Probability and Random Variables
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Lecture - 18
Joint Characteristic Functions

So, in the previous class, we ended with just a brief description of what is called joint characteristic functions. So, today, we will start from there. Maybe there will be a little repetition, but that will only be helpful.

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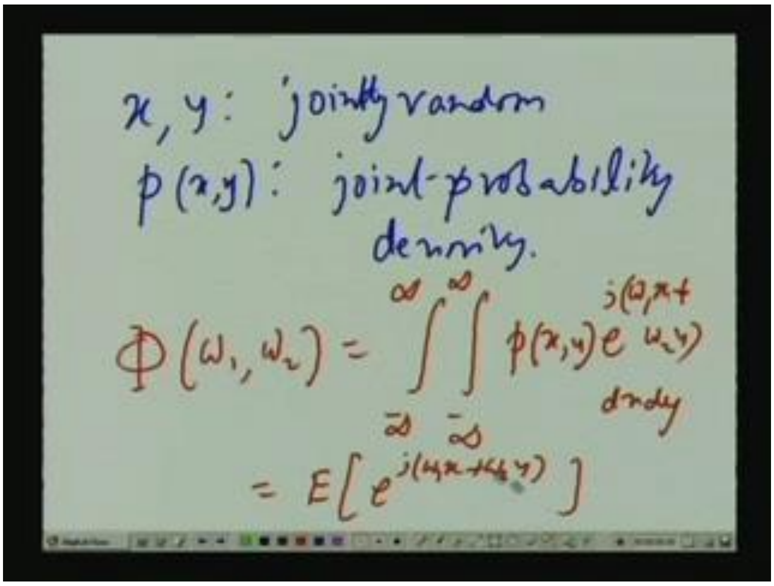

So, here we are given two random variables say x and y – jointly random. Then, we have already seen what is a characteristic functions in the case of a single random variable. So, here we will be simply extending that to the case of two variables. So, here the joint characteristic functions earlier was a function of only one frequency – omega. Now, since there are two random variables: x and y involved, there will be two frequency variables: omega 1 and omega 2; and it will be defined like this. Omega 1 x plus omega 2 y dx dy. You can also see that, this is nothing but the expected value of this exponential. After all, this is a function of x and y. If I want to find out its expected value, I will simply multiply by the joint probability density; integrate from minus infinity to infinity – both with x and y. So, essentially, joint characteristic functions phi omega 1, omega 2 is nothing but the expected value of e to the power j omega 1 x plus omega 2 y. Then, you can also see that I can write this…

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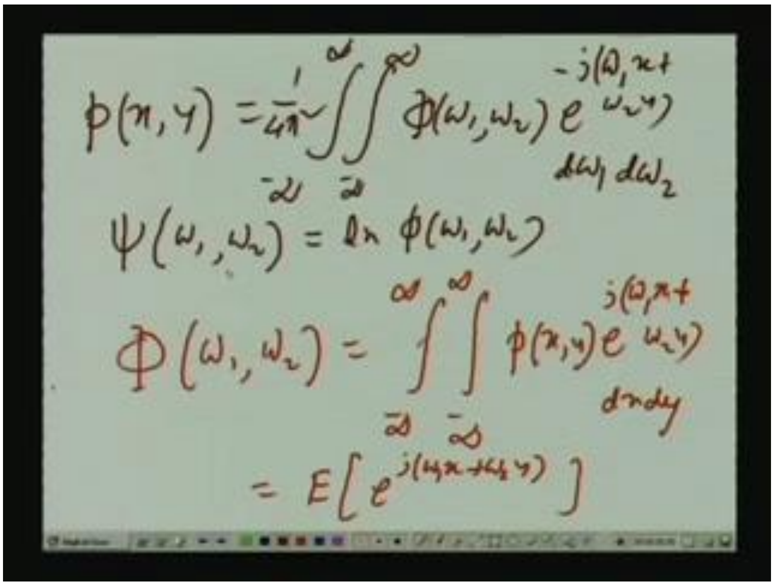

If this is given I can write this as this. This is multiplied left-hand side by 1 by 4 pi square; this also 1 by 4 pi square. Then, you can easily see that, the right-hand side is nothing but inverse Fourier transform of this function p x comma y – inverse Fourier transform of this function of two variables. So, we have got a plus sign here – e to the power plus j. So, it is not minus, because inverse Fourier transform; it is e to the power plus j omega 1 x plus omega 2 y. So, there are two frequency variables. Integral as usual is from minus infinity to infinity. In the case of inverse Fourier transform involving only one variable, we have 1 by 2 pi. But, since there are two variables, it becomes 1 by 4 pi square. If that be the case, then we know that p x comma y also can be viewed as the direct Fourier transform of this quantity on the left – 1 by 4 pi square of… 1 by 4 pi square times phi omega 1 omega 2. So, that means this is a inverse formula, that is, given the characteristic function – joint characteristic function… Now, the minus sign will come – e to the power minus j omega 1 x plus omega 2 y; but the integral will be with respect to omega 1 and omega 2.

Certain things we can see now. Also, one more definition – let me note this 1 by 4 pi square now. This was just for explanation purpose. Along with phi omega 1 omega 2, there is another definition; which also comes out to be useful sometimes. Actually, often phi omega 1 and omega 2 is seem to be – in practical cases, seem to be an exponential function. So, instead of dealing with phi as such, it is sometimes better to take logarithm of this, because if it is exponential later on taking logarithm, we get simpler functions. So, that is called… If I use logarithm of this and that is called second joint characteristic function; second joint characteristic function – psi omega 1 omega 2; which is nothing but ln phi omega 1 omega 2. Remember phi omega 1 omega 2 as such is a complex function; it is not a real function, because it is inverse Fourier transform. Even though p x comma y is real, integral will not be real in general. So, this is complex. So, it is a logarithm of a complex number. So, you have to write it in the polar form and take the logarithm. And you get this psi omega 1 comma omega 2; which is called the second joint characteristic function.

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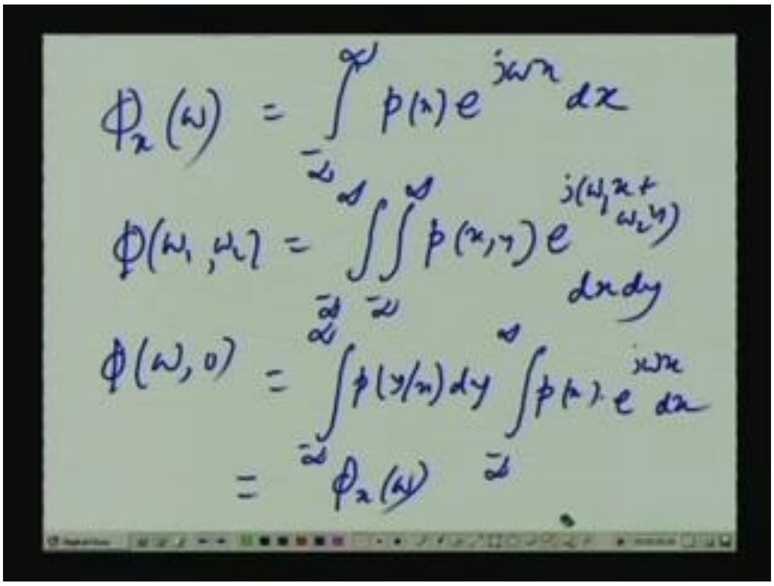

Now, what is meant by marginal characteristic function of x or y? That means just when x is left alone and I am not seeing it altogether with y; then the characteristic function of x, that is, phi x I will say omega. This is nothing but we know… Now, can I obtain this phi x omega given the joint characteristic function phi of omega 1 omega 2? Answer is yes. Let us see what was phi omega 1 omega 2. That was double integral… I am rewriting here. So, what is phi say omega comma 0? That is, omega 2 is 0; omega 1 is omega. So, that means one part is 0. And we can write this integral like this. p x comma y can be then written as p y by x dy and minus infinity to infinity p x; that is, p x comma y is broken as a product of p y by x; that is, p of y given x times p of x. This integral – outer integral is with respect to y, and then p x – we simply have e to the power j omega x and dx.

