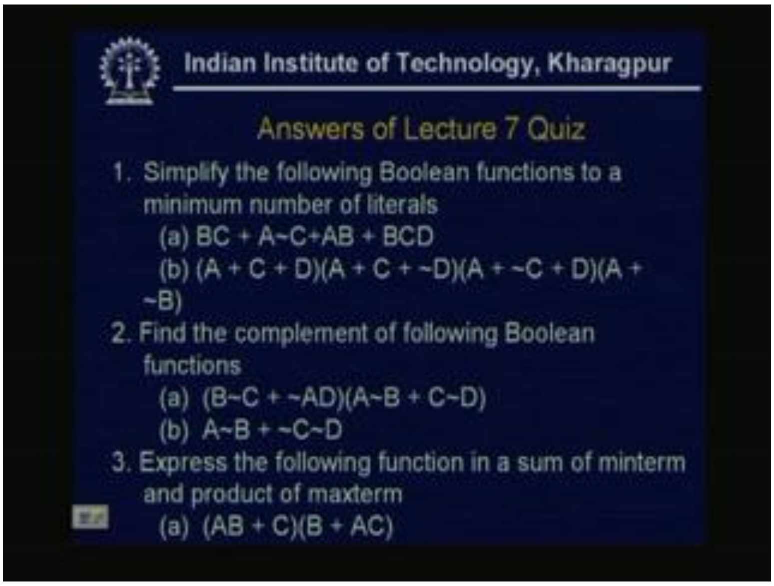
Digital Systems Design
Prof. D. Roychoudhury
Department of Computer Science and Engineering
Indian Institute of Technology, Kharagpur

Lecture - 08
Boolean Function Minimization

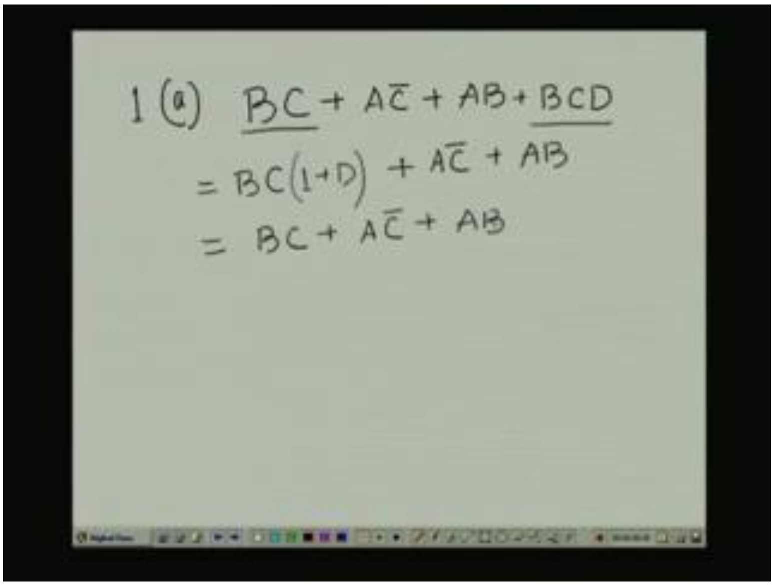
Today, we will read how to minimize the Boolean function, but before that first, we will discuss the answers of the previous lectures quiz.

(Refer Slide Time: 01:00)



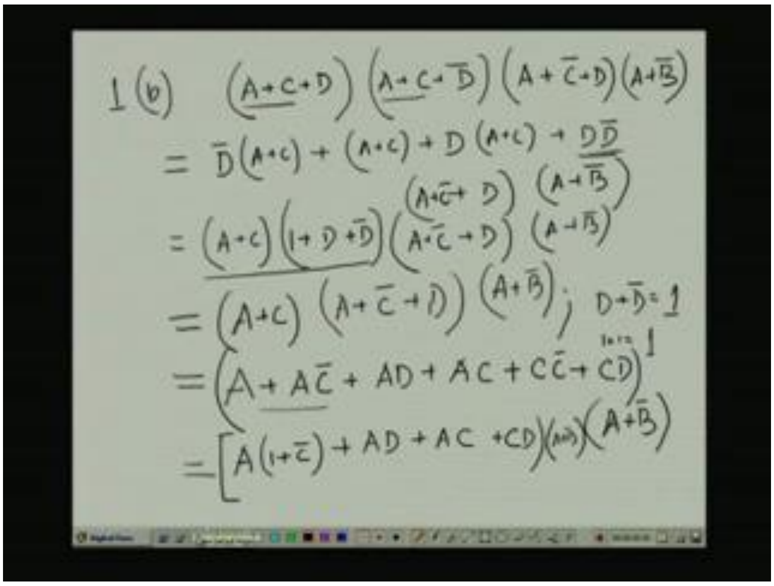
The first question was to simplify the following Boolean functions to a minimum number of literals. First one was BC plus AC bar plus AB plus BCD we will see first how we can simplify that.

(Refer Slide Time: 01:52)



The expression is the one a BC plus AC bar plus AB plus BCD, now if we take BC and BCD, and BC common then it will be 1 plus D and we keep, because no literals are common in between these two terms, so keep as it is. Now, 1 plus D is 1 from the basic postulates, so this becomes BC plus AC bar plus AB, this is simple situation. Now, we see the second one, now the second question was that A plus C plus D, A plus C plus D bar, A plus C bar plus D and A plus B bar, we have to simplify this equation.

(Refer Slide Time: 03:33)



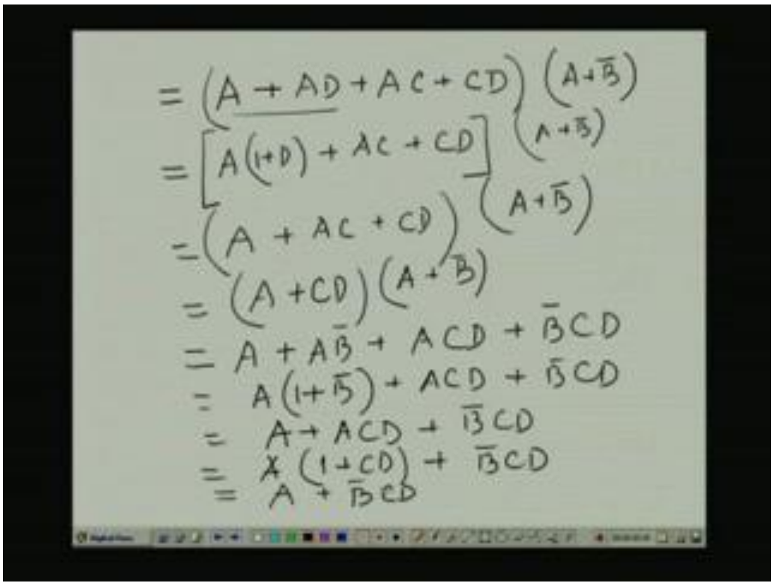
See again, if we take the example one b this is, A plus C plus D, then A plus C plus D bar, A plus C bar plus D and A plus B bar. Now, if we take the first two terms, then A plus C is common and D and D bar are different.

So, this will become if I apply the distributive law then this is D bar A plus C plus A plus C plus D into A plus C plus DD bar and the last two terms keep this is, as it is. So, this will be A plus C bar plus D and A plus B bar, now see that here from the first three terms, if I take A plus C common then it will be 1 plus D plus D bar and DD bar this becomes 0. So, DD bar equal to 0, then the remaining term will be A plus C bar plus D and A plus B bar.

Now, this becomes A plus C because, our D plus D bar is 1 from the basic postulates and 1 plus 1 is 1. So, this becomes this two terms simplified to A plus C, then the remaining terms A plus C bar plus D into A plus B bar. Now, again if we take the distributive law, then this equation becomes A plus AC bar plus AD plus AC plus CC bar equal to 0, then I write that thing CD this is one term and multiplied by A plus B bar.

