Probability and Random Variables
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Lecture - 14
Function of Two Random Variables (Contd.)

In the last class, we started with this topic, function of two random variables. We will continue from there. In fact, I take up the same example, which I touched upon in the last class for some reason.

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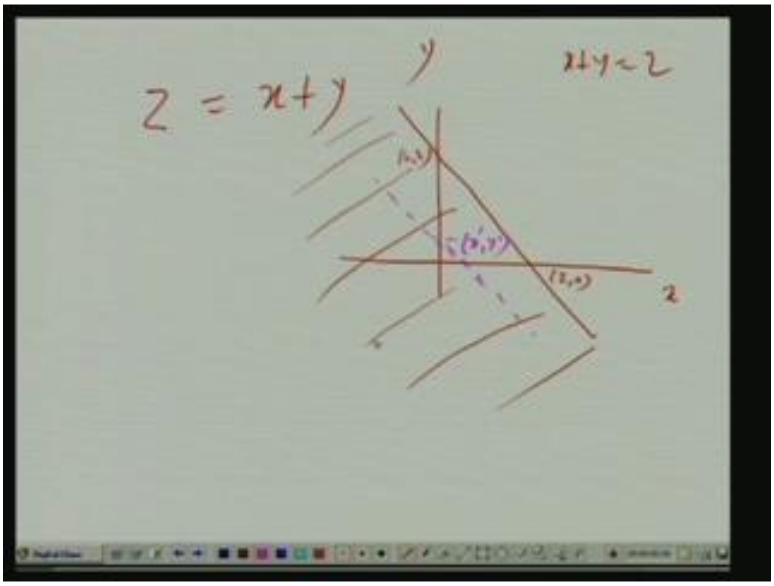

That is, suppose if it is given that, where x and y are two random variables and z is a summation. You have to find out… So, Z is a function of x and y. So, you have to find out the probability distribution and also probability density of this variable Z. Now, how to do that? We recall given any particular Z, we have to find out the probability of Z taking values less than equal to all… Say it is given that some capital Z. So, you have to find out the probability of Z taking values less than equal to capital Z. So, in the Z plane, I will find out the area, which were… The Z plane means at the xy plane and the third axis is the Z axis; in that xy plane, I will find out those areas, where for any given xy, the corresponding small z is such that its value is less than equal to capital Z. So, probability of x and y falling together in those regions – altogether total probability – that will be equal to new probability distribution.

Now, for this… What does this mean? This means that, z equal to x plus y means actually a line like this; if it is x-axis; if it is y-axis. And imagine a third axis vertical from this origin; that is for z. Now, this line – if I take this xy plane – x plus… This line is characterized by the equation x plus y equal to some z; that means z – this point is z, 0; and this point is 0, z. So, this intercept from origin to this point – this is equal to z and this intercept is also z. And as you slide this line further to the right, the value of z increases because this intercept increases. So, actually, it is like a slanted plane; it is like a slanted plane going like this. As we move x and y both, you get a bit of higher and higher values of z. So, for a fixed z, this equation corresponds to a particular line; another z – another line, like that. So, given a particular z, you have to find out the probability of x, y falling in… Given a particular z is associated with this line. This line is corresponding to what? x plus y equal to Z for some given z. So, I have to find out the probability of x, y – probability of random variable Z rather taking values less than equal this z given z. That can happen only if as you can see, I consider this region, because here if you take any point; here if you take any point x, y; x plus y equal to…

Now, if you put it as constant – x plus y; what may be the constant value that will correspond to particular line? And obviously, that line will be to the left of this line. The value of z will be less here; that is, if you take any point say; suppose you take any point here; if it corresponds to say x prime y prime; obviously, x prime plus y prime – that will be less than z, because x prime plus y prime – if you call it a constant say z plus; that will correspond to a line like this. And this is z plus, this is z plus. Obviously, that is less than z. So, in this xy plane, I now want to find out the area, the region, where for any xy pair, the summation x plus y is less than or equal to the given z. Obviously, now it is clear that, the area will be given by this shaded region.

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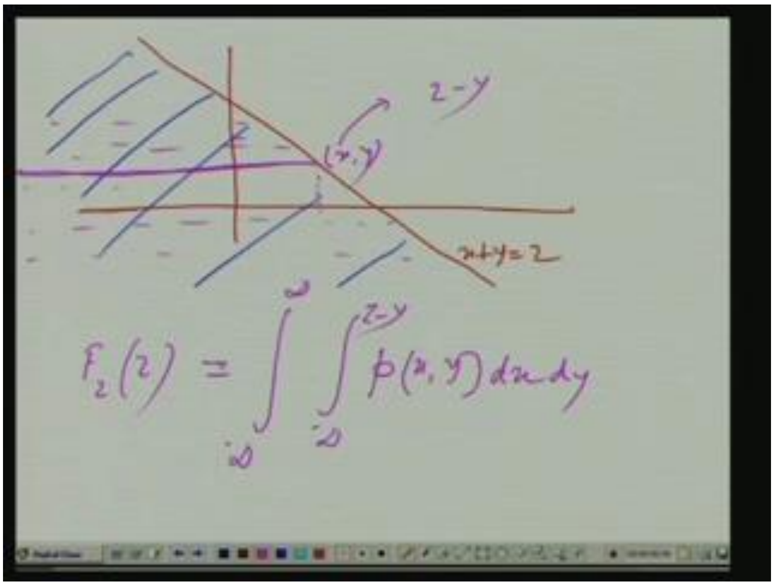

Let me then redraw it. This is x plus y equal to z equation. And this is that shaded area. I have to find out the total probability of this pair x, y falling within this zone. What is the total probability that x comma y – this pair is lying within this zone? That total probability been… I mean that will be same as the probability of the random variable z taking value less than equal to the given z. Now, how to go about it? There are many ways you can find it out. I will first take a line. This line means if it corresponds to x comma y… Then, obviously, for this line, y is fixed. And what is x? x actually is equal to z minus y. I write it like this. Then, only if we always write x as z minus y and then y, then only it will be clear; it will imply that, I am looking at these points – x comma y on this line, because on this line, for a given y, if the x is z minus y… For a given y anywhere, if the x value is z minus y; then obviously, that point is lying on this line. So, I will be…

The way I will proceed is I will be taking the probability; I will be holding y fixed first. And what is the probability that x takes values from minus infinity up to here. Once I find it out, then I will move y from minus infinity to infinity. Then, that will be spanning this entire region. It was for a fixed y; then y will be brought down. So, I will have more and more lines. And then brought down further; brought down here; here like that. And again brought up, brought up. So, the entire area will be spanned; that means, what I will be doing… First, there will be an integral. Or, let me… First, there is an integral; p is the joint probability of xy. But, here y is fixed to a particular value y. And x is moved from minus infinity to y; minus infinity to this x; and x means actually z minus y. And then I have an outer integral with respect to y. That will go from minus infinity to infinity dy. So, you put the dy here. So, entire integral is a function of z only. And that will be here equal to F z z. This z is the given z.

