Adaptive Signal Processing
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Lecture - 41
Singular Value Decomposition (Continued)

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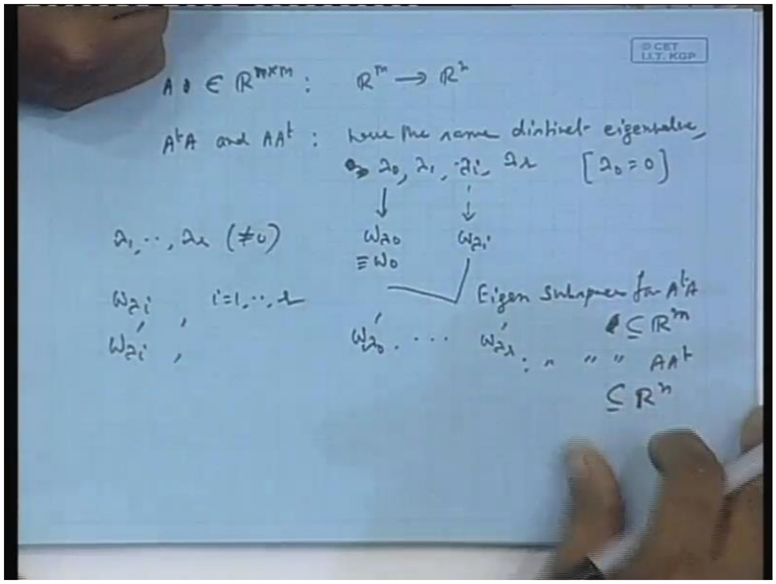

Let us see, what we done so far. A was a matrix of the size, n cross m. It was mapping of Rm to Rn; we have seen. We have seen all these. Then, we have also seen that A transpose A, if we have seen many things, I am only narrating a few, which we did in the end of towards the previous class, because I cannot go and narrating all the properties. A transpose A and A A transpose; they have the same distinct eigen values. Say, 0; I am keeping the position of 0, in fact, I would call it lambda 0; lambda 1 dot, dot, dot, say lambda r and lambda 0 could be 0.

Corresponding eigen subspaces w lambda 0; which I denote as w 0 and in general, if it is lambda I, this is w lambda i. These are eigen subspaces. That is, subspaces for A transpose A. Of all A transpose A, it takes a vector from this, Rn cross cross m, the eigenvector also is here. So, corresponding eigen subspaces also is here. So, this belongs to, this is all, these subspaces are part of Rm. Our notation was w lambda 0 prime, dot, dot, dot, w lambda r prime; these are eigen subspaces for A A transpose. These are contained in Rn, and what we have shown that the eigen values are same; the distinct eigen values are same, both for A transpose A and A transpose, and not only that, we are showing actually, that if you consider say lambda 1 to lambda r, that is non zero eigen values. Consider any w lambda i, that is, i equal to 1 to r.

Consider the corresponding w lambda i prime, then both have same dimensions, which means, for any non zero eigenvalue lambda i, the multiplicity is same, whether it is an eigen value for i A transpose A, or A A transpose. Those two dimensions are same. That is what we are trying to prove last time, and that is where we stopped. May be, we take it up now.

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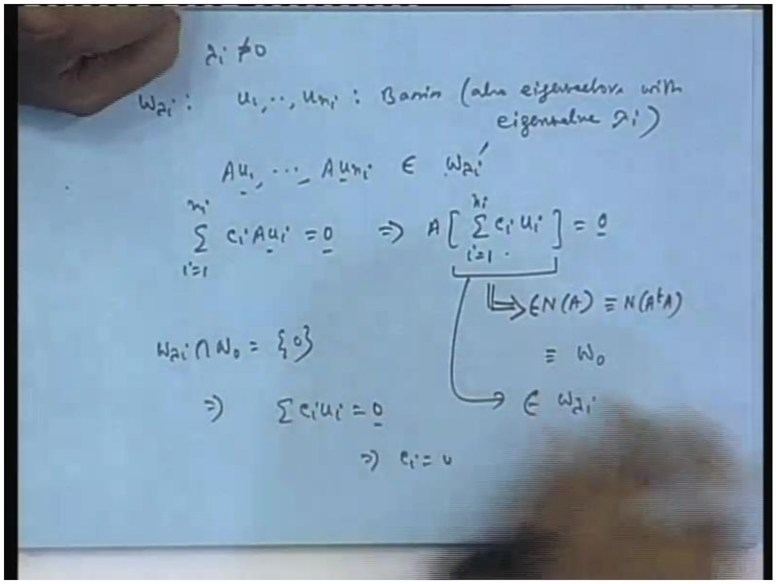

