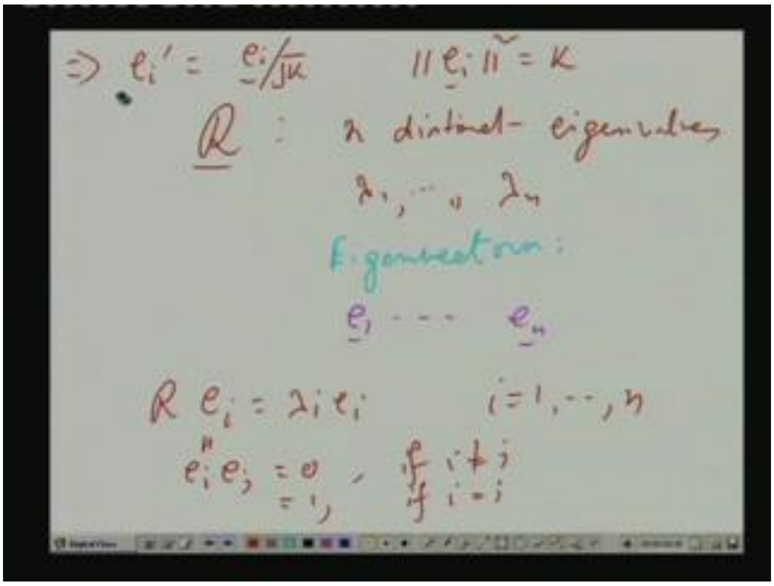

Suppose, we come back the Eigen vector Eigen value part. Suppose R has got n distinct. You can ask me I am taking n distinct Eigen values, what happens if they are not distinct? That part I will not prove. Eigen values lambda 1 dot dot lambda n. Now, lambda 1 to lambda n as we know, they are all real and if R is positive definite, then Eigen value is greater than 0 in magnitude, all right. Now, as a corresponding Eigen vectors, I am denoting by an underscore here. So, I have the set R ei is lambda i ei i equal to 1 to n. We have already seen the Eigen vectors, they consist of complex entries, but they are orthogonal to each other, that is ei hermitian ej is 0 if i and g are not same.

You have seen if i not equal to j and equal to can be made 1, if i equal to j actually ei, sorry I am very sorry. Actually if i and j are same, then it amounts to just mod of ei square. So, that will be a real number may not be 1, but if the real number is k, you take the positive square root of k and divide each entry of ei by square root k and call it ei prime. Then, obviously ei prime will have non-equal to 1, that is if you have got this equal to it is not 1, but equal to say k, then you take the positive square root of k, define a vector ei prime as ei vector by k square root k, that is each entry of ei vector is divided by square root k. Then, you can easily see the non-square of ei prime is equal to 1 because there will be division of k by k. So, this is called normalization procedure.

See you can have Eigen values normalized, they are orthogonal. When two Eigen values are, when the corresponding Eigen values are different, but when they are same, when i and j are same, the ei H ei is turned out to be 1. It can be made to be 1. This is a thing, then what I do?

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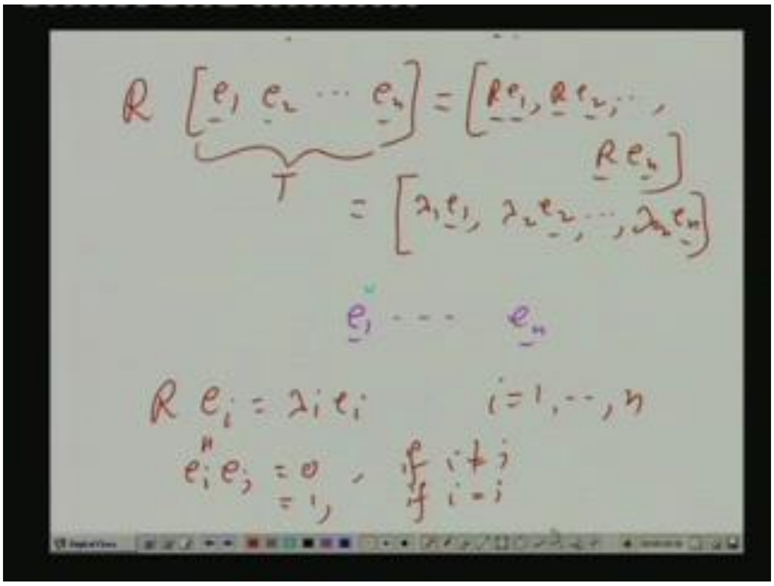

I write R and then, a form matrix e 1 e 2 dot dot dot en and what do I get is R e 1. So, first column of this matrix you can call this matrix to be T. So, what is the first column of the matrix? The first Eigen vector, what is the second column? Other matrix second Eigen vectors like that. So, what is the product? These matrix R times the matrix T. First column will be R e 1, then R e 2, then R en and since e 1 to en, they are Eigen vectors. You will have lambda 1 times e e 1 lambda two times e 2 dot dot dot lambda lambda n times en. Since time is short, I will make it a bit fast. This you can see you can easily write like this.