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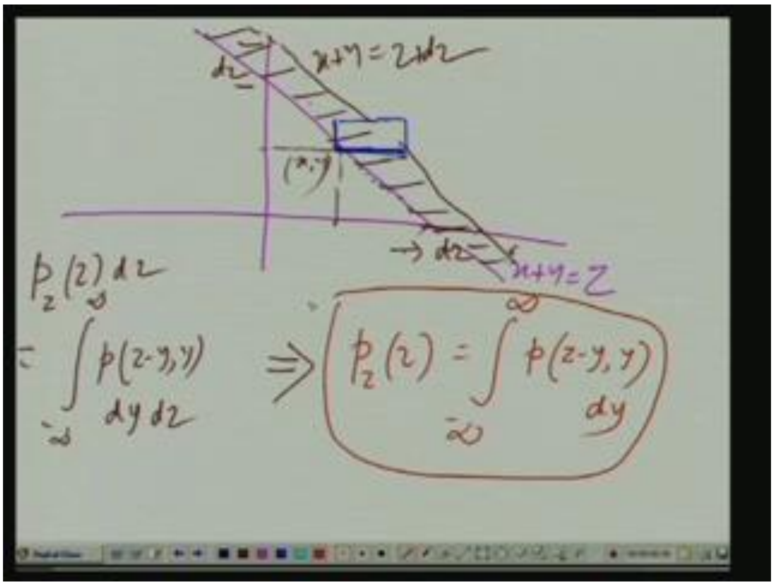

How about the probability density? For probability density, I have this basic line given. This is for x plus y equal to z. And then I will take another line. This is for x plus y is equal to z plus dz. This much is dz. This is dz. Same here; this is dz. So, what is a probability of z taking value between z and z plus dz? If I can express it as some function of z times dz, then that function is a probability density. So, I have to now find out, I have to now concentrate on this area. And I have to find out the probability of this xy pair falling in this… – total probability of xy pair falling within this second zone, because the moment xy pair falls within the second zone, I have z, which is x plus y. I have z either lying here or here or here – any points; means it is lying between z and z plus dz.

Now, what is that? So, we consider a particular point say x, y. What is this? I have to find out an infinitesimal area. Here it might happen to be mean; but actually as you know dz, dx – they are infinitesimally small – infinitesimally small. So, this will be very small. What is this? In fact, this will be dz. This will be what? Dx. And what is dx? Say this y is fixed for this line at on… Here I have got… But, this would go up to this. What is this length? This length is clearly dz. This and this are same. Here y was fixed; x is fixed; x plus y is z. Here x plus y is z plus dz. But, value of y remains unchanged whether here or here. So, only x changes from x plus dx; that gives rise to z plus dz; that means this dx or dz – they are same. So, this is dz and this is dy. So, I have got dz dy. And I will take xy here; then another here at this point, at this point, at this point and all these points; so this line.

Find out these probabilities. These are elementary probabilities – infinitesimal probabilities. And sum them on this line. To emphasize that, I have on this line, x… I have to somehow bring this equation – the relation between x and y, that is, x plus y on this line equal to z. So, as I replace x by z minus y; that is, for a given y, x is z minus y. So, I am on this line. On this line, I will move y on this line from all very closely spaced points. And at each point, I will find out this area – infinitesimally small area. And the probability of x, y taking values within that range will be given by this; and I will sum them. That will be the total probability of the xy pair falling within the certain zone; which means p z z dz is nothing but total probability – y moves from minus infinity to infinity – dy dz. You can cancel dz from both sides. This leads to the following. This is the relation.

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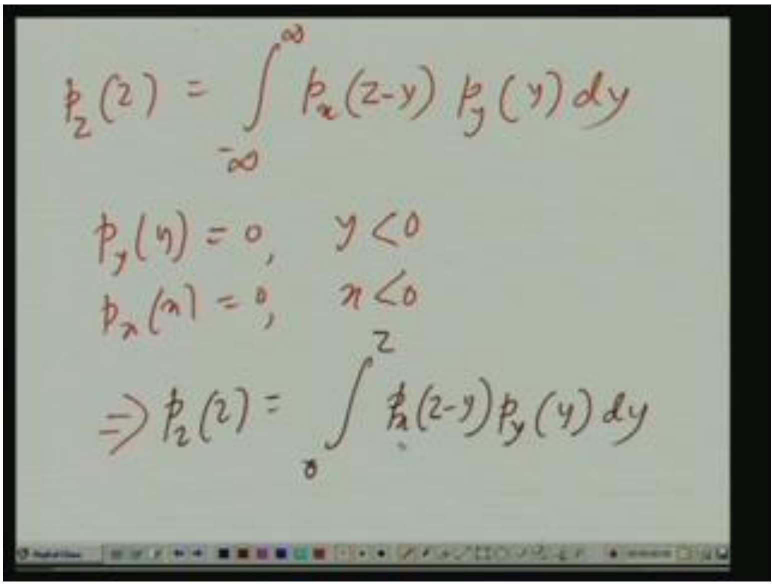

Now, comes something very important. Suppose x and y – they are also given to be statistically independent. In that case, the joint probability will be what? Product of individual probabilities. If that be the case; in that case, what happens to p z z? It just becomes an integral like this – p x… The subscript x now denotes that I am dealing with the probability density of x alone – and p y y dy. All of you know that, this is a convolution integral – famous convolution integral, which occurs in the signal and systems theory, rather in analog system theory; where, in the description of linear time invariant systems, you describe it by a simple response, so that the output will be given by nothing but convolution like this – between the input sequence, input function or input signal, and the impulse response of the system. It is the same thing; which means we can make use of all good properties of convolution here to carry out this integral.