Suppose, I consider lambda i, a particular lambda not equal to 0, w lambda i say find out a basis say u ah 1, dot, dot, dot, uni, basis of these. But there also eigen vectors, because they belong to W lambda I, eigenvectors with eigenvalue lambda i. Then, we have seen that A u 1, dot, dot, dot, A un i, they all belong to, because they are all then eigenvectors of A A transpose, with the same eigen value lambda i. That means they all belong to W lambda i prime, because if u 1 to uni, that the eigenvectors of A transpose A with eigen value lambda i, then Au 1 to Auni; they are the eigenvectors of A A transpose, but with the same eigen value lambda i. So, they belong these, belong here. In the last time we proved this and let me prove it again. This, if u 1 to u n i; they are linearly independent, they are, because they following basis, then Au 1 to A u n i; they also are linearly independent, which is easy to prove.

Suppose, I form a combination like this, i equal to 1 to ni and equate it to 0; you can easily write it as A ci ui, equate it to 0; that means, this vector belongs to null space of A. This vector belongs to the null space of A. Null space of A is same as null space of A transpose A, we have seen it already, null space of A is is same as null space of A transpose A, which is also same as the eigen subspace with 0 eigen value; that is w 0. But since, lambda i not equal to 0, we have W lambda I, intersection with W 0 is only 0, because lambda is non zero and we already know that eigenvectors, belonging to different eigen, distinct eigen values, are linearly independent of the respective subspaces; they are mutually disjoint; they only intersect at origin; that is at 0.

Since, lambda has taken to be non zero, this is 0, and look at this vector summation; ci ui. Since each ui belongs to W lambda I, this summation belongs to w lambda i. But at the same time, since, A times this summation is 0, this also belongs to W 0. Whereas, this belongs to this, also this belongs to W lambda i. That means, this belongs to the intersection between W 0 and W lambda i. But that is only 0, which means, this vector has to be nothing else, but the 0 vector, and you went to uni; they were linearly independent; this only implies ci equal to 0. That means, Au 1 to Au ni; they are also linearly independent. Question is, do they span this entire space? They of course span a subspace of W lambda i prime and the dimension of the subspace is ni.

But if they also span the entire W lambda i prime, then I can easily say that W lambda i prime has the dimension ni, which is same as the dimension of W lambda i, which means multiplicity of lambda i; both for A transpose A and both for A A transpose are same. So, I have to see whether these vectors Au 1 to Auni, they span the entire subspace W lambda i prime or not. To see that, let let us take any vector belonging to W lambda i prime. We will see, whether it can be written as a linear combination of this ni number of vectors.

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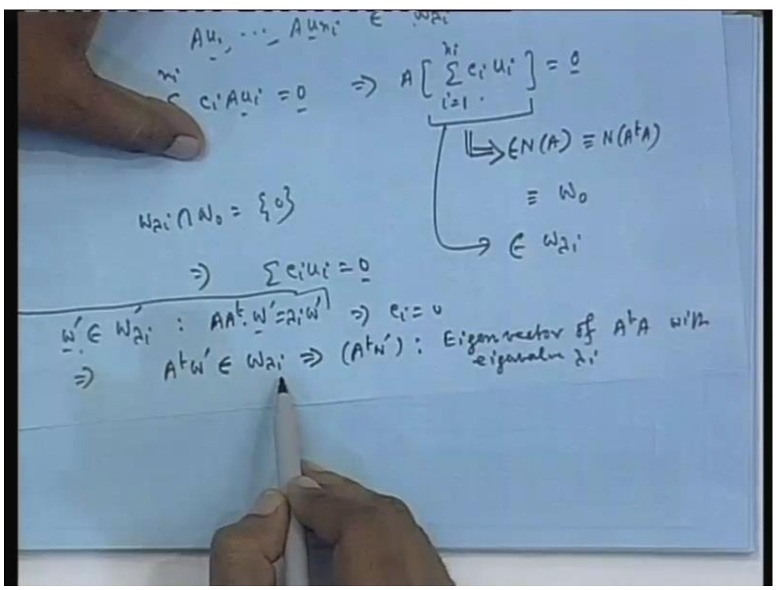

Llet us take a vector say W prime belonging to W lambda i prime. If W prime is here, it is an eigenvector of A A transpose with eigen value lambda i, that is A A transpose W prime is lambda i W prime, isn't it? This means the W prime actually, is an eigenvector of A A transpose, that is why it is in the eigen subspace. That is, these two were since an equivalent, the W prime belongs to this eigen subspace means, the operator A A transpose has W prime as one of the eigenvectors, with the eigen value lambda i.