Now, this is outer integral is 1, because given for any particular value of x, total probability of y is taking values from minus infinity to infinity is 1. And the inner integral is nothing but the marginal characteristic function phi x omega. So, phi x… This is given by phi x omega. So, phi x omega is nothing but phi omega comma 0. So, if we are giving the joint characteristic functions, put omega 2 equal to 0; take omega 1 equal to omega. Whatever you get, there is a function of omega alone and that is the marginal characteristic function of x. By the same token…

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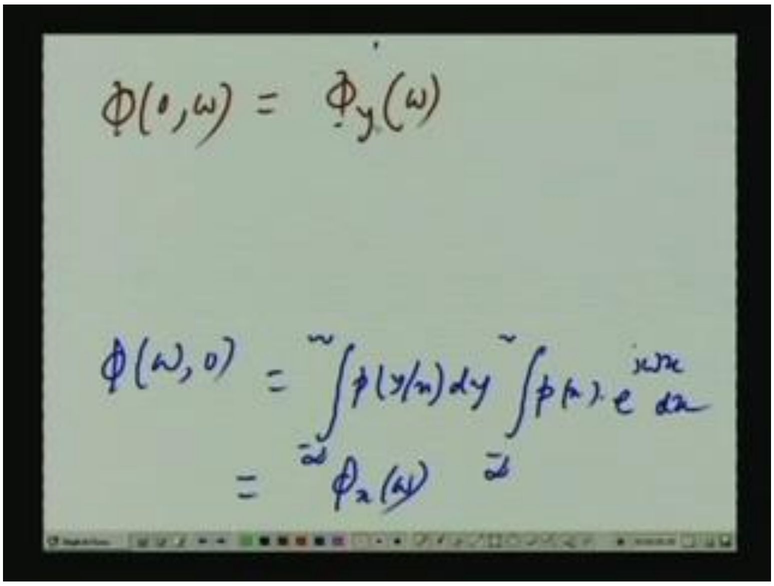

By the same token, I will not do it; but you can easily verify phi 0 omega; that is, omega 1 is put to 0, is equated to 0; omega 2 is taken as omega. That will give rise to the marginal characteristic function of y; that is, given y alone, what is this characteristic function?

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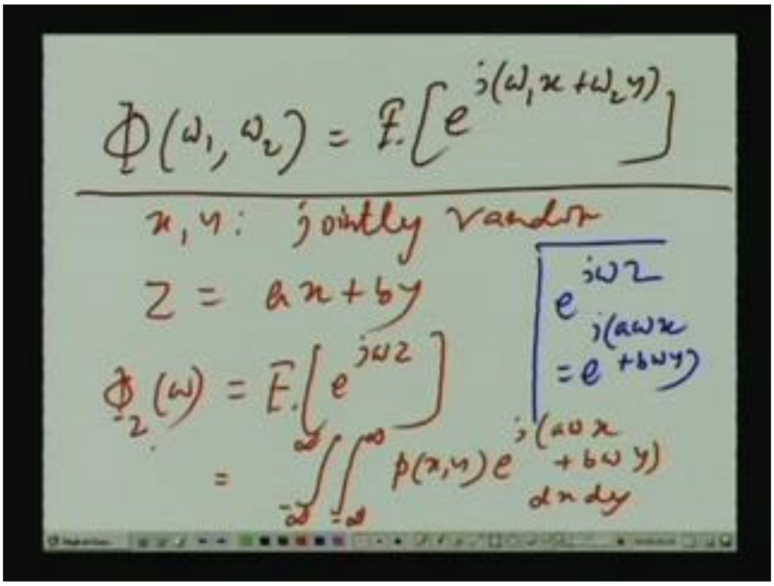

Next, we have seen one thing that, it is nothing but expected value of… This we have seen. Now, suppose we are given that, x, y: jointly random and Z is a linear function of x and y; it is a function of x and y; where, you are considering again; but it is a linear function. Something like say ax plus by. Obviously, z is random. So, it has its own characteristic function; maybe we can call it phi z omega. And what is phi z omega? It is expected value of e to the power j omega z; take this again. Here we are dealing with just a random variable – single random variable Z. x and y – they are jointly random; but z is evaluated in terms of them. And z is a single random variable. So, it has its own characteristic function; which is a function of only one variable – phi z omega. And what is it? By definition, it is expected value of e to the power j omega z. But, you know that, here we can write e to the power j omega z is nothing but e to the power j a omega x plus b omega y. So, this is a function of x and y.

And, we had seen earlier that, expected value of a function of x comma y is what? We simply have to take that function multiplied by the joint density p x comma y and integrate. So, that means this is equal to what? So, here you see we are not finding out the joint characteristic function of x, y; we are simply given that, they are jointly random; we concentrated in straight on another variable z, which is a linear function of x and y, that is, x plus by. And we are trying to find out the characteristic function of this z, not joint characteristic function of x and y, but simply the characteristic function of z. But, while doing so, we find that, it is nothing but expected value of e to the power j omega z. And e to the power to j omega z actually is a function of x and y like this; which means this characteristic function can be obtained by simply multiplying this by taking the expected value of this function; that is, multiplying that function by p x comma y and integrating; that is, p x comma y… Now, if you take a omega and call it omega 1; similarly, if you take b omega and call it omega 2; then you see this is nothing but… By definition, this is nothing but the joint characteristic function of x and y at frequency omega 1 given by a omega and frequency omega 2 given by b omega. So, phi z omega is nothing but… Let me erase some part.