Now, see from the first term this is a nice equation and, if I take two terms at a time, say A plus A plus C bar from here, if I take A common. Then this becomes 1 plus C bar and say I keep as it is, all the term AD plus AC plus CC bar becomes 0 plus CD and then, the A plus B terms A plus B bar.

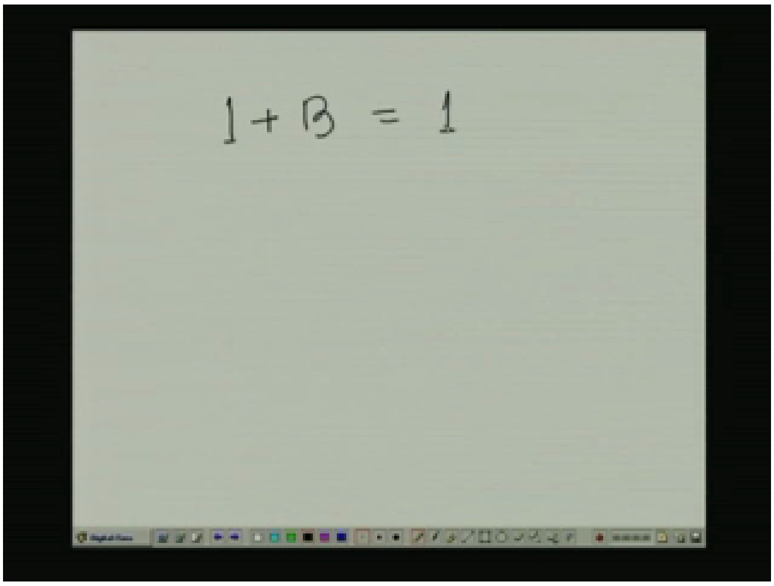
(Refer Slide Time: 07:55)



Now, see 1 plus C bar is 1, so this becomes A 1 plus C bar is 1, so this becomes A plus AD plus AC plus CD, this into A plus B bar. Now, again I will take A common, so this becomes 1 plus D plus AC plus CD A plus B bar. Now, from here again 1 plus D equal to 1, so this becomes A plus this term AC plus CD A plus B bar.

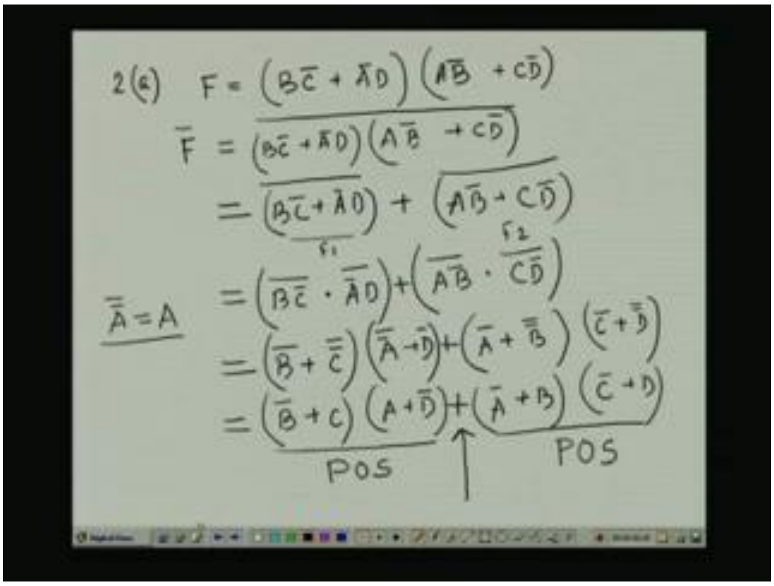
Now, applying the same rule A plus AC becomes A, 1 plus C equal to 1, so this becomes A plus CD, so this become A plus CD into A plus B bar. Now, if I again apply the distributive law, so this becomes A plus AB bar plus ACD plus B bar CD, now from here A 1 plus B bar plus ACD plus B bar CD. From here 1 plus B bar equal to 1, so again A plus ACD plus B bar CD. Now, again if I take A common, this becomes 1 plus CD plus B bar CD, now see 1 plus CD whatever this term become CD is this equal to A, so this becomes A plus B bar CD.

(Refer Slide Time: 10:24)



So, mainly for the, this simplification we have used the basic postulate that 1 plus B equal to 1, we have utilized this formula and we have simplified that equation. Now, we see that our second one, so this for the second one we have to take the complement of the Boolean functions. Now, one by one we say that how we can with the help of the postulates and the theorems, how we can take the compliment, here particularly the De Morgan's law we have to use.

(Refer Slide Time: 11:18)



So, we see, so the question number 2 a is BC bar, again let me see this is BC bar plus A bar D, BC bar plus A bar D into A bar D plus AD. Question number 2 a, BC bar plus A bar D into AB bar plus CD bar, say this is my function F, now I have to compute the complement. So, if I take the complement F bar then, it will be the whole bar; that means, BC bar plus A bar D into AB bar plus CD bar, the complement of the whole thing.

Now, if we apply the De Morgan's law or the Duality Principle, then what will happen, actually the two operators will interchange; that means, plus becomes dot, dot becomes plus and the complemented literals become the un complemented one and the vice versa. So, complement this thing, now we replace the two operators; that means, plus will be replaced by dot and dot will be replaced by plus and the complemented variable that will be replaced by the un complemented one and vice versa.

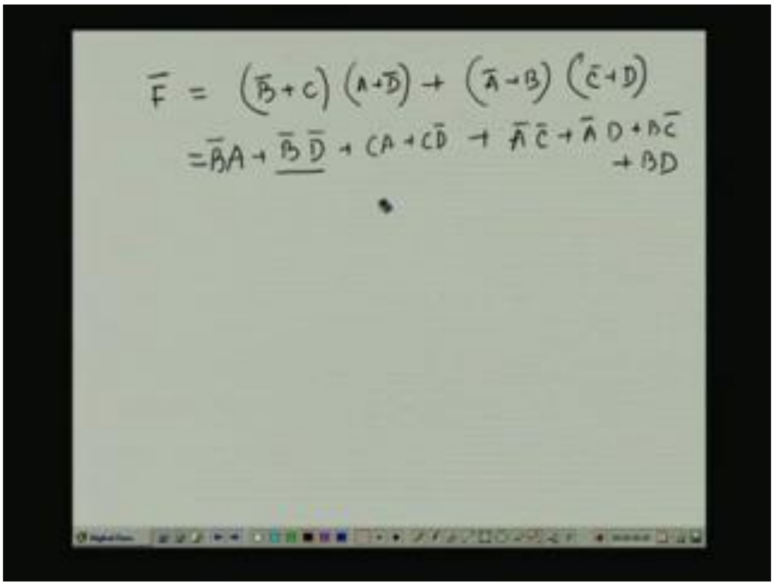
So, the first part that this becomes BC bar plus A bar D complement of that, this dot becomes plus and then again AB bar plus CD bar complement of that. So, if we take these as the F 1 function and this it is F 2, then again this is similar to F 1, F bar is F 1 complement and plus F 2 complement. Now, if we apply the same De Morgan's law, then what will happen that the first term becomes that BC bar complement of that dot A bar D complement of that, it will be the first term similarly this will be AB bar complement of that dot CD bar complement of that.

Now, we apply the same rule and this becomes B bar, this dot becomes plus and this become complement of C complemented of that thing, into dot that again A bar bar plus D, similarly this term becomes A bar plus B bar bar dot C bar plus D bar. Now, see that our complement of A and if we take complement of that thing; that means, A bar bar equal to A itself, this is our involution principle.