But we have also seen, that if W prime is an eigenvector of A A transpose, then A transpose W prime will belong to W lambda i, meaning, A transpose W prime is an eigenvector of A transpose A with eigen value; these are all done in the class, previously, same eigen value lambda i. This means, if you pick up any vector W prime, belonging to these eigen subspace W lambda i prime, and apply the operator A transpose over it, what you get is a member of W lambda i. Therefore, it can be written as a linear combination of u 1 to uni, is it not? Because they form a basis of W lambda i.

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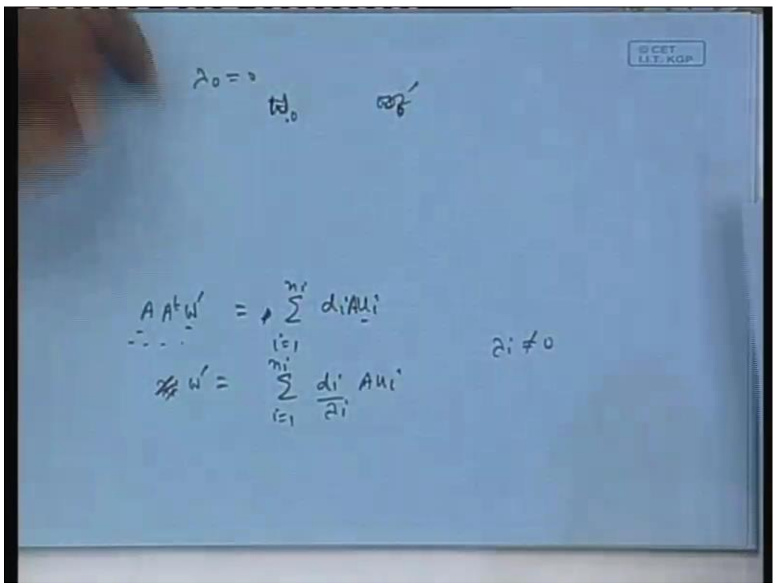

This means, I can write A transpose W prime as some kind of linear combination, di, those vectors ui, i equal to 1 to say ni, because this belongs to W lambda i. If it be so and you know if i apply A over this and therefore, A over this and A can be brought here, what do I get? On the left hand side, I get this W prime i. My starting assumption was, you see A A transpose W prime was lambda i W prime. So, lambda i W prime is equal to di Aui. Important thing is lambda i not equal to 0; this is very important. Therefore, I can take out lambda i from here, this which means, any vector W prime belonging to this eigen subspace, can be written as a linear combination of this Au 1 to Au ni, as shown here. That means, these linearly independent vectors form a basis of W lambda i prime, which means W lambda i prime also has dimension ni. That means, both the subspaces have the same dimension, or lambda i has the same multiplicity ni, whether it is for A transpose A or for A A transpose;, so that is proved.

Mind you, only this is true for those subspaces, for which lambda i was not equal to 0. Because then I could say here, I could prove that they are linearly independent, by saying that, I mean, intersection between the two subspaces is 0. This I could make use of, only when, W lambda i was different from w 0 and intersection was 0, when it comes to the special case lambda 0, which is actually 0. That time you can easily see, on one hand you have got W 0; another hand you have got W 0 prime. But both have the same dimension. Yes, because may be, I come back to this issue little later; this is not relevant here. So, lambda W 0, I leave aside.