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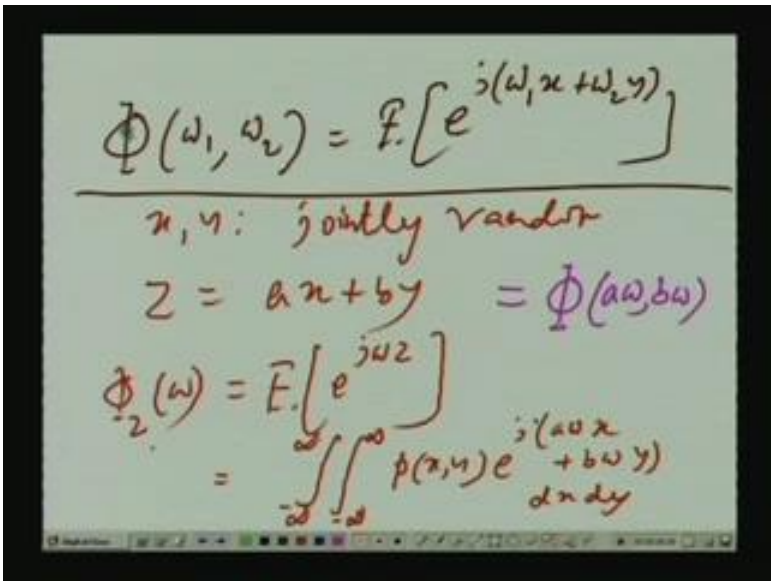

This is nothing but the joint characteristic function, but evaluated at a omega, b omega. This gives rise to an interesting observation; let us do that.

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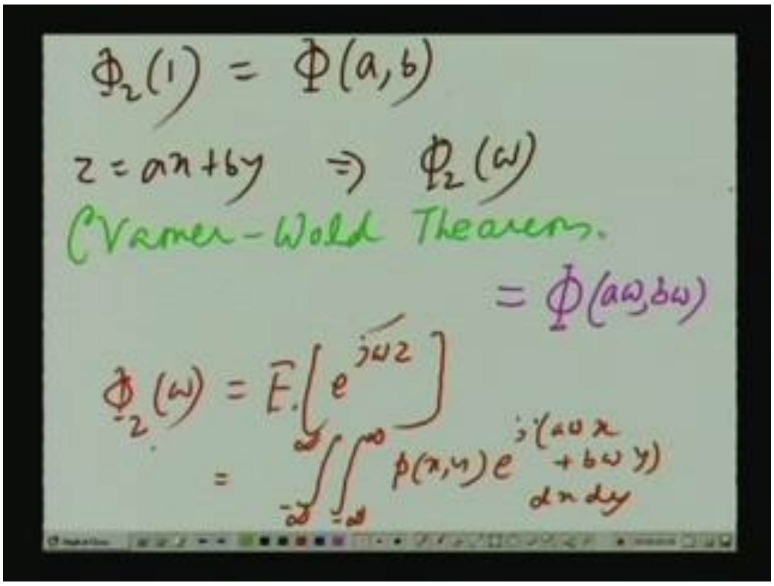

What is phi z 1? That is, if omega is 1, it is nothing but joint characteristic function at a comma b. So, two frequencies you can set: one is a, another is b – the joint characteristic function at those frequencies you obtain from phi z at omega equal to 1. Now, suppose we are considering remember z equal to ax plus by. And then we find out say phi z omega. Now, suppose it is given that, this phi z omega is known to us for all possible a and b, in fact, as a function of n. Suppose… I repeat again phi z omega – that form is known to us for all possible a and b; that means for each a and b, I can simply take the corresponding z, find its characteristic function; put omega equal to 1. That phi z 1 will be nothing but the joint characteristic function of x and y at the chosen a comma b. Frequency to omega equal to a; omega 2 equal to b.

If you want to find out the joint characteristic function at some other frequency – maybe a prime and b prime; what you have to do? You find out new z as a prime x plus b prime y. For that z, again our assumption is that, this characteristic function of z is known for all possible a and b. So, phi z omega is again known even though we have now a prime and b prime. And with this, new phi z omega, again replace omega by 1 and you get the joint characteristic function at a prime and b prime and so on and so forth. So, I repeat… This means that, if the characteristic function of z is known, z is of this form: ax plus by; but the characteristic function is known for all possible a and b, in fact, as a function of a and b; that means this joint characteristic function at all frequencies also can be evaluated. Joint characteristic function of x comma y – x and y at all frequencies can be evaluated, because you tell any frequency – maybe omega 1, omega 2; I am repeating – maybe omega 1 omega 2…

So, consider z now with a as omega 1, b as omega 2; for it also, we know the corresponding characteristic function. Take that; just put omega equal to 1. That will immediately give you phi of omega 1 omega 2 and likewise. But, we also know that, characteristic function is related to the probability density. This means that, if the probability density of z is known, z is of this form; but if the probability density of z is known for all a and b, that is, as a function of a and b; then of course, we know the characteristic function phi z for all a and b. And therefore, we know the characteristic – joint characteristic of x, y for all a, b. And that means we know the joint density of x and y for any chosen a comma b; that means if z is of this form and if the probability density of z is known as a function of a comma b, that is, for all a, b; and from that information, we can find out the joint density p x comma y for any x, y. Alright. This is called Cramer-Wold theorem – Cramer-Wold theorem.

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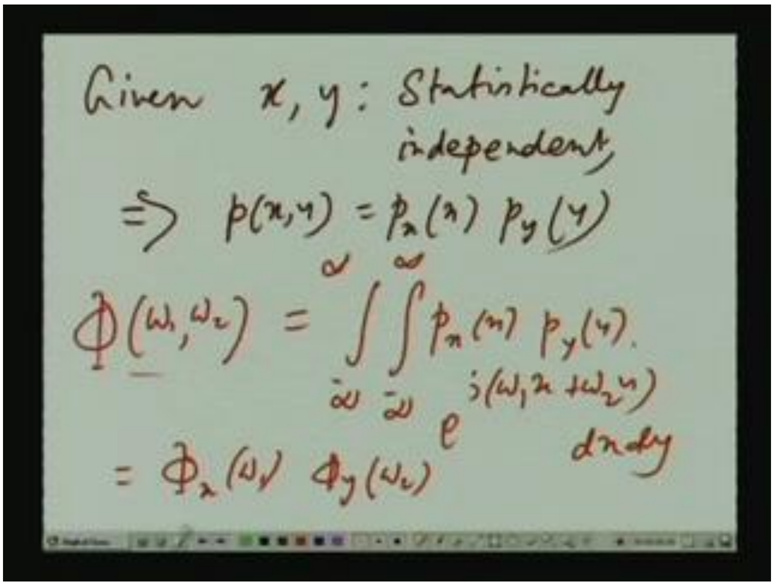

Next, you see one more interesting thing. Suppose it is given that, x and y – they are statistically independent. Given x, y – statistically independent; meaning… In that case, you can see what happens to phi omega 1, omega 2. We worked out a very simple though. Instead of p x comma y, we simply write as p x x p y y and then e to the power j omega 1 x plus omega 2 y dx dy. You can separate the integral into two parts: one involving x alone; another involving y. p x x e to the power j omega 1 x dx. That will give rise to phi x omega 1… I repeat again if you take out p x of x, that is, probability density of x and e to the power j omega 1 x and dx; take it under one integral from minus infinity to infinity. That will give you phi x omega 1.