One such property is that, if it is a Fourier transform, you do not have to carry out the integral; it just becomes the product of the two Fourier transforms. So, it gives you the Fourier transform of p z z. Here if it is further given that, these functions p y y is 0 for y less than 0; here also p x x – 0 for x less than 0 if it is given. In system theory, we say that, these are causal signals; that is, they do not have any component to the left of origin. If the probability density is here as such; then that look like some causal signal; that is, for any y or x taking values – taking negative values, the corresponding probabilities are 0. Then, this integral becomes further simplified. If y cannot start from minus infinity, because from minus infinity up to 0, this value is 0; 0 multiplying this is 0; all of them are finite. So, 0 multiplying any finite number is always 0. So, integral has to start from 0. On the other hand, as y increases from 0 and goes up and up and up to z, it is okay. But, once it crosses z, the argument here becomes negative. And for any negative argument for this probability density, the corresponding probability is 0. So, y can go only up to z. After that, this becomes 0. So, this integral becomes very simple.

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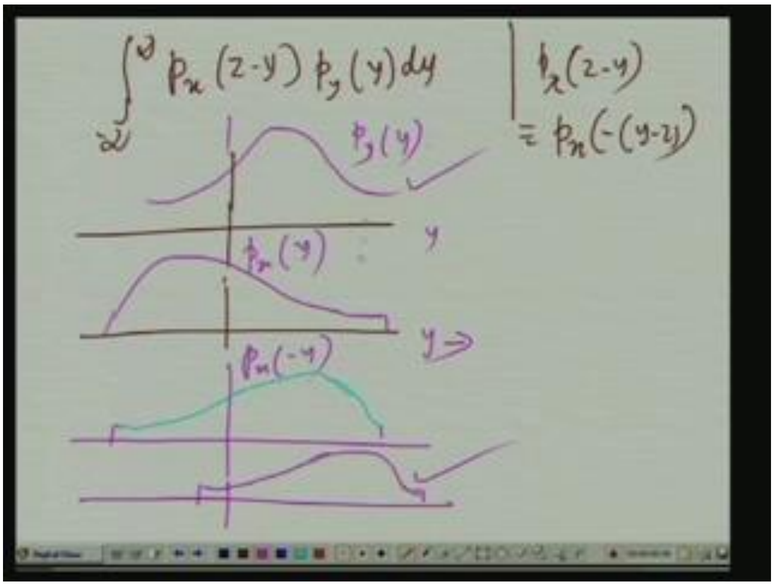

How to carry out the integral? You all know how to carry out these integrals graphically; that is, I rewrite here. You have to evaluate this function. You plot p y y. This is… This is y. Remember I can always write it like p x z minus y can always be written like p x minus of y minus z. So, that means if p x x is given to be like this; if suppose this is p x x or p x – maybe y; this is a function. So, I am making the same variable for both. Just look at it mathematically as if like a function of time, another function of time, or function of space, function of space like that. And function of some variable y, function of it is the same variable y, because after all, the integral we are integrating with respect to y alone. Then, we reverse it to make it p x minus y.

So, it becomes like this. And then here it means this is further shifted – further you know delayed by z. So, depending on whether z is positive or negative, you shift this function; push it to the right or left by an amount z. That takes you to here. Maybe I make it like this. Now, you consider this function and this function, because they have to be multiplied in this integral. The product function is the integrated – means the area under the product function is evaluated. So, you multiply this and this; this function and this function; find out where they are overlapping; just multiply the two; get the product function; and then find out its area. That is how it has to be found out.

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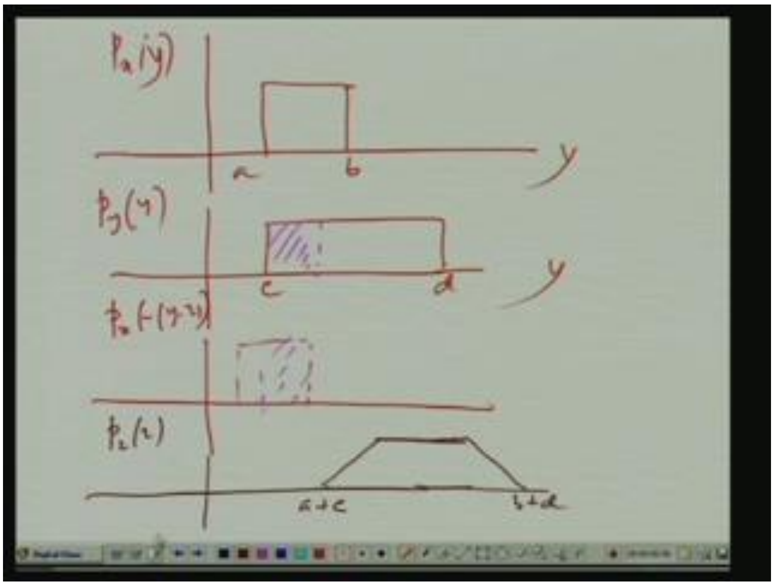

Next, for example, suppose it is given that, both are uniform. Taking your permission, I replace it by y just to make the same variable in both. Then, I have to reverse it. So, reverse it means a goes to minus a; b goes to minus b. And then if I multiply the two without any shift, I find there is no overlaps; it is 0. If I delay it further to the left, shift it further to left, no overlapping in it. So, it will become 0. Only when I shift it far to the right; from minus a, bring it to c; then by how much do I shift it to the right? c minus bracket minus a. So, c plus a. So, only when z becomes equal to c plus a, the overlap between the two starts. And as we further push it to the right, the overlap between them increases. The area then becomes a linear function. It is like this. Initially, it was on this side. Now, maybe it is like this; it is here. So, only in this zone, there is overlap between the two.