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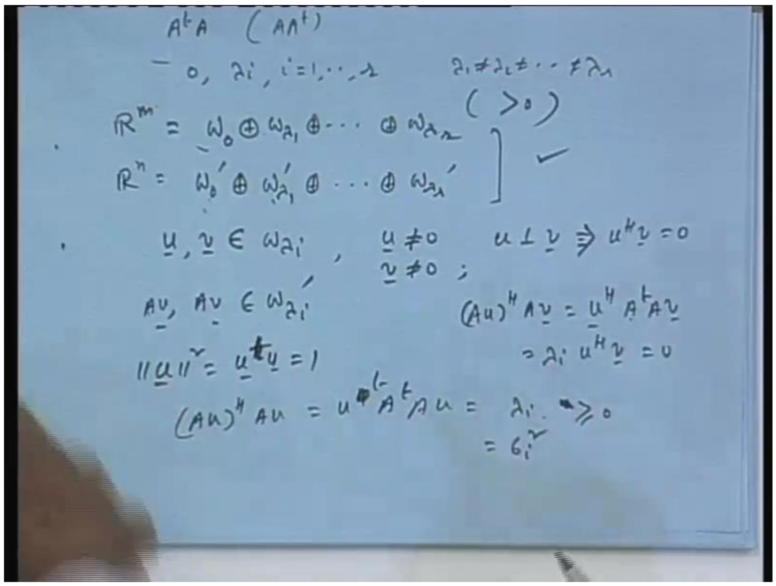

Now, one more thing; consider these matrices A transpose A, also A A transpose. They are all Hermitian means, symmetric; in this case, real symmetric matrix, isn’t it? But in general, Hermitian matrices and not only that, they are non negative negative matrices, which we have seen. So their eigenvalues are real and either 0, or if not 0, they are strictly positive, this we know. Further, we had seen, if eigenvalues are 0 and then lambda i, i equal to 1 dot, dot say r, lambda 1 not equal to lambda 2 dot, dot, these are the distinct eigenvalues and in this case, we also know, they are all greater than 0, because it is non negative, suppose, this is given. Then we had made use of one property throughout this course; Adaptive Signal Processing course; that these matrices, Hermitian matrices are actually, any Hermitian matrix; this is a special kind of Hermitian matrix; they are diagonalizable.

That is why, I could diagonalizing them, and there, we know that eigenvectors belonging to distinct eigenvalues, they are orthogonal; that is not a problem, that we have proved. But we have further assumed that they are diagonalizable. We made use of it in this course, and we did not prove it actually.

We might prove it; it may be towards the end of this class. But the fact, that I am going to stay is this; what is the space on each A transpose A works? The vector space is Rm. Rm is actually, can be written as, because it is Hermitian, Rm can be written as W 0 direct sum W lambda 1, direct sum dot, dot, dot, direct sum W lambda r, and the other one, A A transpose; it works in w ra; it works on Rn; W 0. This, we can prove and this means, these operators are A transpose A is diagonalizable, that is, you can find out a set of, in the case of linearly independent vectors, why when linearly independent; you can make them orthonormal. You can take in the set of orthonormal eigenvectors or basis vectors for W 0, for W lambda 1, for W lambda 2, and when we append them, this forms an orthogonal or orthonormal basis for this.

Using that, using them we can diagonalize the matrix. We have seen it already in the class. But how this is true; we did not prove. We make use of this result, that this vector space is diagonalizable. Because, if the spaces can be written like this, we have seen it already, 2 to 3 lectures back, that if the vector space can be written like this, that is the disjoint sum of, that means, this direct sum of eigen subspace is corresponding to the distinct eigen values, then it is diagonalizable. Why diagonalizable? That also, you have seen, how to diagonalize the matrix operator; that also you have seen. So, that thing applies here. But, why it is so? That is why it is, if the operator is Hermitian, the corresponding vector space can indeed be written like that, and therefore, the operator is diagonalizable; that we have not proved. What you know without saying all these, we have made use of this fact in this course. But I thought, may be, it will be proper to give a proof towards the end of this Adaptive Signal Processing course, which I will do little later, but let me make use of this fact.

This means, one more thing you see, W lambda 1, W lambda 1 prime; both have the same eigenvalue lambda 1, and same dimension, may be, n 1. W lambda 2, W lambda 2 prime; both have the same dimension, may be, n 2 and same eigenvalue lambda 2, so for W lambda W lambda prime and all that.