So, this becomes B bar plus C similarly this becomes A plus D, A plus this is actually complement A bar bar plus D bar, A plus D bar, this becomes A bar plus B, this becomes C bar plus D. And actually, we missed the plus here this is 1 plus it remains at it is, now further we can simplify this two terms because, if I want some standard form; that means, either it will be sum, sum of product forms or the product of sum forms.

See, it is neither the sum of product forms or the product of sums because, if I take only the first two parts, this part this becomes the product of sums similarly this becomes the product of sums. This is one POS this is also one POS, but this is 1 plus exists here, so this is not any standard form. So, if I want to make this as a standard form, then again we have to simplify further, we have to simplify it further.

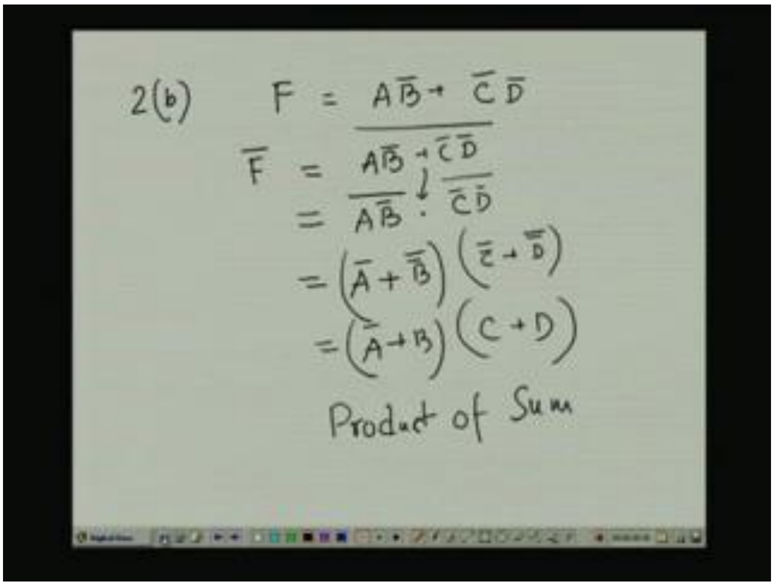
(Refer Slide Time: 17:03)



So, this becomes a complement becomes, then if I put again the term B bar plus C A plus D bar plus A bar plus B and C bar plus D. Now, if we this, apply the distributive law, then this becomes B bar A, plus B bar D bar, plus CA, plus CD bar, similarly this becomes A bar C bar, A bar D, plus BC bar, plus BD. Now, this becomes B bar D, B bar D bar, plus AD bar, so these two terms I keep as it is and then B bar A.

If I take this term again, this is a sum of product forms, this is a sum of product forms, but this can be further reduced, if I apply sum of the minimization rules and it can be further simplified. Now, we see the second one, that 2 b that AB bar plus C bar D bar, we see that how it, it can be, we can take the complement.

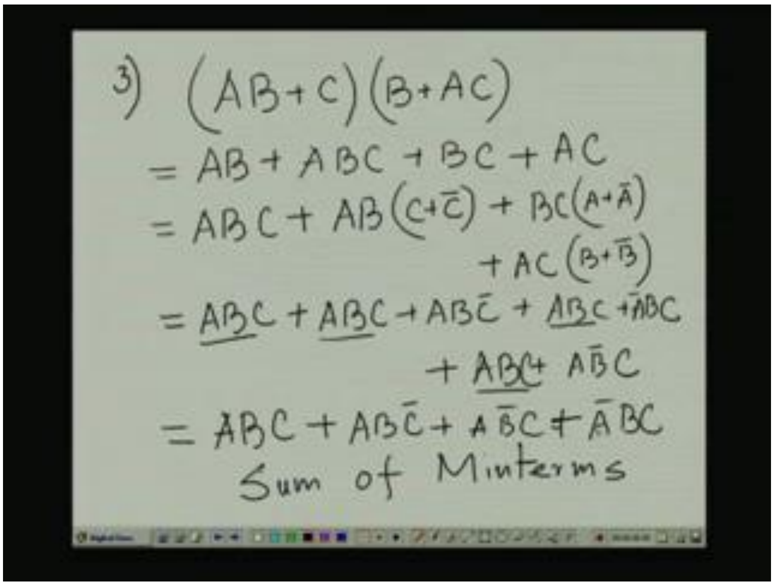
(Refer Slide Time: 19:44)



So, this becomes the 2 b is, AB bar say F equal to AB bar plus C bar D bar, similar way if I take the complement, then it will be AB bar plus C bar D bar whole complement. Then this becomes AB bar complement of that plus, this plus will be replaced by dot and then C bar D bar complement of that. Then this is equal to A bar and dot becomes plus and this is B bar bar then this becomes C bar bar plus D bar bar.

Now, this is A bar plus B into C plus D, now see this is my product of sum form, is a product of product of sum. Now, we see the question number 3, question number 3 is express the following function in a sum of min terms and product of max terms. In the last lecture we have defined the min terms and max terms and the sum of min terms and product of max terms. So, now we have to represent AB plus C in to B plus AC, in that way see how we can how we can do that thing.

(Refer Slide Time: 22:00)



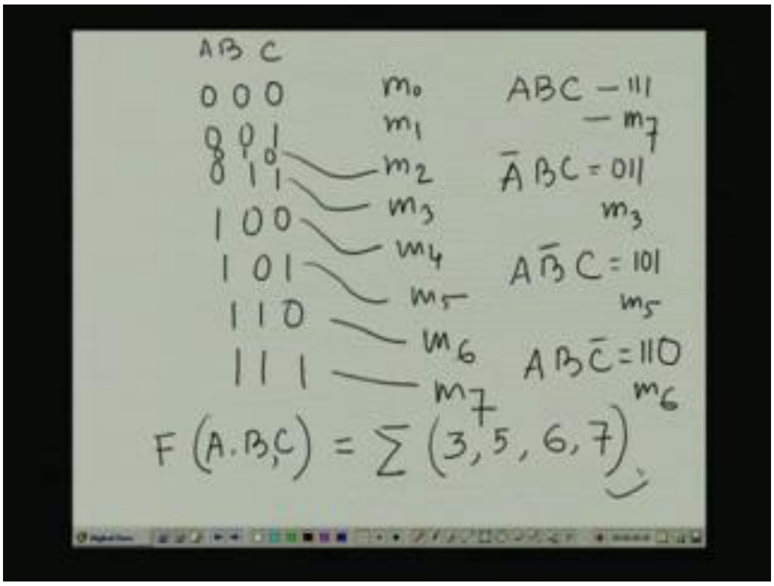
So, this is question number 3 and the problem is that AB plus C into B plus AC, first we take the sum of min terms, now again here we have to apply the postulates and the theorems. Now, first if we apply the distributive law, then it becomes AB plus ABC plus BC plus AC this becomes AC, now I will keep ABC as it is. Now, see AB I can write AB into C plus C bar because, C plus C bar is nothing, but one similarly when I take BC I will add A plus A bar and ACB plus B bar.

So, just notice that when I am taking a term and I am adding some term or taking the product, then I am taking the remaining literal which is not exists in the term. See here AB, so the remaining literal C was not here, so I introduced C in such a way that actually C plus C bar, C plus C bar is 1. Similarly, A plus A bar is 1, B plus B bar is 1, so I have when A was not there then BC into A plus A bar similarly AC into B plus B bar, so that I have introduced.

So, how it becomes then this is ABC plus, now again if we take ABC plus ABC bar plus this is again ABC plus A bar BC plus ABC plus AB bar C. Now, see all ABC this becomes ABC because, x plus x equal to x, so again from the first postulate it becomes ABC and the remaining terms are will be there ABC bar plus AB bar C plus A bar BC. So, this is my sum of min terms because this is a three variable function, it was a three variable function, so these are the min terms, the sum of min terms.