Similarly, if you take out p y y e to the power j omega 2 y and dy; put under another integral from minus infinity to infinity. That will give rise to phi y at omega 2. So, that means if x and y are statistically independent, then the joint characteristic function is simply a product of the marginal characteristic functions or individual characteristic functions. The reverse also is true. If the joint characteristic function can be broken as a function product of two functions: one of omega 1 and another of omega 2, because one you can then view as the marginal characteristic function of say x, another marginal characteristic function of y. So, that means if phi omega 1 comma omega 2 can broken like this, then it means that, x and y are statistically independent; that is, p x comma y can be broken as a product of p x x and p y y. That also is very easily seen. This time we will be using the inverse formula.

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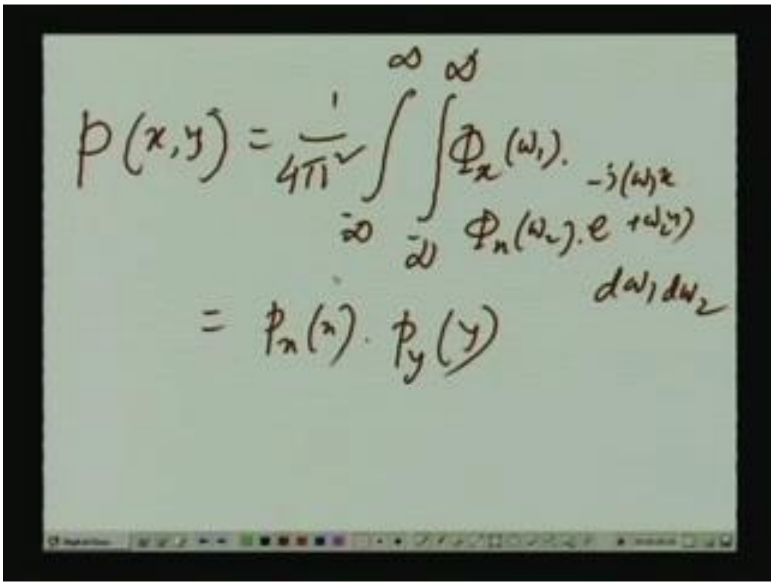

Now, the joint characteristic function I simply write as a product of two functions: phi x omega 1 phi x omega 2 e to the power minus j omega 1 x plus omega 2 y d omega 1 d omega 2. And as before, phi x omega 1 and e to the power minus j omega 1 x – they come under one integral. Phi x omega 2 e to the power j – it will be minus j omega 2 y – come under another integral. And they give rise to 1 by 2 pi – 1 by 2 pi; 1 by 2 pi with one integral; 1 by 2 pi is another integral. So, you get from one integral p x x; thereby, the inverse formula related with the marginal characteristic functions; similarly, from the other one also – this. So, joint density is a product of marginal densities; which means x and y are statistical independent.

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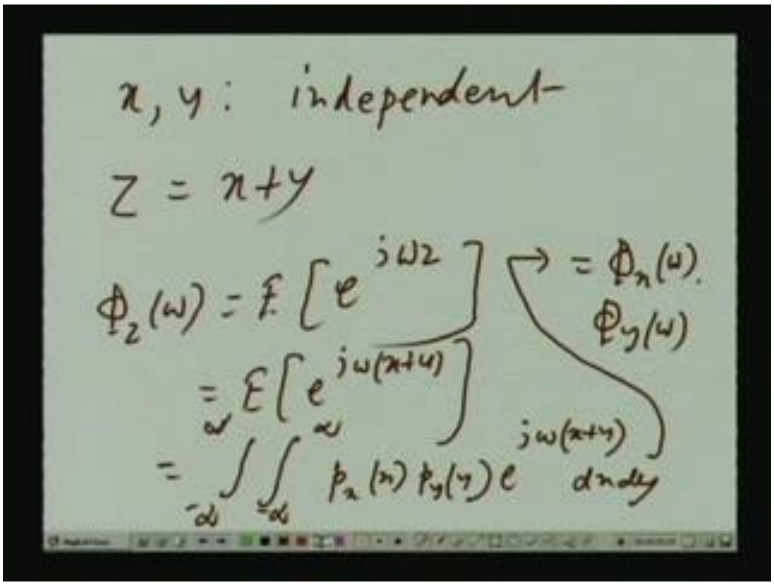

Now, one interesting observation that, suppose it is given that, x, y: independent; why independent? I mean statistically independent; that is, the joint density is a product of two marginal densities. And you construct a function z as x plus y. What happens to the probability density of z or what happens to the characteristic function of z? Now, phi z omega we know is expected value of e to the power j omega z; which is same as e to the power j omega x plus y. It is a function of x and y. So, this means that, we simply integrate this quantity by multiplying by p x comma y. But, p x comma y is p x into p y, because they are given to be statistically independent. As before, you separate the integral into two integrals: one is with x p x x e to the power j omega x; another is p y y e to the power j omega y. From one, we get marginal characteristic function phi x at frequency omega.

And, from the other, we get marginal characteristic function of y at a frequency omega; that means this becomes equal to… This then becomes equal to phi x omega times phi y omega – same omega. So, phi z omega is phi x omega into phi y omega. But, you see we are in the Fourier domain. After all, phi z omega is nothing but inverse Fourier transform of one probability density; phi x omega also inverse Fourier transform of one probability density; phi y omega also inverse Fourier transform of one probability density. So, if they are multiplied here; then corresponding densities will be what? They will be convolved. We know that, if two functions are convolved, then their Fourier transform is a product of the Fourier transform… If two functions say are convolved and you take Fourier transform of the convolution; then it becomes the product of the Fourier transform of the two functions: convolution in one domain gives rise to product in other domain.

So, here I am getting a product in the inverse Fourier domain, because… I repeat again what is phi z omega; that is, inverse Fourier transform of function p of z. Phi x omega is inverse Fourier transform of again another function p x x; and phi y omega is another inverse Fourier transform of p y y. That is the probability density of y. And they are being multiplied here to give rise to phi z omega; which means…

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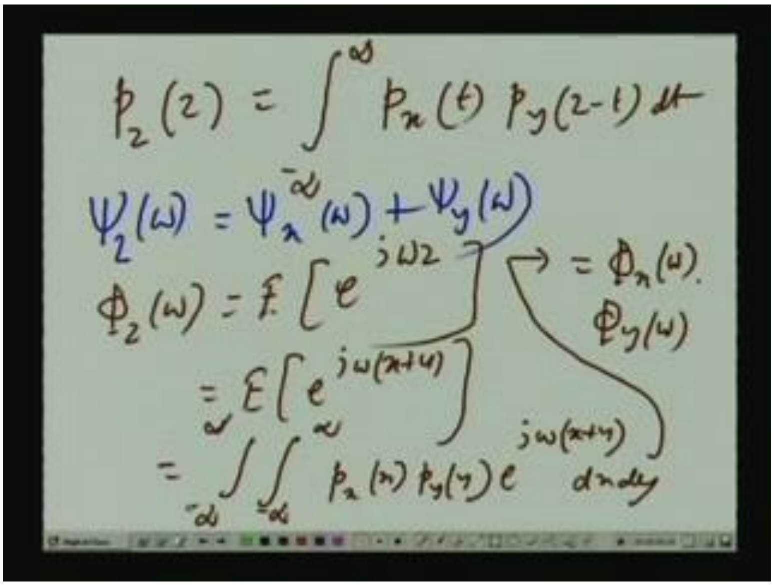

Which means p z z is a convolution. You can take some variable – any variable say t and p y z minus t dt. So, it is convolution. And just one more thing; phi z omega is phi x omega times phi y omega. So, if you consider the second joint characteristic function psi z omega; which is nothing but logarithm of this; then into logarithm of a product is summation of the two logarithms. So, this becomes ln of this phi x omega plus ln of phi y omega. And ln of phi x omega is psi x omega; and ln of phi y omega is psi y omega.