But, overall dimension here is n; overall dimension here is m. So, W 0 has dimension which is m minus within bracket, n 1 plus n 2 plus dot, dot, dot plus nr. W 0 prime has dimension here, as n minus, again n 1 plus n 2 plus dot, dot, dot, r. So, the dimension of these two spaces are not same, you know, W 0 and W 0 prime, but these are the null spaces we have seen these are the range spaces of that operator. These are the things we have.

If this be the case and one more property, this is one thing we are assuming. We have not proved; we will prove subject to that, we accept that using this eigen subspaces, Rm and Rn can be written like that, which means, they are diagonalizable, we know. Second thing, another property, before you going to the SVD, a representation is that, suppose you take u v belonging to W lambda i, and u not equal to 0; v not equal to 0; but u is orthogonal to v; meaning u Hermitian v is equal to 0, inner product is 0. Suppose, it is given, u and v both belong to that just two non zero mutually orthogonal vectors, belonging to W lambda I, then we know Au and Av; they also belong to W lambda i prime; that is the eigen subspace for A A transpose with the same eigenvalue, lambda i. Question is, are these also orthogonal?

You can see, if you take Au Hermitian Av, what you get, you know, u Hermitian and now, Hermitian in this case will be simply transpose, because we are dealing with all real cases only; A transpose Av. But A transpose A is the operator, v is taken from W lambda i. So, v is an eigenvector of A transpose A. That means, you will simply get lambda i times uH and v and uHv is 0; so 0.

That means, if two mutually orthogonal vectors are taken up, and they are mapped to W lambda i prime by this operator A, they also remain mutually orthogonal, which is a very interesting result. It means, that, and this is true; not only for lambda i greater than 0; this is true for any lambda i. That means, if you take up from one orthogonal basis for each of this W lambda 1 or W lambda 2, and map the basis vectors by using this multiplication by A, you get another orthogonal basis for the current respective vectors, you know, for the respective eigen subspace like, W lambda 1 prime, W lambda 1 prime, like that, which means, this A operator maps one orthogonal basis of W lambda 1 to another orthogonal basis of W lambda 1 prime; so same for W lambda 2, W lambda 2 prime, dot, dot, dot, or W 0 to W 0 prime. So, one orthogonal basis of one eigen subspace maps to another orthogonal basis of the corresponding eigen subspace, with the same eigen value. This is another interesting property.

Instead of taking this, suppose, u is such, that norm square of u or norm of u, that is, uHu, suppose it is 1. That is its unit norm, and if I take Au, is it of unit norm? Answer is, no. If you take this norm square, this is uH A transpose, in fact, all the H should be transpose, according to me; because we are dealing with real cases; Au and A transpose Au, u is an eigenvector, because u belongs to w lambda i. So, A transpose Au is simply lambda i times, u transpose u, but u transpose u or uHu, you can put transpose also, that is 1, so this is lambda i and lambda i is, if i am taking this lambda i from this place, in fact, lambda is greater than equal to 0; you can call it sigma i square. That means, if you take u as a unit norm vector from W lambda i, and map it to W lambda i prime by multiplying by A, that Au vector is not unit norm; its norm is lambda i or less sigma i square; sigma i square is actually, the sigma singular value.

Now, we come to this thing, the main is (( )) decomposition theorem, but well, in this journey, you have proved a lot of things for you benefit. I mean, we could have skipped, but I want to get into these mathematical details, because they come every now and then in our any kind of discourse on statistical signal processing or in adoptive signal processing and those kind of things. That is why, I have included some proofs and also to complete this, because I do not want to leave anything unproved, and use those results. That is not my habit, so that is why I am proving everything.

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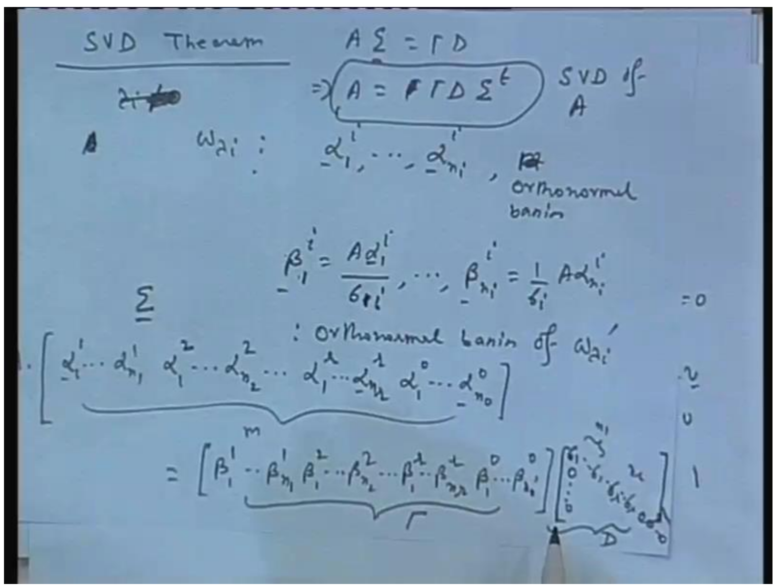