Now, we see how we can write this thing as a sum of product of max term, now before that another thing, another way we can represent this thing, say this terms as ABC, ABC bar AB bar C A bar BC.

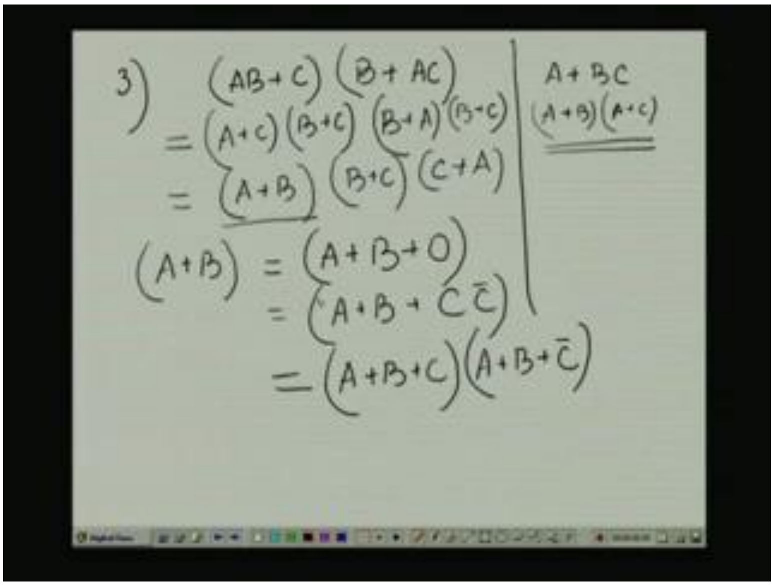
(Refer Slide Time: 26:00)



So, I can tell that if the min term wise, we can write say 0 0 0, 0 0 1, 0 1 1, 1 0 0, 1 0 1, 1 1 0 and 1 1 1. So, last day we defined this as the terms as m 0, m 1, another term was here 0 1 0. So, this is m 2, this is m 3, m 4, m 5, m 6 and m 7. Now, if I take the variables are ABC, then actually ABC means that my m 7 terms, ABC means 1 1 1 means this is actually m 7.

Now, A bar BC, A bar BC means 0 1 1, so this is nothing but m 3, similarly, AB bar C means 1 0 1, that is m 5 and A B C bar is 1 1 0, that is m 6. So, in the function given, the terms present are m 7, m 3, m 5, m 6, so now, we can write in this way, that the function is F of A B C and this equal to as this is a sum of min terms, so this is sum of that 3,5,6 and 7. So, I can also represent in these form, so this is also the sum of min terms form, now we see the product of max terms.

(Refer Slide Time: 28:36)



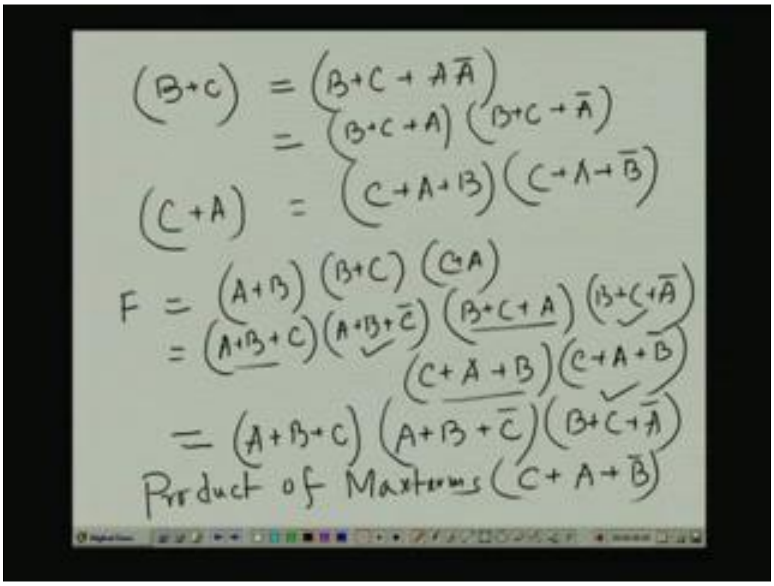
So, if it is a product of max terms, then again question number 3, that AB plus C into B plus AC, now see here, I can AB plus C, again I can use some theorem. That actually this becomes we remember that A plus BC means, if we apply the distributive law, say this becomes A plus B dot A plus C. Now, if I write apply this theorem here, then I will be getting this is A plus C into B plus C.

Similarly, the second term becomes two product terms B plus A and B plus C, now B plus C is same, so this becomes A plus B, B plus C into C plus A, now I have to represent this as a product of max terms. Now, we see, because already this are this come as a product of three terms, so now, we see that what will be my, the first term say A plus B.

Now, what I can do and in product term, because C is not there, the literal C is absent in the first term, so I have to introduce C. So, how I am introducing, A plus B now here as A plus 0, as it this becomes 0. So, now zero I can replace that A plus B plus C C bar because C was not here, so I have to introduce C and in the, this way I can take this thing.

So, now again if I apply the same rule, then this becomes A plus B plus C dot A plus B plus C bar. So, the only the first term becomes the product of two terms, similar way I can see B, what the B plus C and C plus.

(Refer Slide Time: 31:17)

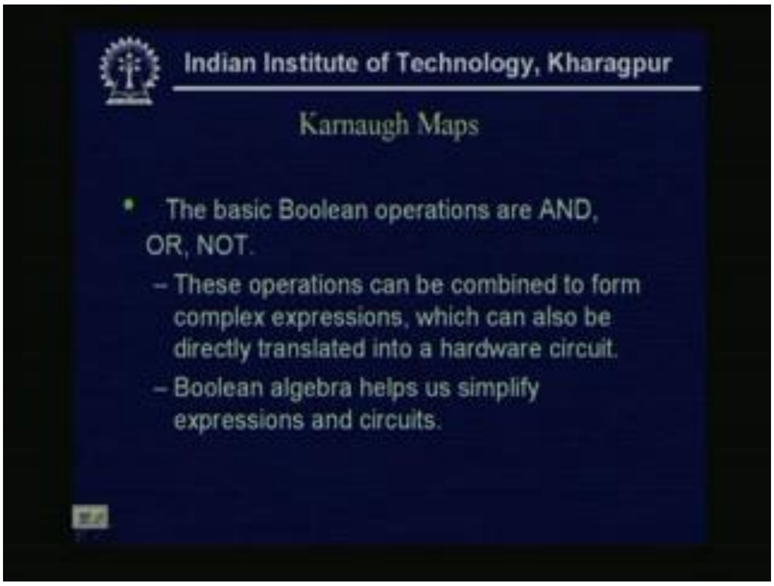


So, how B plus C become B plus C as here A is absent, so I can introduce A thus in the same way as A bar equal to 0. And this becomes B plus C plus A into B plus C plus A bar, similarly my C plus A becomes C plus A plus B into C plus A plus B complement. Now, my F will be that A plus B into B plus C what was there earlier, C plus A this becomes A plus B plus C, A plus B plus C bar this is for the first term.

For the second term B plus C plus A B plus C plus A bar, for the third term C plus A plus B C plus A plus B bar. Now, see here A plus B plus C, A plus B plus C, A plus B. So, x dot x equal to x itself, so again from the postulate, the basic postulates I get A plus this becomes A plus B plus C and the remaining three terms will be there.

So, this becomes A plus B plus C bar, then B plus C plus A bar and this becomes C plus A plus B bar, so this is my product of sum forms, product of max terms. So, we have discussed the answers of the last day lectures quiz, now today we will start that, today’s lecture. Now as already we have seen that, we will see the Boolean function minimization today.