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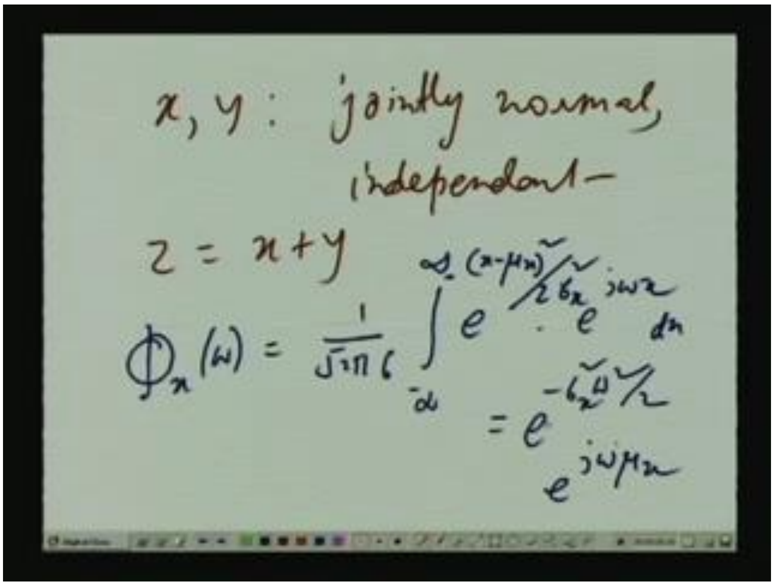

Now, you consider an example. Suppose x, y – they are given to be jointly normal and independent. Mind you in the case of independent also means uncorrelated. And you form z equal to x plus y. Then, if x and y – they are jointly normal means x also is a normal variable, y also is normal variable. x is Gaussian; and the marginal density is also Gaussian. x has Gaussian density; y has got Gaussian density. And jointly, they have a Gaussian form with correlation 0. They are statistically independent. So, they are un-correlated. The correlation coefficient will be 0 here. This is given. So, question is if x and y – they are individually Gaussian and they are summed; they are statistically independent of course – meaning un-correlated also; and they are summed to form z; then what is the probability density of z?

Now, we have already seen this earlier; where, z was in fact not only x plus y – whatever say linear combination of x and y – x plus by. And we have shown earlier also that, this is Gaussian. But, we will now show the same thing using the characteristic function concepts; which is a very simple way of doing things. We know… We remember that, earlier we found out say phi x omega. Earlier – few lectures earlier actually, we considered a case where x was a zero-mean Gaussian random variable. We are finding out this characteristic function. And that time we considered this integral – 1 by root 2 pi sigma – minus infinity to infinity e to the power minus x square by twice sigma square. So, this is the density function and multiply by it j omega x. And that term we found out. We said that, this integral will be nothing but e to the power minus sigma square omega square by 2.

Now, there will be slight change. Instead of having a zero mean, we want to have a mean of mu – say mu x. So, that means instead of x square, we should have x minus mu whole square. So, that means there will be some change here – x minus mu x whole square by twice sigma square. What will be the result? Result will not be this. What will be the result? Now, you can make a substitution x minus mu x equal to say x prime. So, integral limits – limits will remain same from minus infinity to infinity. Of course, integral was with respect to x. So, dx and dx prime was same. Only thing is x is to be then replaced by x prime plus mu x. So, an extra term – e to the power j omega mu x will come up; that is all. So, an extra term e to the power j omega mu x will come up. This is the characteristic function – marginal characteristic function of Gaussian random variable with mean mu x and variant sigma square.

Now, we have got two random variables: both are individually Gaussian; and therefore, jointly Gaussian – x and y. x has a mean mu x and variance sigma x. So, in fact, we call it sigma x now. And y also has a mean mu y and variance sigma y square. And x and y are statistically independent – means they are un-correlated. x and y are added; you get z. What is the probability density of z? We just do it by using characteristic function. So, that will give you an idea about how characteristic functions are useful. We know that, since x and y are independent – statistically independent and z is x plus y, the characteristic – marginal characteristic function of z will be simply the product of the two marginal characteristic functions: one is of y; one is of x, another of y.

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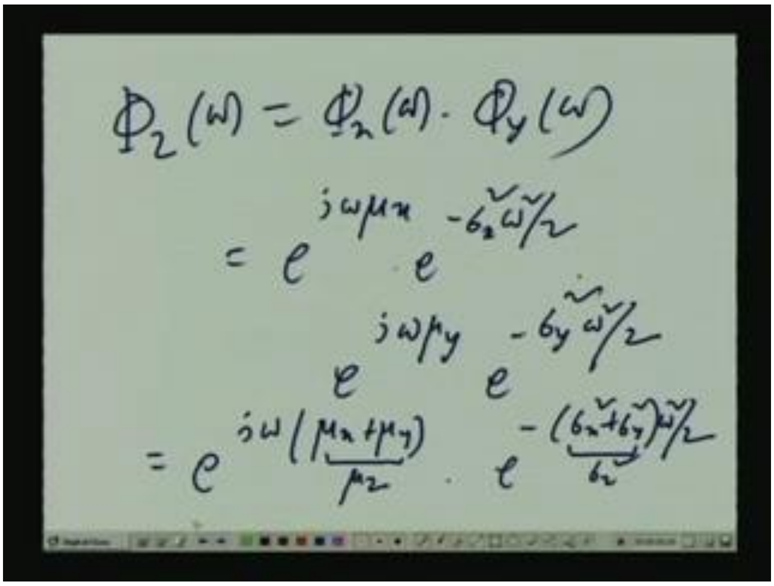

So, we know that phi z omega is nothing but phi x omega times phi y omega. But, we know their form. This will be… And similarly, this one will be… Which means we can take omega common. By the way, whenever you have situation like z equal to x plus y; we all know what is the mean of z; that is simply the summation of the two means: mean of x and mean of y. So, mu z is nothing but mu x plus mu y. So, irrespective of whether x and y are uncorrelated or Gaussian or whatever, mu x plus mu y is always equal to mu z. So, this is nothing but mu z. And similarly, whenever you have got z equal to x plus y, then what is the variance of z? Once again it is simply the summation of variance of x and variance of y; irrespective of whether x and y are uncorrelated or not, whether they are Gaussian or not; so, sigma x square plus sigma y square is sigma z square.

And now, you can easily see... Let us take this form – e to the power j omega mu z times e to the power minus j sigma z square omega square by 2. We have already seen this kind of form that, when there is a Gaussian random variable is mean – some mu and variance sigma square, then only the characteristic functions takes this form – e to the power j omega into mu times e to the power minus sigma square omega square by 2. So, obviously, you can see now that, z also is Gaussian with mean mu z and variant sigma z square. From this, it follows easily.