Now, let us come to SVD theorem. We will assume the same A matrix on all those, I am not rewriting them. Suppose, we are assuming same A matrix, n cross m and all those notations, everything continuing. If I can write SVD theorem, I am not bringing any new notation or notions. Now, suppose A is a given matrix, and I am considering say W lambda I; lambda i, I am taking to be greater than 0; not equal to 0. Or maybe, it is not required. I am taking just lambda i; W lambda i. From W lambda i, I form a set of orthogonal basis vectors using the same notations. Each of them has unit norm, this is orthonormal basis. This is an orthonormal basis vector, basis set is given to me. So, if I use A matrix multiply each of them, pre multiply, I will get an orthonormal basis for W lambda i prime, that you have seen; that will be orthonormal. But they are, the norms of each vector will not be 1, but will be equal to plus lambda i or sigma i square.

That means, if I choose say beta i 1 as simply A times alpha i 1, but divided by square root lambda I, positive square root lambda i, which is sigma i, sigma 1. In this case sigma i, because i th eigenvalue dot, dot, dot, beta i ni is nothing, but I am doing the same multiplication. But to make them unit norm, I am dividing them by sigma i, because we have seen that you should multiply them by A, they maintain orthogonality, but unit norm is lost. So, you have to divide by the corresponding singular value or square root, positive square root of corresponding eigenvalue lambda i, which I denote by sigma i.

Then, they become unit norm and in this case, this forms not only orthogonal, orthonormal basis of W lambda i prime.

Now, suppose I form a matrix like this. A times suppose I put like this. I first start with eigenvalue 1, not 0; eigenvalue 1. I will order like this; eigenvalue 1; then eigenvalue 2; dot, dot, dot, eigenvalue r.

That eigenvalue 0 case, will be taken up at the last. So, first consider W lambda 1 corresponding orthogonal basis vectors, you write alpha 1, 1; dot, dot, dot, alpha 1 n 1. Then, take alpha 2 1, dot, dot, dot, alpha 2 n 2, dot, dot, dot, say you take alpha r; rth eigenvalue; this for W lambda r 1, dot, dot, dot, alpha r; Just I am doing nothing; I am just putting the, which orthogonal basis vector side by side, so they are not only mutually orthogonal. Since, the eigenvectors corresponding to distinct eigenvalues are orthogonal, then by themselves. There is an inter set, is an orthogonal basis vector. Because alpha 1 to alpha 1, 1 to alpha 1 n 1; they are not only mutually orthogonal; they are orthogonal to alpha 2 1 to alpha 2 n 2, because the corresponding eigenvalues are distinct, so on and so forth.

So, I go for 1 to r and then I write alpha 0; this is a 0 th case, is just an ordering; there is nothing much in it. There is the 0 th case, I want to separate out, and I write at the last; alpha 0 n 0. Essentially, there is how many eigenvectors, say how many such vectors I find? Same as the dimension of this space, Rm, because of this result, that Rm was, I mean, can be written like this; that A transpose A is diagonalizable, that means, Rm can be written like this. If you take all the orthogonal basis vectors of the respective subspaces, append them; total you get must be equal to m, that is the dimension. That means, this is how many m, total m. You can call these vectors, may be, you can give it a name. Now, let me see what name to give, may be, you can call it sigma.