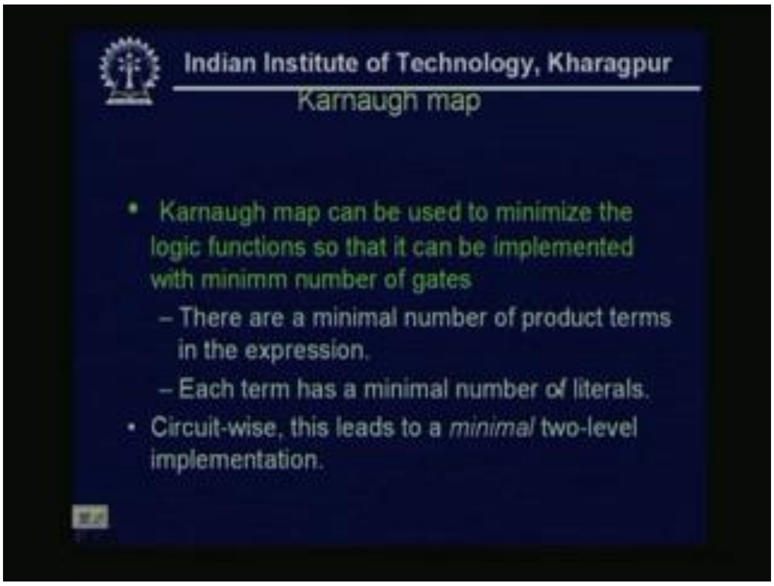
(Refer Slide Time: 33:37)



Now, for that first we have to discuss that Karnaugh maps, and before giving that actual definition again we will recapitulate that the basic Boolean operations are AND or NOT; that means, our complement plus and the dot. Now, these operations can be combined to form complex expression just now we have seen many of the, this type of expression which can also be directly translated into a hardware circuit, that also last class we have discussed.

Now, Boolean algebra helps us simplify expressions and circuits, the theorems and the postulates we have read, you seen that we can always, but as last day I mentioned, that there is not clear cut rule that we have to apply in different way. And today's that the quiz questions we have solved there we have seen, that applying those theorems and postulates, we can simplify those Boolean expressions. Now, today we will read that some rules or some methods and this is called the Karnaugh map.

(Refer Slide Time: 35:06)

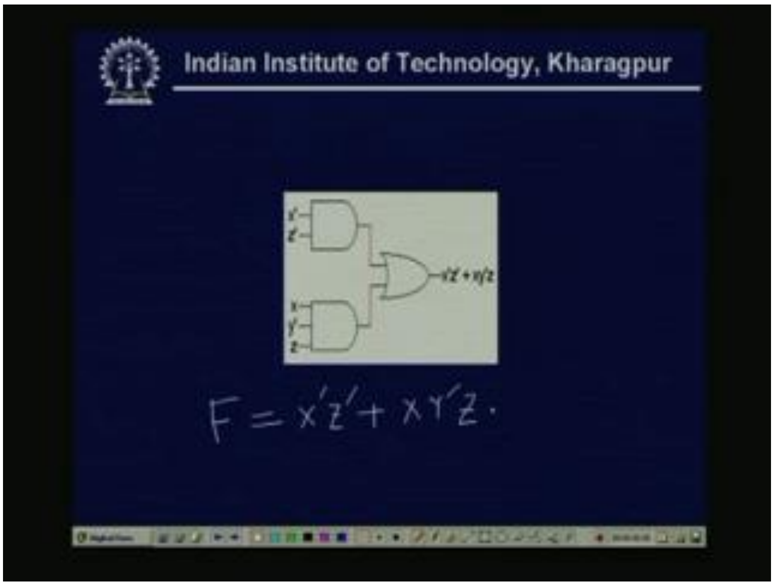


First we define the Karnaugh map, so Karnaugh map can be used to minimize the logic functions, so that it can be implemented with minimum number of gates. So, we will be seeing that it is a graphical representation of the Boolean expressions. If it is in standard forms either it is a sum of min terms or product of max terms, because always the product of max term can also be represented by sum of min terms.

Then, graphically if we represent then there can be there is some rule that we can simplify this expression with minimum number of literals and the operators. So, that with minimum number of gates the function can be realized, now there are minimum number of product terms in the expressions. So, what do you mean by this minimization using Karnaugh maps that minimal number of product terms and each term has a minimal number of literals.

So, just now what we discussed, just applying the postulates and the theorems, then we will, now today we will be seeing a well defined graphical method by which we can minimize the Boolean expression. And if we consider that circuit, because always that we have seen that, if it is a given a Boolean expression, some truth table is possible and if truth tables are possible then always AND OR realization is possible. So, circuit-wise, this leads to minimal two level implementation and that is why, this is very important in the digital circuit design.

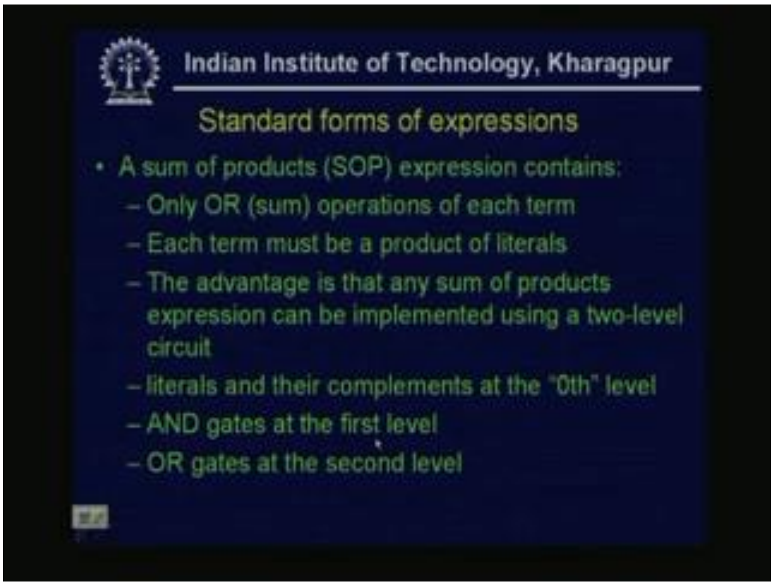
(Refer Slide Time: 37:13)



Now, see that here there is one circuit, simple circuit the two AND gate and one OR gate, and the input of the first AND gate is the x, so what this function realizes. The output Z realizes x dash x dash z dash means the complemented of x and z plus x this is a three input AND gate.

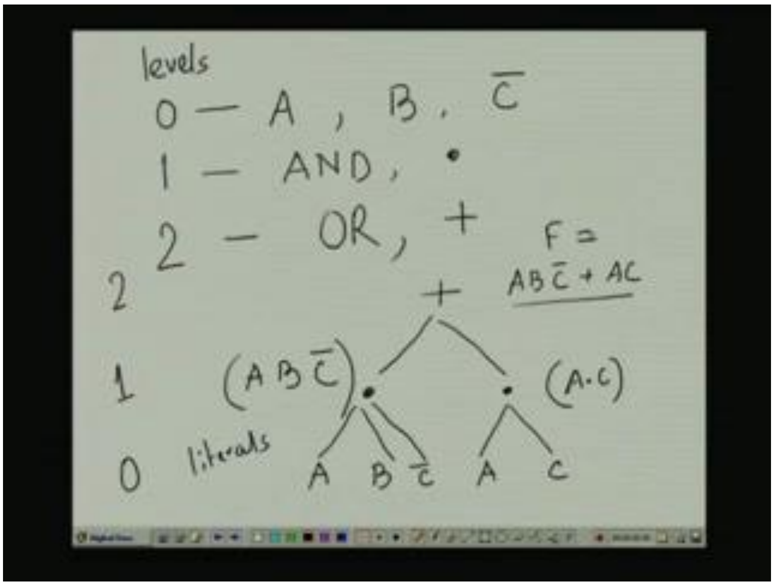
So, x y dash, then z is also a one variable, so we change the output, say this is a function F simple function F, so F equal to x dash z dash plus x y dash z. Now, already we have read last class that standard forms.