As we further push it to the right, the overlapping area increases linearly, because this is increasing linearly. What is area after all? This is rectangle – this height versus this width. And width is increasing; height is fixed. So, it is linearly linear in width of course. So, it goes up. For what value of z this thing starts? At a plus c. That is the amount of shift I gave. From this point onwards, for that value of z, the overlap starts. And the overlapping area increases linearly. After some time, this entire thing comes under this curve. So, area remains fixed. As I slide it further to the right, still area does not change. So, height remains fixed. As I slide it further to the right, some portion of this function starts leaking out; go beyond d. So, amount of the overlapped area – volume of the overlapped area starts decreasing again linearly, because the width – because the area is a linear function of the width; height is fixed. So, it takes the shape of a trapezoid actually. I am not detailing this; it is because I know that, most of you know how to carry out things graphically; how to carry out a convolution graphically. So, it will go here.

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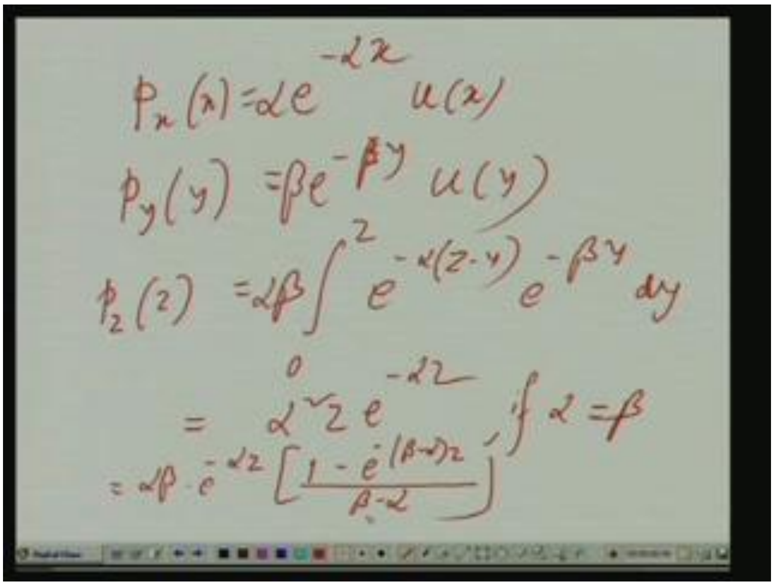

Let us go for another example. Suppose it is given that, u x means unit step function; that is, it is 0 for all negative values of x and equal to 1 for all x greater than equal to 0. Alpha is some parameter to be chosen. So, there are ((Refer Time: 25:24)) probabilities. Probability is 1 – total probability; that is, if we integrate it from 0 to infinity, it will become equal to 1. So, alpha is to be determined like that. And p y y. And obviously, it is given that they are statically independent; that is, we are not considering that joint density in the previous example also, in this example also; it is implicit. Then, what is p z z? Integral will be from… Both are causal.

So, integral goes from 0 to z e to the power minus alpha z minus y e to power minus beta. And this should be y. First, consider the case; where, alpha and beta are same. Then, what happens? If alpha and beta are same, this e to the power plus alpha y, e to power minus beta y – they cancel because alpha and beta are same. e to the power minus alpha z goes out of the integral, because integral is with respect to y only. And what does the integral give you? Only z.

Actually the actual form is alpha e to power minus alpha x and this is beta e to the power minus beta y. So, there will be alpha beta. In that case, you get only alpha beta or maybe alpha square you can say, because beta and alpha are same – alpha square. Then, z comes from the integral limit. If you integrate, only e to the power minus alpha z is here; which you can take outside the integral. Integral is only with respect to y from 0 to z; which gives you z alone. So, alpha square z e to the power minus alpha z. This ((Refer Slide Time: 27:37)) if alpha equal to beta. If alpha not equal to beta, then what happens? If alpha not equal to beta, e to the power minus alpha z goes out; that we have got e to the power alpha y minus e to the power minus beta y. What does it give you? It will give you… If you integrate, you write it as e to the power minus – within bracket, beta minus alpha y. So, if you integrate, a minus comes – 1 by beta minus alpha e to the power minus beta minus alpha y; put the limits. So, in one case, you have got 1; in another case, you have got…

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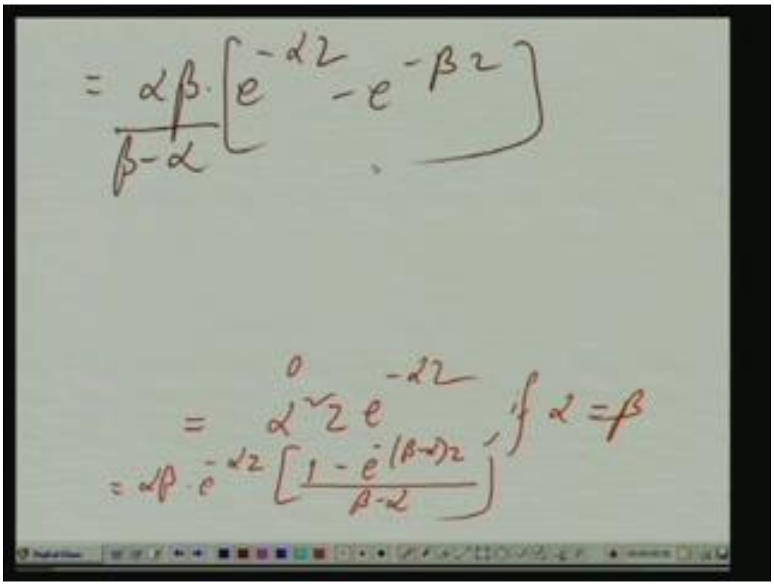

What does it give you?