If you do this A sigma, what you will get is A times this, or A times this, this will be simply, by my definition, sigma 1 times beta 1, 1; sigma 1 times beta 1 n 1; like that. If I form another vector, another matrix beta, first like here, eigenvalue 1, here also, eigenvalue 1,1; dot, dot, dot, say beta 1 n 1. Then again, beta 2 1, dot, dot, dot, beta 2 n 2, dot, dot, dot, beta r 1, dot, dot, dot, beta r nr. I am just putting nr here and then I take vectors that is 0 eigenvalue; beta 0 1, dot, dot, dot, beta 0; it will not be n 0, because I told you; for non zero eigenvalues, the multiplicity is same, but for the 0 eigenvalue, is not. So, you can give it any name n 0 prime. You form a matrix. This is again an orthogonal matrix, because

I mean, first within each eigen subspace, vectors are mutually orthogonal; that is how we have seen, because I mean, that is what I have seen, that is, if alpha 1 to alpha n 1, they are mutually orthogonal; A times them also, they are mutually orthogonal; then since, they corresponding to different eigen values, I mean, this sets also, I mean, whether you pick up somebody with eigenvalue 1 or somebody with eigenvalue 2; they also will be more mutually orthogonal. So, they are orthogonal, not this is orthonormal, but it will be like this; A times alpha 1, 1, just as an example; A times alpha 1, 1 will be sigma 1 beta 1, 1. So, sigma 1, 0, 0, 0, A times alpha 1 n 1. Alpha 1 n 1 will be again, sigma 1 beta 1 n 1. So, sigma 1 will continue for a while; how many times? n 1 times. So, many dot, dot, dot, after a while, sigma r will continue, nr times and then 0s.

So, it is a diagonal matrix where, there will be non zero; there will be positive, strictly positive diagonal entries concentrate on the singular values and then 0s. You can call it say D. Now, since, this matrix I called sigma; this matrix you can call say capital gamma. In short, using short notation, A times sigma is gamma times D, or A is, since sigma is, I mean, all the columns are mutually orthogonal; does not orthonormal. Sigma transpose sigma; sigma is unitary. Sigma transpose sigma will be identity, that is, sigma transpose itself is sigma inverse. If I post multiplied by sigma transpose, it will become identity and here, that means, gamma, sorry, gamma D sigma transpose. This is called SVD of A, and D consist of the singular values. To get the singular values is not very difficult. Within an A, you just form A transpose A or AA transpose, and take the eigenvalues, whichever is the non zero eigenvalues, you find out, total number of non zero eigenvalues, including their multiplicity. Those non zero eigenvalues will form their, this thing, you know, they will be called as singular values.

In fact, in many single processing work, these non zero positive singular values, they actually come from signal, and in a signal plus noise situation, you know, those eigen singular values which are not 0s, but very small; they can be directly as coming from noise actually, and those problems exits. You will see and those are called subspace based signal estimation techniques and all. The entire thing based on this SVD thing. Another thing is that suppose, A is a matrix and in general, say rectangular matrix. You are asked to find out the rank of A, but we know that rank of A is same as rank of A transpose A, and what is rank of A transpose A? Number of, that is, if you have to find out the range space and range space is given by what? That is, this is the range space. Dimension of this will be the rank of A transpose A. What is the dimension? First, you find out multiplicity of lambda 1, then multiplicity of lambda 2, dot, dot, dot, multiplicity of lambda r. So, what are those eigenvalues; non zero eigenvalues; find out their multiplicities; add that will be the rank.

So, simply to find out the rank of A, you need to find out the singular values; that is the eigenvalue of A transpose A; find out the non zero eigenvalues; just add their respective multiplicity; that will be the rank. So, that is how SVD is used for rank determination. There are other applications of ST SVD, but before I mention them, may be, I consider this proof. Because this is something, I assumed for this course and did not prove.

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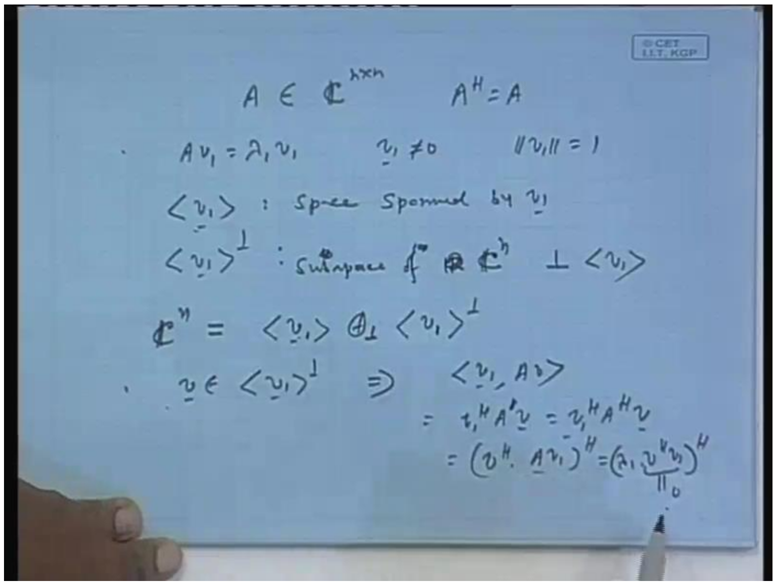