(Refer Slide Time: 38:21)



Now, in the standard forms only OR operations of each term exist and each term must be a product of literals, what we defined as the sum of product, now what is the advantage. The advantage is the sum of product expression can always be implemented using a two level circuit, just now one example we have seen. Then, see this is a standard form, this is a sum of product x dash z plus x y dash z, always this can be represented by a AND OR gates. Now literals and their complements at the zeroth level and AND gates at the first level OR gates at the second level.

(Refer Slide Time: 39:28)

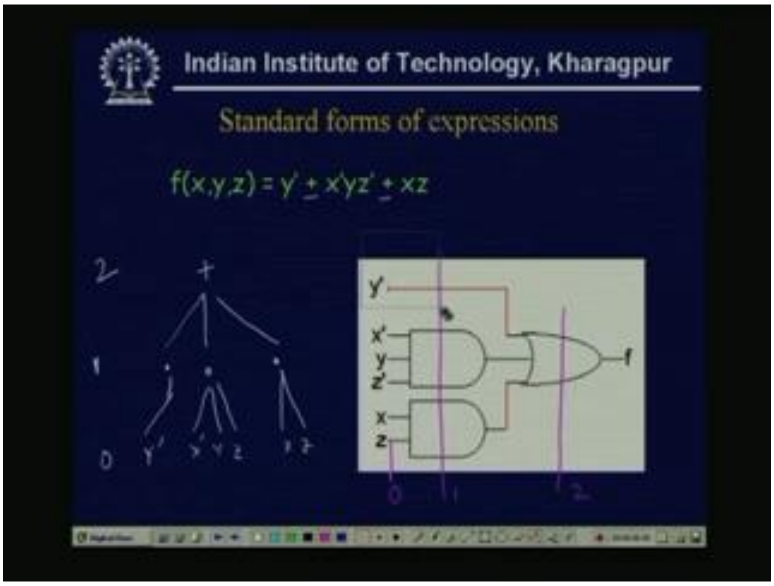


So, if we see some hierarch, say as if we are thinking that say that literals A B C bar, they are at zeroth level, say these are my levels. And the first level say as the AND gates are there or dot operation, in the second level as if OR gates are there; that means, plus, or if hierarchically. We do, then as the this is my and these are my, say it can be not always this, this can be more than two branches, say these are my literals.

Now, see here A B, so I take another one, say C bar here I take A C, so what will be, what it will represent at this level, this becomes actually A dot C, this becomes ABC bar. Now, in the, so this is in the zeroth level, this is the first level and this is in the second level, second level this becomes ABC bar plus AC. So, this is my function F, F equal to ABC bar plus AC, so this is the advantage that always we can represent in this way.

And if we represent in this way sum of product form, then always it can be implemented or it can be realized by the two level AND OR logics. Now, again another standard forms of expression, another example we take.

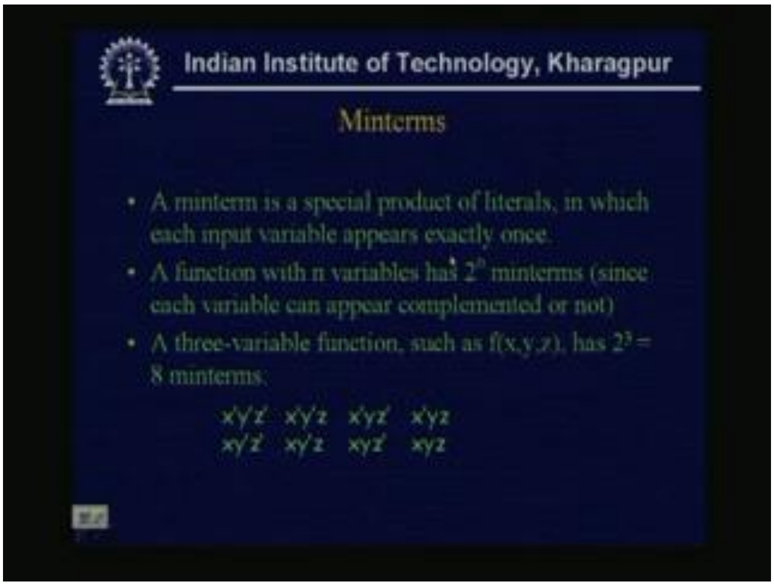
(Refer Slide Time: 41:48)



This is a three variable function x y z and the function is y dash plus x dash y z dash plus x z, see that or just now that example I have given. See, if I think that in the zeroth level, that it is y dash is there, then x dash y z and x z. So, in the first level in the first level, it is AND, see it is only one literal, so as the, it is AND gate with itself, so three ANDs. So, this is my first level, now again these are in the second level this two OR operation is there in the second level, so this is my hierarchy.

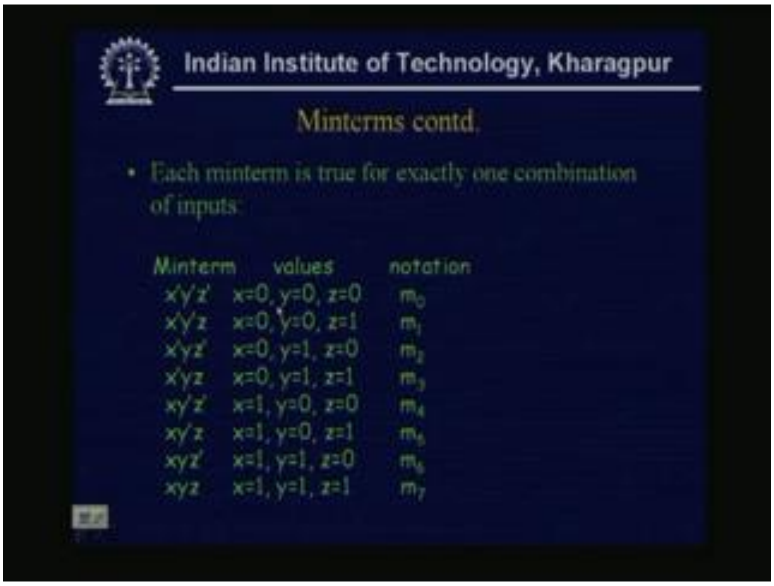
So, if I do this actually the inputs are zeroth level so; that means, its y dash z dash y z, x z there are at the zeroth level, these ANDs they are at the second level, these are the second level. And the third level is that one OR. So, that means, so these are my first level, these are my zeroth level literals, these are my first level. See here only one literal y dash is here, three literals are there, two literals are there and this is my second level OR. So, this is the way I can represent that thing.

(Refer Slide Time: 44:29)



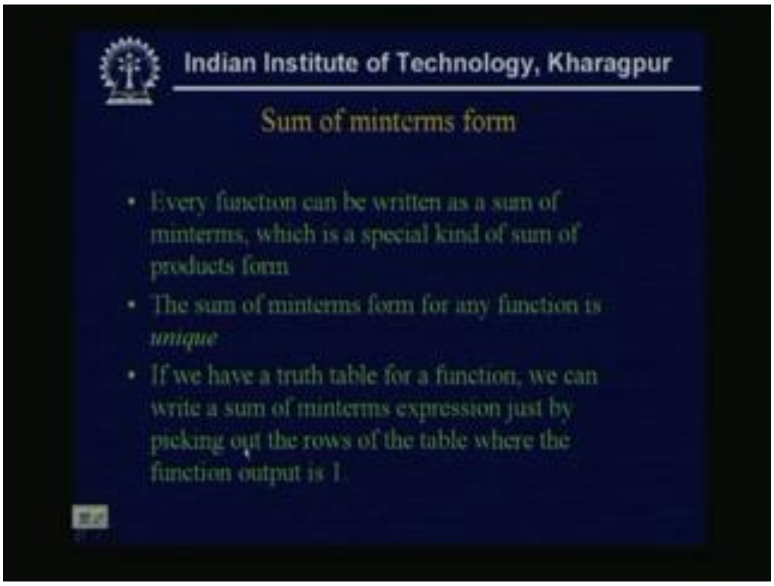
Now, already we have defined the min terms, that this is a special product of literals in which each input variables, appears exactly once. So, a function with n variables has two to the power n min terms, already last day we have read, so three variable function such as f x y z, this is the these are the min terms. So, min terms are A dash B dash C, A dash B, x dash y dash z dash, x dash y dash z, x dash y z dash, x dash y z like all the terms these are the all possible min terms.