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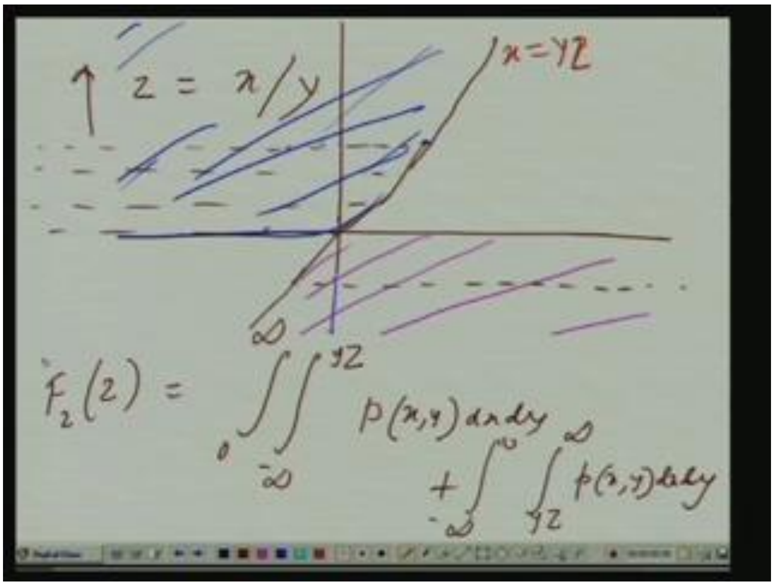

Now, we consider another function. We have taken z equal to x plus y – the simplest – is a very simple case. Now, take z is equal to x by y. Again, it is a function of two random variables: x and y. So, I have to find out the probability distribution of z first and then probability density. The probability distribution of z means given any particular value say capital Z, what is the probability? That small z takes values less than equal to capital Z; that is, z is given. For a given z, it is x equal to yz or y equal to zx by z. Depending on whether z is positive or negative, I get a straight line always passing thorough the origin. If z is positive, it will have slope like this; it will pass through first quadrant. If it is… If z is negative, it will pass through this; it will have a negative slope. I just take z to be positive here for convenience; for just the sake of here actually; it does not change anything. So, what is this equation? This equation is x equal to yZ – some capital Z. I have to now find out the probability distribution of z; that means I have to first identify the areas in this plane; where, for any xy belonging to that area, x by y ratio is less than or equal to this capital Z.

First consider where y is positive. And for our sake, I am taking Z to be… – given capital Z to be positive. You can apply the same logic for the negative Z case. Suppose y is positive. In that case, for anything here in this region, it will take – it will start from here on this line and go to the left. For a fixed x, value of x is becoming lesser and lesser. On this line, x and y are such that x by y equal to capital Z. But, as you go to the left, enter this shaded area, x by y ratio will definitely have value less than capital Z. So, when y is positive, I have no problem. I simply take this zone. And when y is negative, then what happens? If y is negative, then as you see, say y is negative.

So, I am here. This much is x; for any negative y, this is x; this is y. So, that x by y is a positive number – capital Z, negative x, negative y. If I go further to the left, x becomes more and more negative; which means x by y – negative here and negative here – they cancel; the x by y ratio is value, which is in any case positive, because both x and y are negative here. But, the ratio value – it becomes higher than Z – higher than capital Z, because on this line, I had some x and y; x was negative; y was negative obviously, because Z had to be positive. Say x by y ratio remains capital Z; but x is negative; y is negative.

Now, holding y fixed here; if I move x further to the left; x is becoming more and more negative. But, that fact that is negative does not matter, because y is also negative. Important thing is the magnitude of x goes up. So, the ratio – x by y ratio it takes values higher than capital Z. So, I cannot take this zone. On the other hand, if I move x to the right, then what happens? Whether x is negative or finally becomes positive; what happens – the magnitude of this ratio x by y – that becomes lesser and lesser. First, suppose I move x to the right. So, still x is negative, but x by y – this ratio – it is valued; which does not matter; which does not depend on whether they are negative or positive, because it cancels out. Its value obviously becomes lesser than capital Z and that continues as x goes into this quadrant, because Z – it becomes positive x by negative y. So, negative quantity, which is in any case less than equal to capital Z; that means here I take this zone. So, two zones.

Now, I have to find out what is the probability of x comma x and y together falling in this zone and this zone. How to do that? Let me get some space here for my job. First, consider this zone. Here how to proceed? Once again I will take a line; find out the… So, for this line, y is fixed. So, x… What is the probability of xy pair falling within this line for which y is fixed and x is moved from minus infinity to infinity? And once that is done, then I will move y from where? From 0 to infinity. So, this line will be sliding from this axis upwards like this up to this and going further up. That is one part.

So, that part will be what? So, F z actually will have two components. In one component, integral with respect to x will be from minus infinity up to x, and which x? That x which lies on this line. So, for a constant y, what is the value? yZ; Z is first given ((Refer Time: 36:16)) particular y. So, x goes from minus infinity up to yZ dx dy. And then y moves from 0 to infinity. That is one side of the story. And there is another integral for this lower area. Here again I take y; I take x. So, this line. So, I take x from this point on this line to infinity. What is x here? For a given y, again x is yZ up to infinity; and then y goes from minus infinity to 0. As you can see, the entire thing becomes a function of capital Z alone. And there is a probability distribution of Z.

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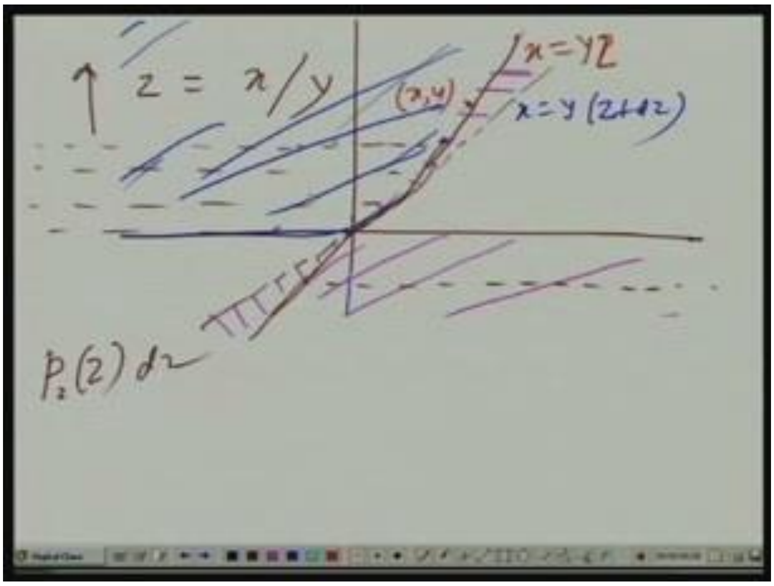