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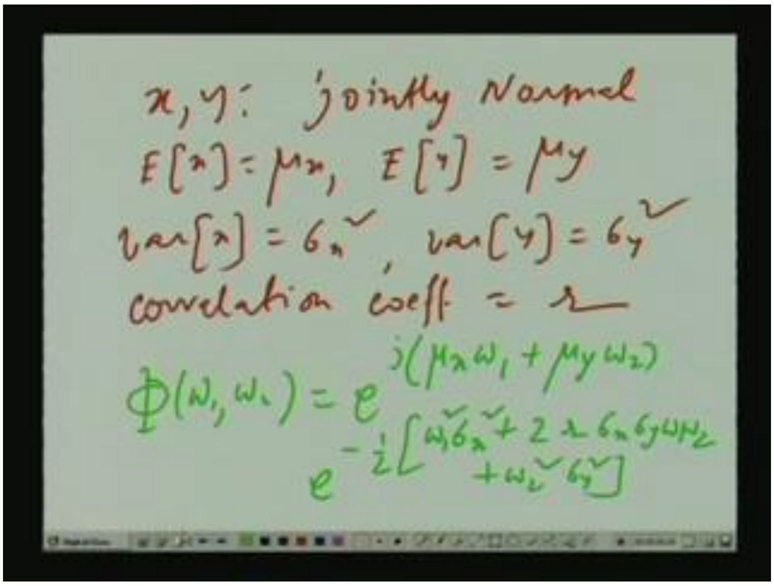

One more example; now, suppose there is no z. It is the different kind of problem, we are simply given that, x, y – they are jointly Gaussian – jointly normal; normal or Gaussian – they mean the same thing as I told you many times. You are given that, e of x is mu x;

there is mean of x is mu x; e of y is mu y. And the variance and all – wherever sigma sigma x square, sigma y square and all that; variance of x is sigma x square. Correlation coefficient – r. Then, you have to show, we have to find out… This is another example. We have to find out what is the joint characteristic function. See if you know the joint density function p x comma y and you know the formula for joint characteristic function, then at least in principle, theoretically, you should be able to find out the joint characteristic function. We have to carry out the integral; where, have to show here that, the joint characteristic function is of this form e to the power j mu x omega 1 is a big formula actually into e to the power minus half omega 1 square sigma x square plus twice r sigma x sigma y omega 1 omega 2 and omega 2 square sigma y square. It is a big formula, but with symmetric. We have to show this.

Now, I tell you if you really want to substitute p x comma y by the Gaussian joint density function and want to carry out the integral, we will be at c. It is very difficult. But, there is a clever way of going around it – getting around that. So, let us take that – the route.

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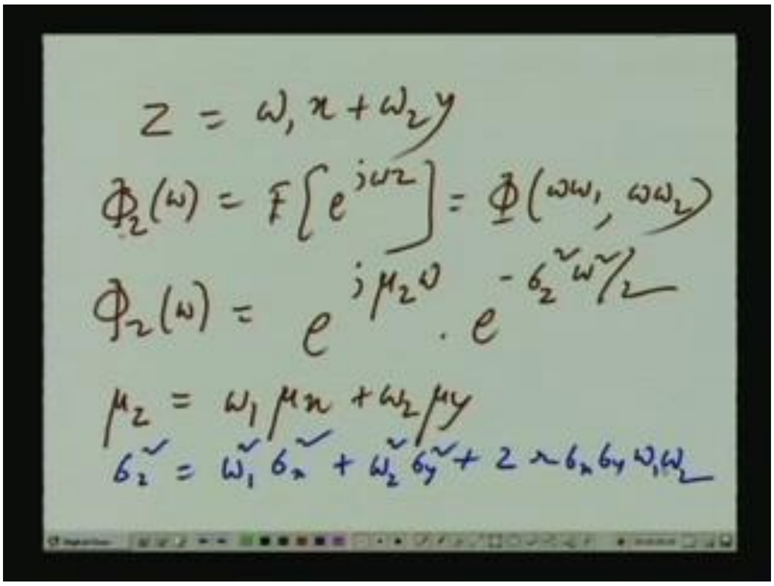

Suppose instead of finding out the joint density directly, we concentrate again on a variable z, which is a function of x and y. And this takes this kind of form z equal to say suppose some omega 1 x plus omega 2 y. Now, we have seen already that, if x and y are jointly Gaussian; then in a linear combination of x and y also is jointly Gaussian; so, you know that, z is jointly Gaussian. So, that means we know that, z is Gaussian. I repeat again if x and y are jointly Gaussian, then we have seen already that a linear combination of x and y is also Gaussian. So, z is a Gaussian random variable. That on one hand we know. On the other hand, what is phi z omega if you take this kind of form? After all, it is nothing but expected value of e to the power j omega z. And replace z by this. And you know you get the joint characteristic function at… After all, if you replace z by omega 1 x plus omega 2 y, then you have one term omega times omega 1 into x plus omega times omega 2 into y. Say you have got two frequencies: one is omega omega 1; another is omega omega 2. And it does becomes the formula for joint characteristic function. We have seen it already. So, in such case, phi z omega – whenever you form any z like this – a linear combination, corresponding marginal density is obtained or can be writable in terms of the joint… If the corresponding characteristic function can be written in terms of joint characteristic function.

I repeat – x and y are given to be jointly Gaussian, but we are not here finding out their joint characteristic function directly; rather we constructed a new variable z as a linear combination of x and y. And you are concentrating on z; finding out its characteristic function – phi z omega. But, phi z omega by this definition is nothing but the joint characteristic function of x and y at a frequency omega omega 1 and omega omega 2, because what is phi z omega? It is nothing but expected value of e to the power j omega z; replace z by omega 1 x plus omega 2 y. So, we have got terms like omega times omega 1 into x and omega times omega 2 into y. And this we have seen already. This becomes nothing but joint characteristic function at two distinct frequencies – two frequencies: omega omega 1 related with x; omega omega 2 related with y. So, that is true for… I mean that is true always irrespective of the particular probability density – joint density for x, y; whether they are Gaussian or whether they are not, this is always true. But, now we know the case that, when x and y are jointly Gaussian, that in the linear combination of them also is a Gaussian random variable; which means phi z omega also will be having this form – e to the power j mu z omega. We have already seen e to the power minus sigma z square omega square by 2.

Now, what is mu z? Mu z is omega 1 mu x plus omega 2 mu y. What is sigma z square? We have to subtract z mu z from z; that means – omega 1 within bracket x minus mu x – omega 2 within bracket omega y minus mu y – they are added and then whole square – expected value. So, one case, we get a term like omega 1 square into sigma x square. Similarly, another term is omega 2 square into sigma y square; but there is a cross term – cross term related to covariance twice. And covariance is nothing but correlation coefficient odd times sigma x sigma y by definition. So, twice r sigma x sigma y; there is a covariance times omega 1 omega 2. Now, we are interested actually… Remember our target; we are interested in finding out phi omega 1 comma omega 2. But, there is an additional variable omega related – omega present. So, why do not we make omega equal to 1. So, if you make omega equal to 1… What happens if we make omega equal to 1? We have e to the power j mu z – e to the power minus sigma z square by 2. e to the power… And that will give rise to phi of omega 1 comma omega 2? What is phi omega 1 comma omega 2? e to the power j just mu z, because omega is 1. And e to the power minus sigma z square by 2. Replace mu z by this expression; replace sigma z square by that expression. And you can easily see that, we are getting the previous result.