(Refer Slide Time: 45:09)



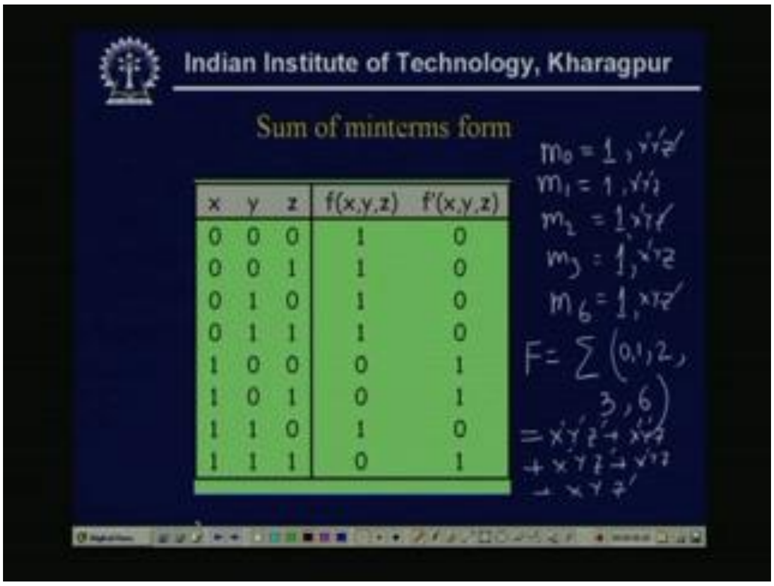
Now, already we have seen, that these if these are the min terms means what do we mean, last day we have discussed that this is the convention, that this is the that x if x equal to 0, y equal to 0, z equal to 0; that means, one literal is 0, then it will be actually complemented. So, x prime y prime z prime or x dash y dash z dash means the three variables are zeros. And these the decimal equivalent of these values is the subscript, so this becomes m 0, similarly we can do this thing.

(Refer Slide Time: 45:51)



Now, the sum of min terms that every function can be written as a sum of min terms, which is a special kind of sum of product form. And the sum of min terms from any function is unique, so this is one property that only in one way I can do this thing. If we have a truth table for a function, we can write a sum of min term expression, just by picking out the rows of the table, where the function output is one, see this is one example.

(Refer Slide Time: 46:24)

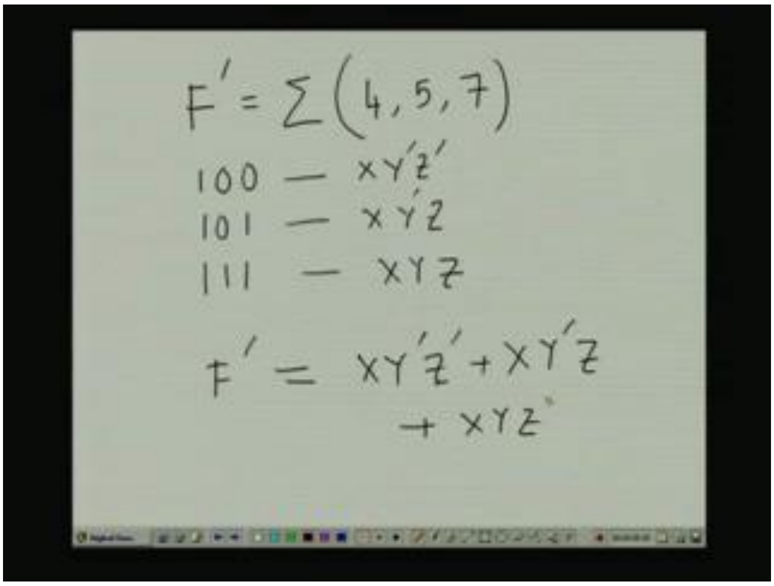


So, these are the min terms that all possible values are taken, this is truth table x y z and these are the output, see there are two outputs the f function and f complement of. So, if say when that m 0 terms, means that x y z all are 0, then it becomes 1; that means, my if I write, that terms then this is my m 0 equal to 1, then m 1 equal to 1, m 2 equal to 1, m 3 equal to 1 and m 6 equal to 1.

So, as it is a sum of min terms, I can write my function is sum of 0 1 2 3 6, in this way we can write. And this becomes that if it is a 0, zeroth term means actually x dash y dash z dash, m 1 means because this becomes x dash y dash z dash, this 1 means this is x dash y dash z, 2 means this is x dash y z dash, 3 means this is x dash y z and 6 means this is x y z dash.

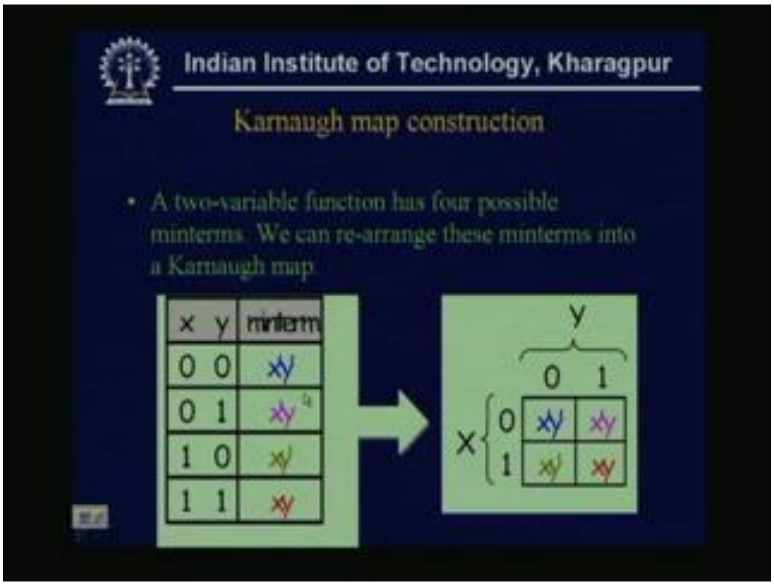
So, as this is a sum of product form, I can write all the OR of this thing plus of this thing. So, this becomes x dash y dash z, plus x dash y z, y z dash, x dash y z, plus x y z dash, there are five ones in the f function, so this will be the thing. Now, if we take the f complement accordingly if we take the f complement, see here only m 4, m 5 and m 7 exist.

(Refer Slide Time: 49:27)



So, if I, m 4 in the expression, see F dash will be sum of 4, 5, 7, so what will be the terms because, 4 means 1 0 0, so this becomes x y dash z dash. 5 means 1 0 1, this becomes x y dash z and 1 1 1 means x y z. So, my F dash becomes x y dash z dash, plus x y dash z, plus x y z.