How about the probability density? That is again very simple. For probability density, that is, I have to find out now what is p z Z. How to find it out? Let z go from capital Z to capital Z plus dz. So, what is the probability of z falling within this zone – these values? Capital Z and capital Z plus dz – that by definition is given by this function times dz. This we find out. How to find it out? That means what is the relation? What is z after all? z is x by y. So, now, instead of z… Or, that is x equal to yZ. Now, instead of capital Z, I have got another line, that is, x equal to yZ plus dz. So, basically, I have to now look for this area, because within this area, for any x comma y, the value x by y will be greater than capital Z and less than capital Z plus dz. That is obvious. So, what is the probability of x comma y falling within these triangular sectors? To find that probability, I take points on this line; maybe I start with a point here say x, y. Let me erase something.

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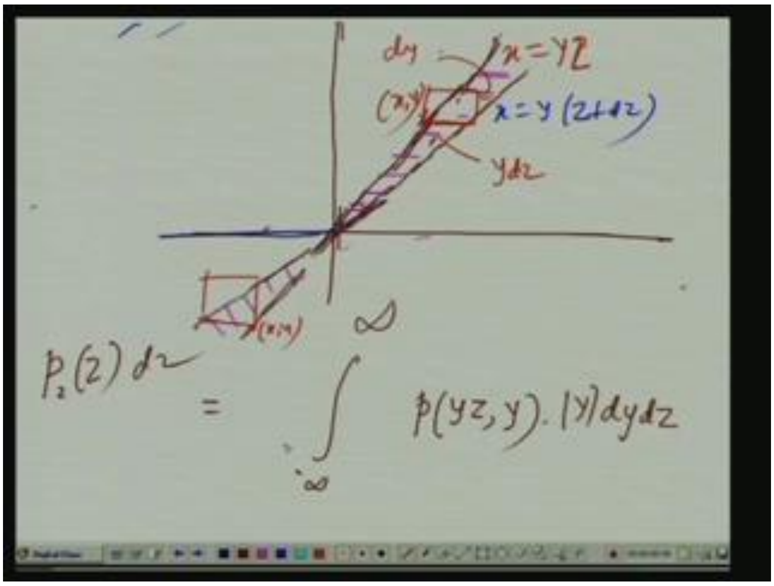

So, let me take a particular point x comma y; and where, obviously, x equal to yZ for this y. Here I take an infinitesimally small area. What is this area? How much is this? y is fixed; but z – earlier I had capital Z for this line; now, here z plus dz. So, what is dx? As z goes from capital Z to capital Z plus dz, x increases from x to x plus dx. So, dx equal to nothing but y times dz. So, this is y times dz. There is no problem here; y is positive; z is of course positive. And what is this height? This height is dy. This height is dy. So, problem is – on this side, if I take a point say x, y; this is the area. Here y dz becomes negative; but you see I am only interested in the length, not in whether it is negative or positive, because I have to find out the area. So, that is why it is better that I take the only magnitude of y; whether y is here or here, just take the magnitude of y and compute the area and then ((Refer Time: 42:23)) probability; that means… What is the probability of x, y falling within this area or this area? That is, p x comma y; but x is yz… – into the area. Area is what? I put mod y now, because y could be negative here, but area is positive. y goes from minus infinity to infinity. So, I am giving you this. Through this exercise – this, I am trying to develop the skills in you just to how to have… I mean a given such function of two random variables, how to proceed.

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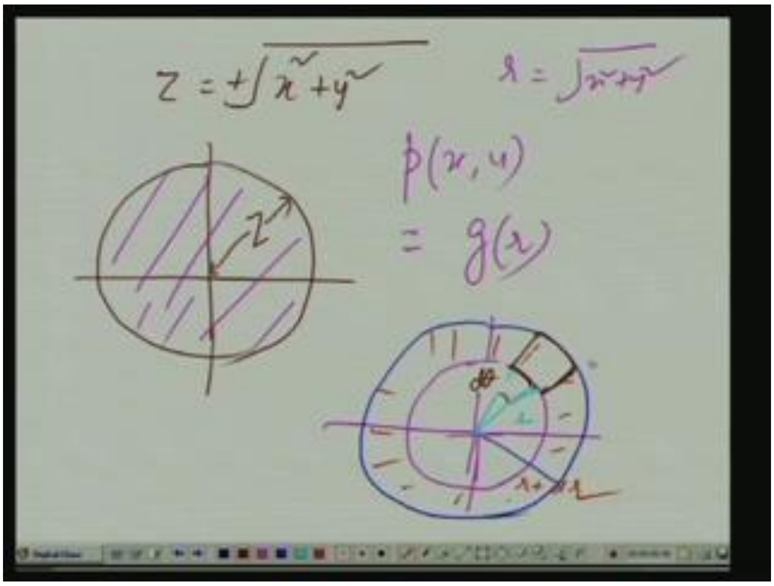

Take one more example. For any given capital Z, in the xy plane, this corresponds to line; it is a circle of radius capital Z; that means what is the probability distribution, how to find it out? That is, given any capital Z, take a circle of radius Z and the probability of x, y falling in this. That will be the probability distribution of Z. What will Z be? Now, suppose it is further given, that is, x, y; their probability densities are such that, they are circularly symmetric; which means p x comma y – joint probability density does not depend on x or y, but depends on square root of… It is a function of just the distance of the point xy from the origin. That is a function of r; where, r is square root of x square plus y square. We have seen this earlier; that means, if you take a circle of radius r on each point on the circle, the probability density is fixed; we call it gr.