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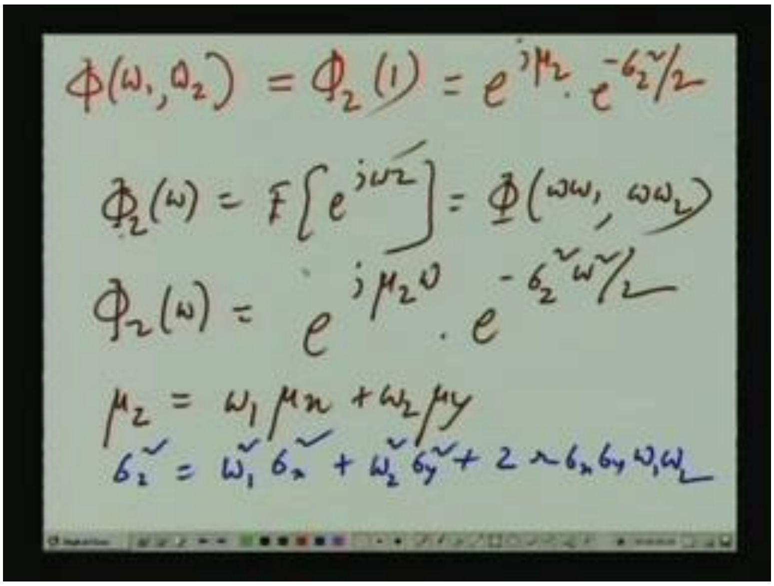

That is, I repeat again; what is… What is this? This is nothing but phi z with omega equal to 1; which means e to the power just j mu z omega equal to 1 and e to the power minus sigma z square by 2. We have already found out mu z; we have already found out sigma z square; just put them back there. You can easily see that, we are getting the expression that we have written earlier. So, often you see I am just constructing this new variable z as a linear combination of x and y helps in reducing the complexity of the problem.

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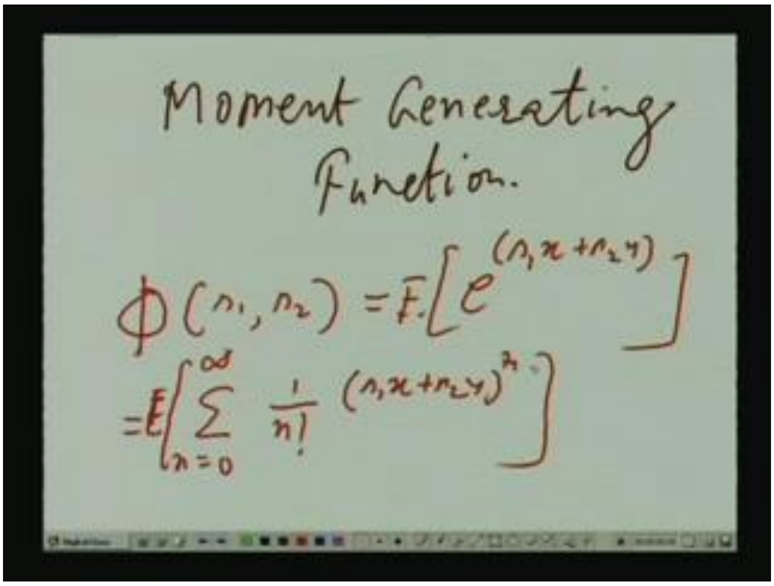

Next, we define a moment generating function. Maybe we use a different symbol, because phi we have already used. So, maybe I put a bar, because it is not characteristic function. It is phi bar s 1 and s 2; s 1 and s 2 are real numbers. We define like this. In fact, we can… There is no need to put bar there, because the definition is this – e to the power s 1 x plus s 2 y. Now, earlier we had seen that, what is characteristic function? It is a function of omega 1 omega 2. It is nothing but expected value of e to the power j – within bracket, omega 1 x plus omega 2 y. There is just a slight change in notation; j omega 1 has been now, is called s 1. So, it is not real; it is not necessarily real. And j omega 2 is replaced by s 2. But… So, we still may retain the same notation phi; phi s 1 comma s 2. So, this is called moment generating function.

Now, this is an exponential function. So, we can expand this exponential into a series form. And then using linearity of the expectation of ((Refer Time: 47:53)) apply the e pointer on each term of the series. So, what is the series form? Series form we know; e to the power x is nothing but 1 plus x plus x square by factorial 2 plus x to the power 3 by factorial 3 plus dot dot dot dot plus x to the power n by factorial n plus dot dot dot dot. Instead of one variable, you have now in the argument this whole quantity – s 1 x plus s 2 y. So, summation is from n – 0 to infinity 1 by factorial n. And this whole quantity s 1 x plus s 2 y whole to the power n – expected value. Then, I… As I told you, I will use the linearity of this expectation of pointer e; I will push e on each term of this series and factorial n; n is constant, not random. So, e will apply directly on these because x and y – they are jointly random. So, that means this can be written as…

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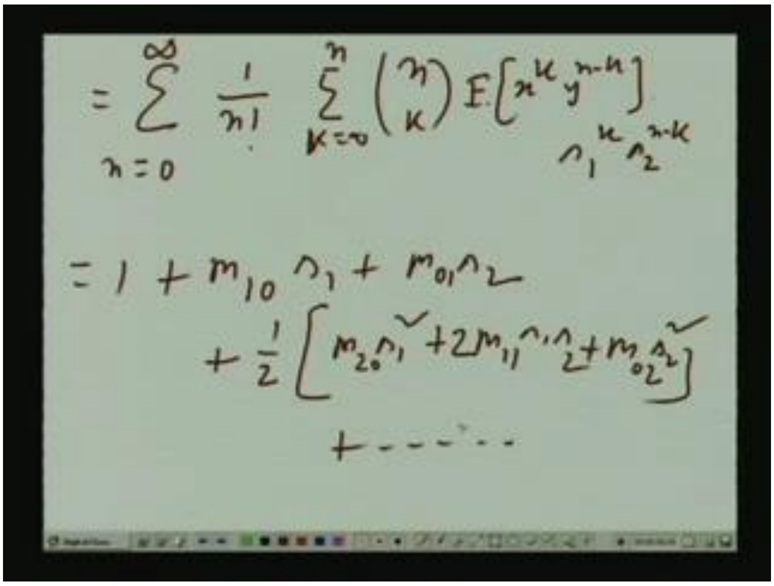

And, you see I can always expand this s 1 x plus s 2 y whole to the power n by the binomial formula; that means this entire thing then becomes… For each n… For each n, we have got one term – s 1 x plus s 2 y whole to the power n; and we are breaking it with the binomial series. So, there are n plus 1 terms. So, that binomial series given by this… – by the binomial series is given by this inner summation. Expected value directly works on x to the power k y n minus k. And then you have got s 1 to the power k s 2 to the power n minus k. And you can now easily see; let us start with n equal to 0. If n equal to 0, you have only 1 coming, there is no other term. So, 1. Then, consider n equal to 1. For n equal to 1, this is 1; but k equal to 0 to 1. So, there are two terms. In one case… And in any case, 1 0 – that will give us to 1.