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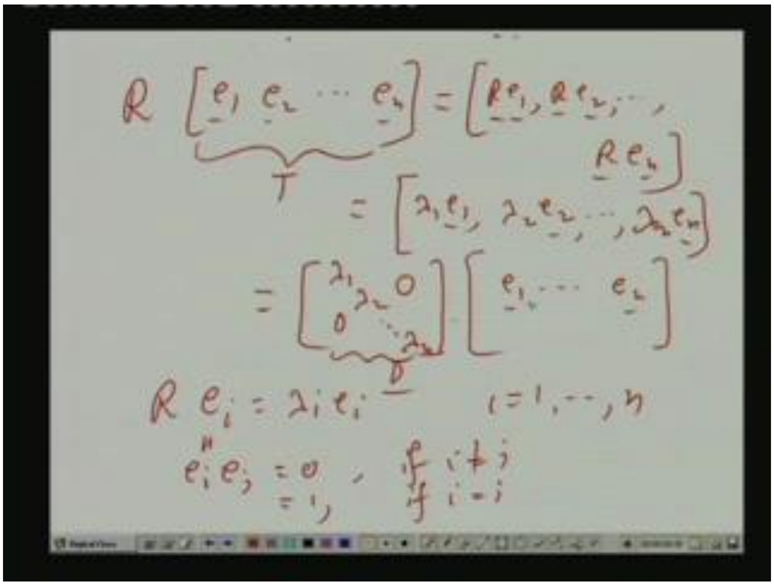

You can verify lambda 1. You can have a diagonal matrix and remember, it is a diagonal matrix with real valued diagonal entries, and diagonal entries can be positive R is given to a positive definite. You can call it D matrix times by Eigen vectors e 1 to en. You can easily verify.

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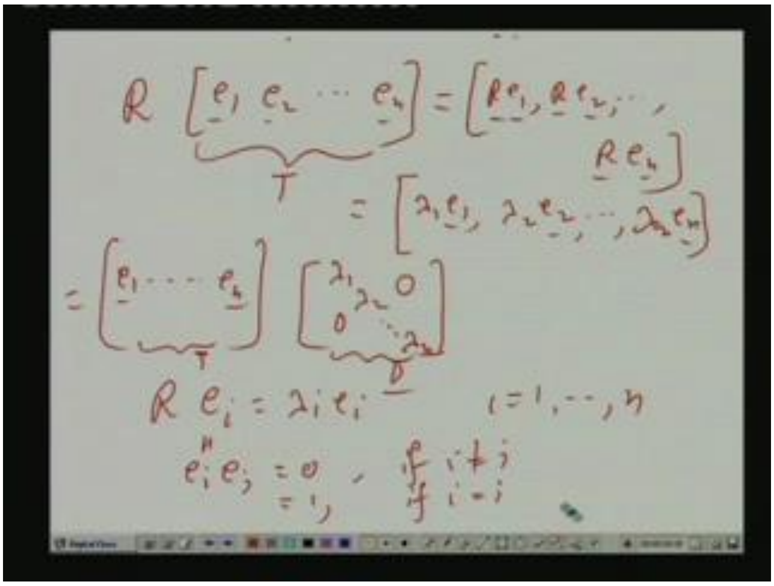

See the first row of this lambda 1 and all zeros you multiply, sorry it is the other way that is e 1 which is nothing but T matrix. You can easily verify, because I mean this is task for you because today time is short. So, I cannot get into these elementary things.

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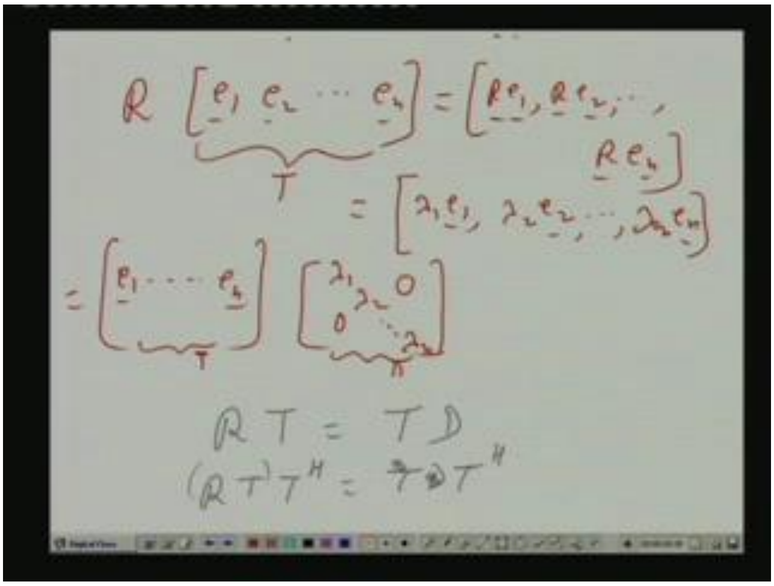

RT is TD. So, if I now do this RTT hermitian, then I get TDT hermitian, right. I am post multiplying R T by T hermitian. So, here also the right hand side, they bring in T hermitian. Now, what is TT hermitian? That is very interesting to see and that is what we will conclude with, but before that what T hermitian T is...