Now, let us consider a circle here of radius r. And this angle is… And then you take another circle of radius r plus dr. Now, what is the probability of Z falling… that is, what is the probability of x comma y – that pair falling in this zone. That means small z taking values between r and r plus dr. What is the probability? So, we consider this point. At this point, the probability density is gr; but that is true all over the circle. So, we find that elementary area here. We try to find out the elementary area. How to find out the elementary area? Take another line. Then, let this angle be d theta; this is d theta. So, what is this length? r d theta. And you take this. What is this length? This length is dr. Since dr d theta – they are infinitesimally small; this area actually… I mean this is almost like a rectangle of lengths – of width what? r d theta one end and dr another end. So, the area is r d theta dr.

Then, what is the total area of this strip? Here r is fixed, but theta goes from say 0 to 2pi. Here actually the probability of x, y falling within this zone will be gr; then what is this area? gr is at this point – gr. Then, we are taking an area. What is that area? r d theta dr. But, I am interested not in just infinitesimally small area, but in the entire strip. For the entire strip, I hold r fixed. Considering such areas, I get an area; again an area; like that. So, I integrate with respect to theta and theta from 0 to 2 pi. What does it give?

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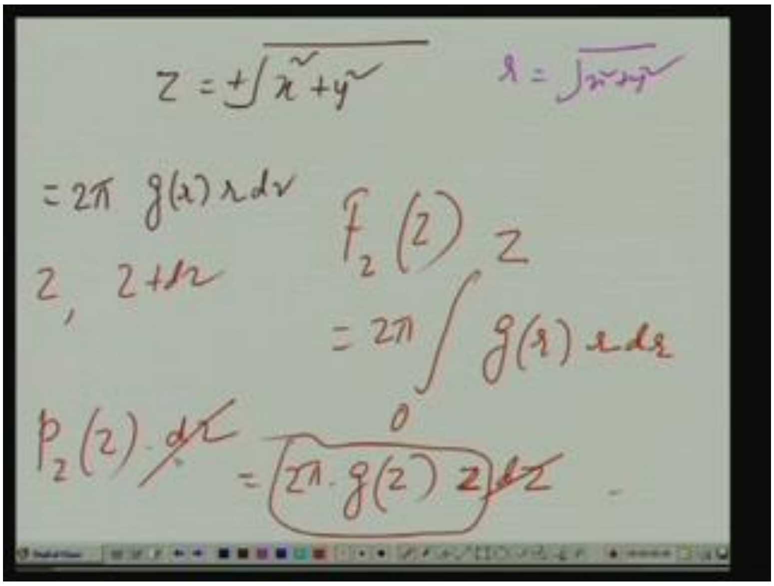

What does it give? This gives 2 pi g r r dr. That means what is F z capital Z? It is the probability that… I mean here we found out the probability of z taking values between r and r plus dr. Now, we have to find out the probability of z taking values; where, r goes from 0 up to capital Z. Then, that entire area given by the circle of radius capital Z will be covered. So, 0 to capital Z 2 pi. So, this is the probability distribution. As you see, if you integrate it, it becomes a function of capital Z only. So, it gives you the probability distribution of Z. And how about the probability density? Either you can define this density with respect to z. So, it becomes just 2 pi g z z dz. Or, you can use the previous argument that, what is the meaning of after all… – which means that as z goes from capital Z to capital Z plus dz, what is the probability of small z falling within the zone between capital Z and capital Z plus dz? That should be given by our definition, this function. Then only, this will be called the probability density of the random variable z. But, what is it? As this goes from capital Z to capital Z plus dz, We have seen it corresponds to that strip between two circles – angular strip. There the probability is 2 pi g – r here is capital Z. So, 2 pi g z. Then, r is again z. And dr means dz. And you can cancel dz from both sides. These are functions – 2 pi g z z. So, we have given you some examples of function of two random variables. So, in the next class, we will continue from here. We will go to define characteristic functions and other things – moments and all related to two random variables.

So, thank you very much.

Preview of next lecture
Probability and Random Variables

Lecture - 15
Correlation Covariance and Related Innver

So, as you know, we have been considering the case of joint statistics involving two random variables. We found out the joint probability density and joint probability distribution of the functions of two random variables also – two such cases. More cases are… More examples are possible. More examples are possible. But, I did not consider; I think you have got enough idea about how to proceed.

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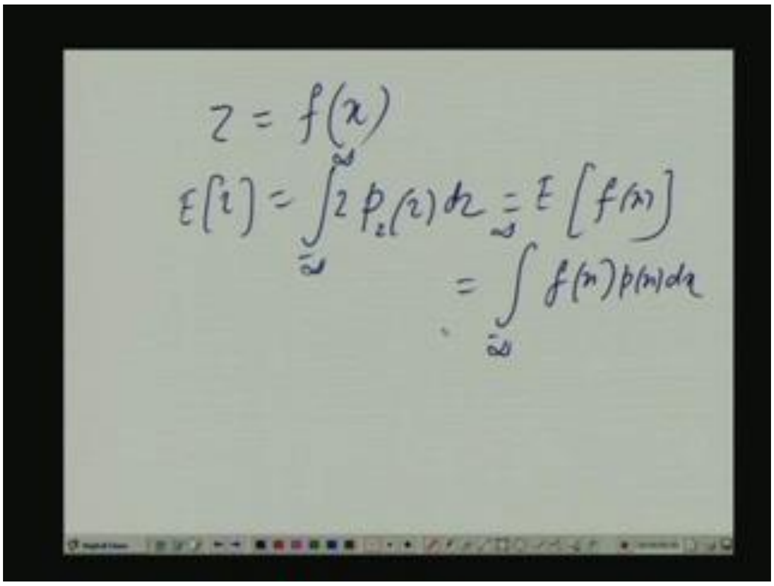

Now, we have to do something similar to what we did in the case of function of single random variable. One of them was things like this. If z was given to be a function of random variable x; then we have shown that, E of z; which is nothing but… We have shown that, this is nothing but E of f x; which is multiplied by p x dx. We have also proved it. Now, here there is a function of single variable. We will now consider the case for two variables. A similar result will come up and we will prove it. I will not say that, I will prove it, but I will just develop argument in favor of the result along analogous lines and that will be enough.