(Refer Slide Time: 50:23)



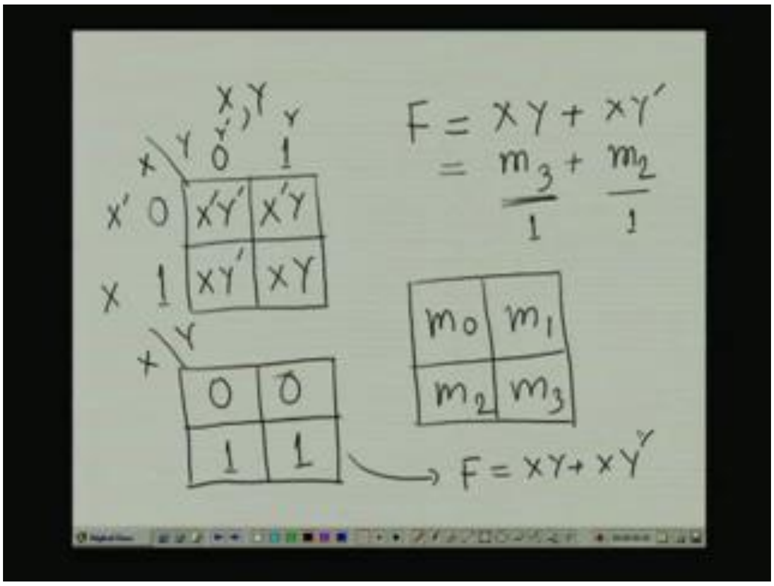
So, now we see how we can construct the Karnaugh map because, we defined that this is a graphical representation of where we can put that Boolean expression, if the expression is represented as a sum of min terms. So, a two variable function has four possible min terms, say these are the four possible min terms x bar y bar, x bar y, x y bar, x y, now we can rearrange this min terms into a Karnaugh map. Now, see that if these are all possible combinations or we can tell this is a rearranging of truth table, this is the truth table that 0 0 0 1 1 0 1 1. And if the, it will be 1, it will in the second column it will be 1, if the function is such; that means, that particular min term will be 1 for this inputs.

Now, say this is a table, now this table I can rearrange like that, say this is my x is x is taking for 0 or 1, value x can take 0 or 1 value. So, now these table I have rearranged as the; that means, this these are four combinations I have kept two by two form, say this is a two by two table, instead of this table I am taking two by two. So, x can take either 0 or x can take either 1.

Similarly, see here y can take either 0 or y can take either 1, so y can take either 0 or y can take either 1. Now, see when x y is 0 0, the min term is x bar y bar. So, here in this table, say x is 0 y is 0; that means, x bar this min term actually x bar y bar means m 0, this is my m 0. And if it is x equal to 0 y equal to 1; that means, my 0 1 means x bar y, so this becomes x bar y.

Similarly, if x equal to 1 y equal to 0, then this becomes x y bar min term, so this is x y bar, now if it is x equal to 1 y equal to 1; that means, 1 1 this becomes x y. So, actually these are the positions of the min terms for all possible combinations, now we are at the, this is my Karnaugh map. So, this is nothing, but the rearranging of the truth tables, just to keep the min terms, otherwise the function becomes 1, now we can easy, what we can do, that we can also write in this way.

(Refer Slide Time: 54:14)



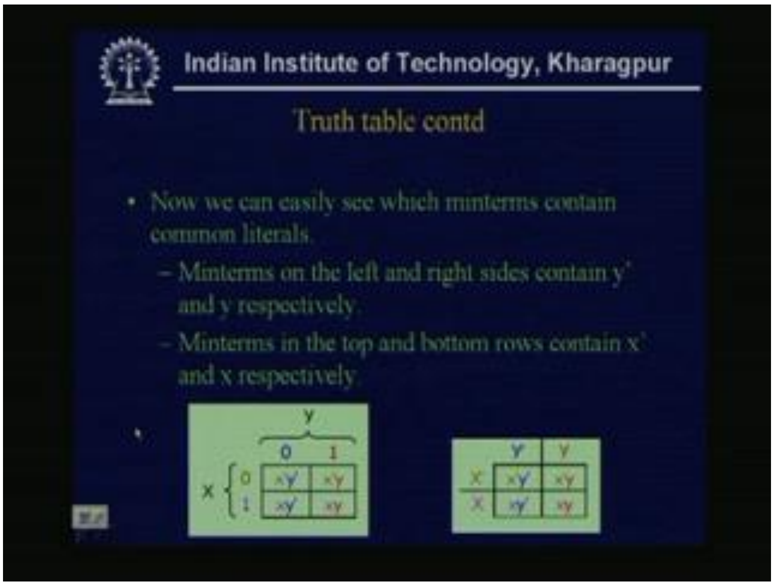
Say, that if it is a two variable again say x and y, this is two variable function, so how we are the Karnaugh maps we are writing because, x can take two values either 0 or 1. Similarly y can take, so if you put y here and we can put x here, then y can take also 0 and 1 value. So, just now what we have seen actually, this is my y bar, this is my y, this is my x bar, this is my x.

So, this becomes x bar y bar, this is the position of the min term x bar y bar, this is the position of the min term x bar y, this is the position of the min term x y bar and this is my x y. Now, say I have one function F equal to say x y plus, say x y bar so; that means, what we have, this is first this is a sum of product form.

So, the min terms are, this is x y means 1 1 means, this is m 3 and this is 1 0 means 2; that means, m 2 these two min terms are there. Now, how this Karnaugh map will be there then. Say actually this is the position of m 0, this is the position of m 1, this is the position of m 2, this is the position of m 3. So, here we have seen that the product term; that means, this product term or the min term for this will be 1 and the function here it will be 1.

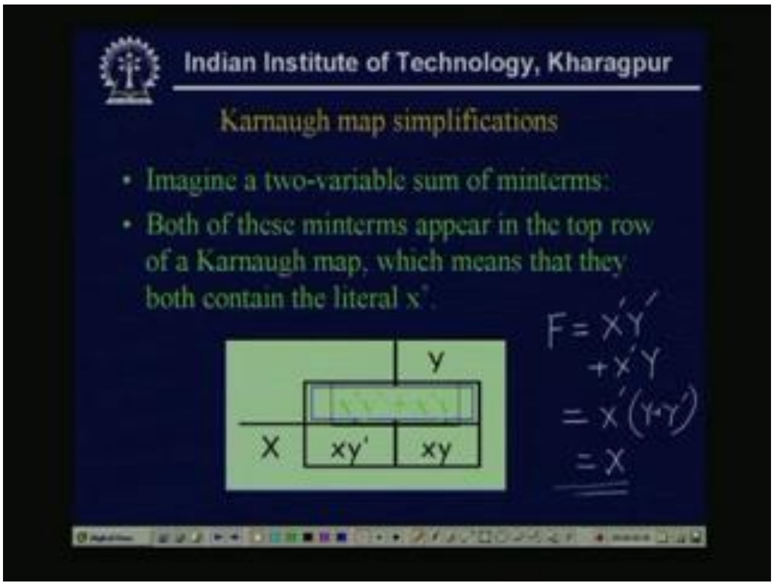
So, that means, m 3 and m 2, that last two will be 1 1 and remaining portions will be 0 0. So, this is my, this is the representation of this function. So, this is the Karnaugh map representation of this function, F equal to x y plus x y bar.

(Refer Slide Time: 57:30)



Now, we can easily see which min term contains common literals, see min terms on the left and right hand side contains y bar and y, just now we have explained. And min terms on the top and bottom rows are x bar and x, so this is the just now we explained that these are the positions of the min terms. Now, how we simplifier with this simple example, we see that how we can simplify this thing.

(Refer Slide Time: 57:52)



So, again that two variable function if we take and see that the four terms or the four min terms are x dash y dash and x dash y and x y dash and x y. So, both of this min terms appear in the top position; that means, if I take a function F. So, this becomes x dash y dash plus x dash y, and see here if we can take x dash common then y plus y dash becomes 1 and so, this becomes x. So, in this way I can, minimize that function. So, today we will finish here and next we will discuss that, how we can draw the Karnaugh map for the large number of variables or large number of literals.

Thank you.