In one case, we have e to the power – expected value of x to the power 1 y to the power 0. And in other case, expected value of x to the power 0 at y to the power 1. So, E of x times s 1. You have got y to the power 0 here. So, we will have s to the power 0. So, that will give rise to m 1 0 by our previous definition. The other one will give rise to m 0 1 s 2. Then, we consider n equal to 2. For n equal to 2, what we have – 1 by factorial 2 of course. So, that is 1 by 2. Then, one term… About x square and y to the power 0 s 1 square. That will give rise to m 2 0. And this will be 1, because 2 r… 0 here will give rise to 1; factorial 2 divided by factorial 2 into factorial 0; that is, 1. So, m 2… m 2 0 s 1 square. Another term will be when k equal to 0 and n is 2; that is, e of y square s 2 square. And once again, this will be 1; k is 2 now. So, 2 – factorial 2 divided by factorial 2 into factorial 0 is once again 1. So, that will give rise to m 0 2 s 2 square and then twice… where, k is 1. Now, we have this 2 comma 2 1; will be k 1 – will become 2. We have factorial 2 divided by factorial 1. So, twice e, and now x to the power 1 y to the power 1 s 1 to the power 1 s 2 to the power 1. So, m 1 1 s 1 s 2, so on and so forth. We will stop here today. And from this, we will derive some new results and take up some examples in the next class.

Thank you very much.

Preview of Next Lecture

Lecture - 19

Joint Conditional Densities

So, in the previous class, we are discussing these joint moments, rather this moment generating function. We start from there again. There will little overlap. But, there is nothing wrong if there is overlap with the last phase of previous day's lecture.

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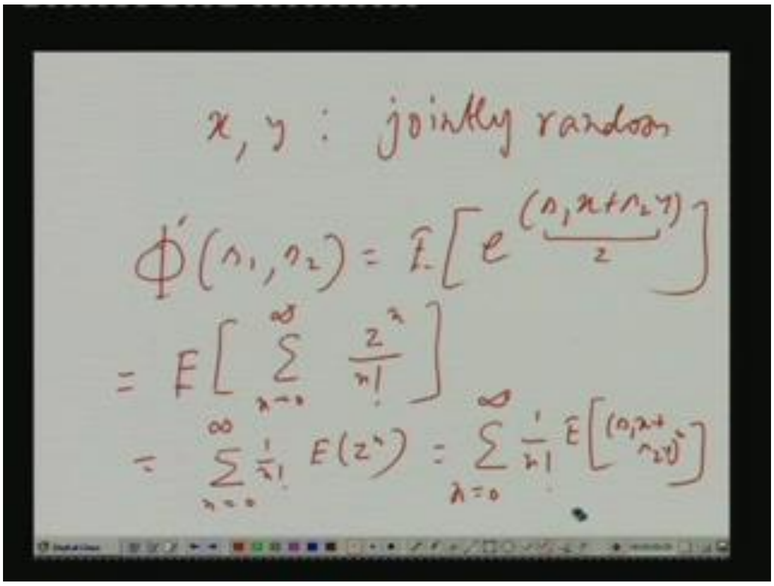

So, we are given two random variables as before x, y – jointly random. Then, we define this function phi; maybe we can put a phi prime. I will tell you why I am putting phi prime s 1, s 2 as the expected value of e to the power s 1 x plus s 2 y. Now, listen; earlier, we had a situation; where, s 1 was equal to j omega 1; s 2 was j omega 2. And the entire thing was called the joint characteristic function. It was written as phi of omega 1 omega 2. Since j omega 1 is replaced by s 1 here, and j omega 2 is replaced by s 2, and j is missing on this side, I am giving a new name phi prime. That is the only difference. This is called the moment generating function. We have already seen what is the moment – joint moment; that is, x to the power r, y to the power k; its excepted value of this product is called the joint moment of order k plus r equal to say n.

Now, this function will help us in getting those moments of various orders. To understand that, let us first do this. This is an exponential. So, s 1 x plus s 2 y; where we call it z. So, e to the power z. We expand e to the power z into a power series. We all know what the power series is. There will be summation of terms – an infinite summation actually. And then expectation is a linear operator; you can apply expectation on each of the terms in the summation separately. If you do that; that is, first, we have expected value n equal to 0 to infinity z to the power n factorial n. This is the exponential series. And then I will apply this expectation operator on each term in the summation. Factorial n is a constant. So, it remains outside. Then, what happens? That is, E… Now, we know what is z; z is this factor. So, replace z by its actual form s 1 x plus s 2 y whole to the power n. Now, s 1 x plus s 2 y whole to the power n is actually a binomial series of n plus 1 terms. Again, that can be… So, I can expand this term.

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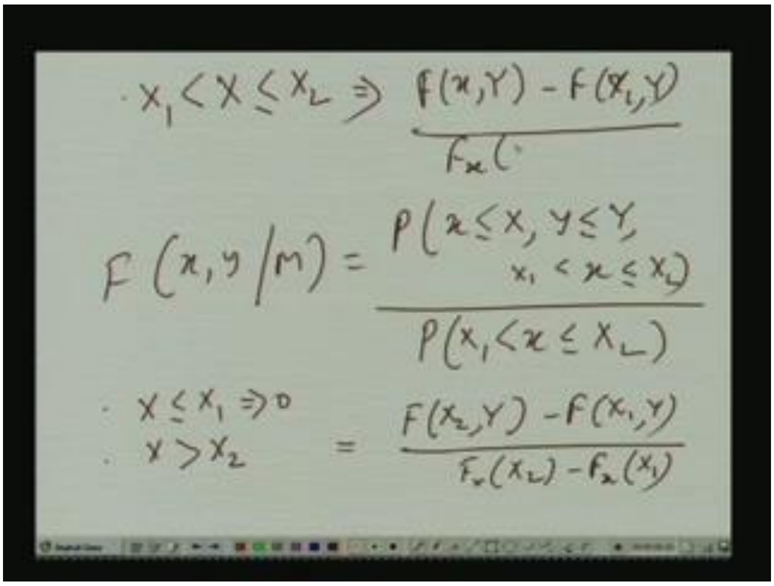

Now, to find out density, you will see one thing that, in one case, we have got 0. So, density will be 0 because derivative after all. In the other case also, if you have to find out the… there is a constant thing. After all, joint density means what? I mean we have to take partial derivative del square F del x del y. There is no x here; it is constant. So, derivative will be 0. Only here there is x; only here there is x. So, only here you will get this density. And what will that density be? This is independent of x. And here del square f del x del y will give rise to p of x, y. So, here p of x, y divided by whatever you have here – this constant. In other two cases, it will be 0. So, I stop here today. That is all for today. And in the next class, we start from here.

Thank you very much.