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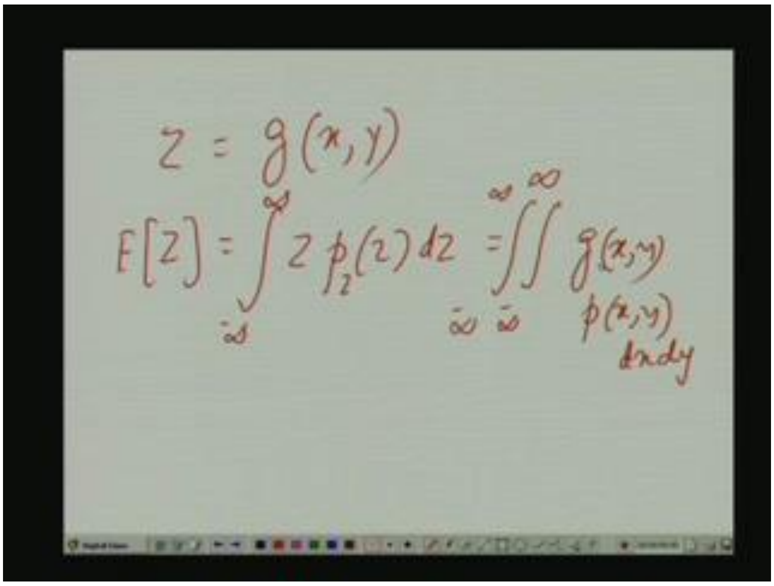

So, we have given this. Maybe… where g is a function of two variables – two random variables: x and y. Then, this will be nothing but… This will prove… The proof actually… That is, it is an expected value of g of x, y. The proof actually is purely along analogous lines. And thus I will show it by an example.

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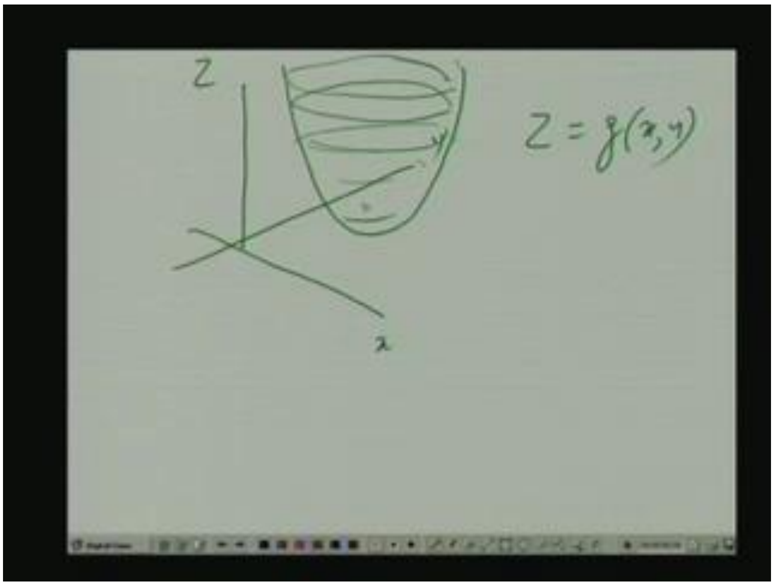

Suppose this is z-axis; this is x; this is y. And the function is like this. This is basically a surface. It is a bowel shaped curve. These are the contours actually. So, this is a surface, is given by z; that is, z, which is g of x, y. Actually, it is nothing but a function; where, for any x, y, the corresponding value of z lies on the surface of this bowel – bowel-shaped surface.

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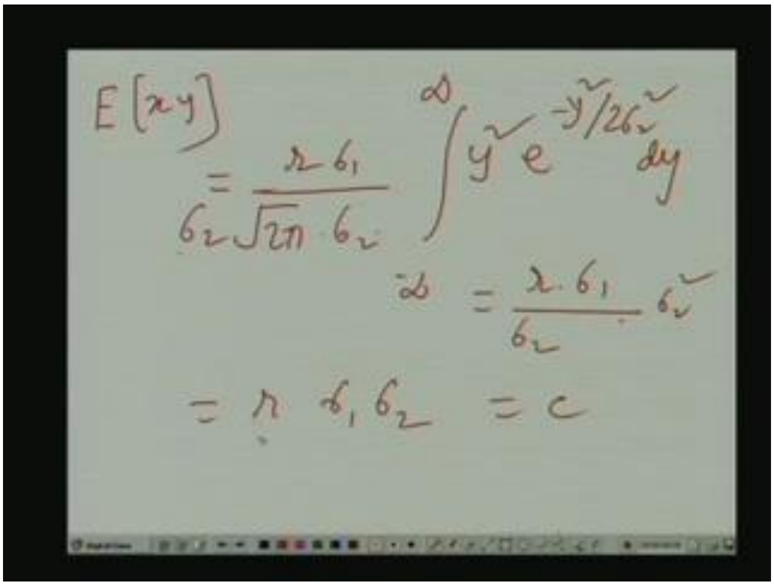

Sigma 2 square; that variance is nothing but sigma 2 square. In fact, I forgot to mention sigma 1 is corresponding to the sigma 1 square is the probability, is the variance of random variable x – sigma 2 square is the variance of random variable y. So, this will give rise to sigma 2 square. So, r sigma 1 as it is and one sigma 2 left out and sigma 2 square comes up. So, we get r times sigma 1 sigma 2. And what is e x y? It is same as c also, because means are – mean of x and mean of y were taken to be 0. So, co-variance is nothing but as we know, e of x minus mu x; but mu x is 0 times y minus mu y; mu y is 0. So, it turns out to be e of x times y only; which is c. So, c is equal to r sigma 1 sigma 2. So, what is r? It is nothing but c by sigma 1 sigma 2; which is the definition of our correlation coefficient. So, we now justify our previous assumption that, this constant r, which occurs in the joint density function for jointly Gaussian random variables is nothing but the correlation coefficient between those.

So, today, we will not proceed further. We will stop here. But, in the next class, we will consider joint moments; like earlier we have taken moments for single random variable. We are moving derivative functions; then you have characteristic functions and all that. Similarly, here also we will take the joint moment involving two variables. And then from that, we will proceed to joint characteristic functions. This will in the end take us to what is called central limit theorem and all that. So, that is all for today.

Thank you very much.