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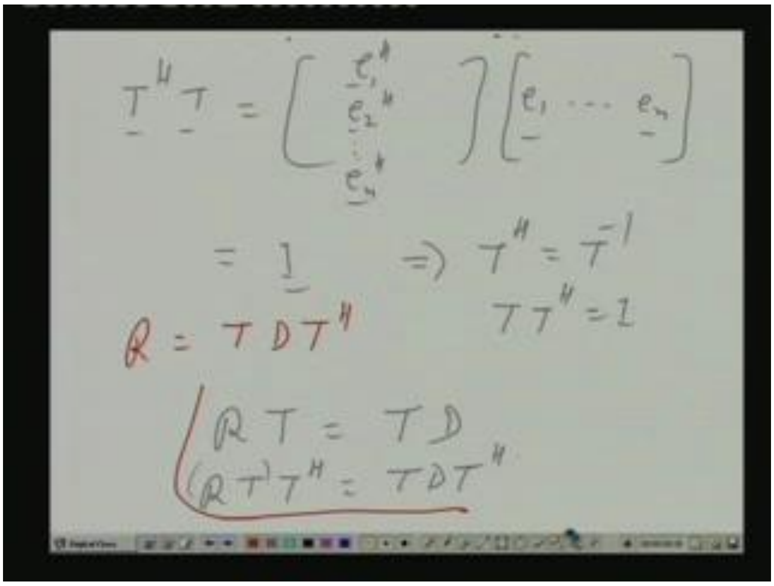

T consist of first column of T was Eigen vector e 1 1 hermitian transposition that I first column will become first row. See you will have e 1 H with conjugation column vector becoming row vector. So, there is transposition and each element conjugated. So, together hermitian transpose transposition to e 1 H, then second column becomes second row. So, e 2 was a column vector. Now, it becomes e 2 H row vector and elements conjugated and then, en H.

If you multiply this with T, so e 1 en you can see first row times first column that will give a mod even square which is equal to 1, but e 1 H times e on H times e 2 will be 0 e 1 H times. E 3 will be 0 e 1 H times en will be 0. Similarly, e 2 H times e 1 is 0. Only e 2 H times second column e 2 will be 1 and likewise. So, this will become identity matrix. So, TH means TH is nothing but T inverse which means TTH is also i. So, TTH is i here which means R is nothing but T D TH. So, I am just concluding immediately.

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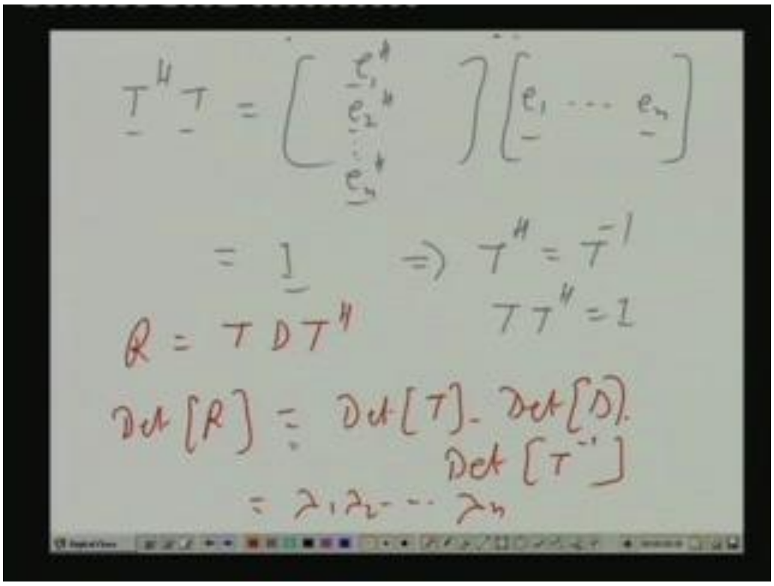

So, what is determinant of R is nothing but determinant of T, determinant of TH that is determinant of T times determinant of D times determinant of TH which is nothing but T inverse. As these two cancels, determinant of T determinant of T inverse cancels, so you get back determinant of D. What is determinant of D? It is the product of Eigen values lambda 1 lambda 2 dot dot dot lambda n. As you have seen if the Eigen values are firstly real, so determinant of R is always real. That is done and if they are all positive, if the Eigen matrix is positive definite, then each is positive. So, determinant is positive. So, that is all for today. We continue from here in next class.

Thank you very much.