I assume, I now prove the diagonalizablity of Hermitian matrixes, but time being short, I might run through it. Suppose, you are given a matrix A and now, I will make A to be Hermitian and in general, complex, to make it more general. So, its C may be n cross m; A Hermitian is A; that is A is Hermitian matrix. Now, form Av 1 as lambda 1 v 1, that is, I form is, I try to find out is eigenvalues on eigenvector, I get at least 1 eigenvector, say call it v 1; that is v 1 not equal to 0 eigenvector. Not only that; you normalize the eigenvector, so that, its norm is 1, you find out this. Then by this, I denote space spanned by v 1 vector, and by its orthogonal compliment, spanned by, I mean, it is what subspace of, sorry, Cn, the space which is orthogonal to this. What will be the basis of this; that is if you take v 1 and then consider this space and by Gram-Schmidt orthogonalization, get n minus 1 orthogonal basis vectors.

Then, consider those n minus 1 orthogonal basis vectors and take their span. That span will be this. This is all the standard stuff, this we have done. So, I will not spend too much time on this. In this case, the total vector space, Cn can be written as a direct sum, rather I would say orthogonal sum; because one is the orthogonal of other and now, you consider v element of this. This means, if I take now. I will show that, if you take v element of this, and on that you apply A matrix; that is Av, Av also will belong to this. That is, this subspace v 1 orthogonal v 1, I mean, this orthogonal subspace, that is orthogonal compliment of the span of v 1, which is invariant under A. That is, any vector pick from the subspace, on that if you apply A, you get back something from there. That is very easy. That is, if you take Av, if you take the inner product between say this given v 1 and Av, what you get is v 1 H Av, but A and AH are same. So, I can write it as v 1 H AHv; this means vH Av 1 all together H, and Av 1, we know is lambda 1 v 1, so lambda 1 vH v 1 H. But, because of orthogonality, this is equal to 0, because v 1 belongs to this space span by v 1, and v v is taken from its orthogonal compliments, so this is 0. That means, if v belongs to these; that is v belongs to the space which is orthogonal to span of v 1, then Av also belongs there. Because inner product between Av and that mother vector v 1; this is 0. This you have seen.

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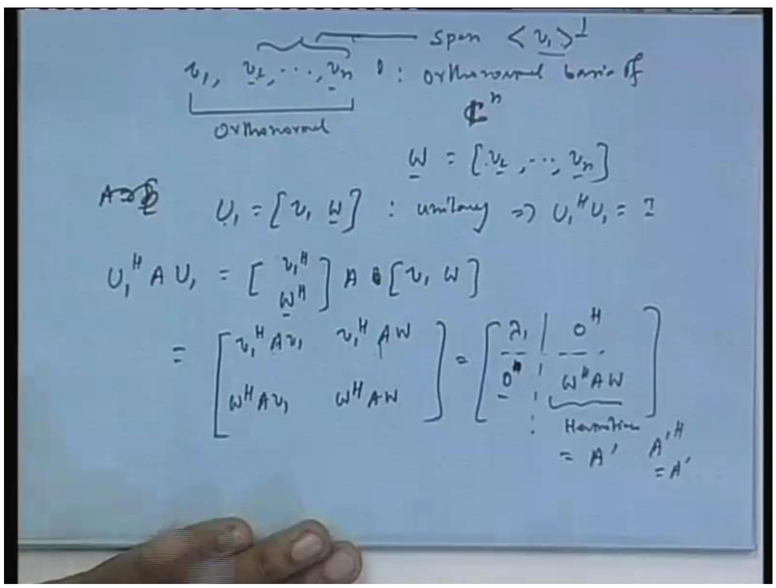

Now, suppose, I form, we are already given v 1 and by this say Gram-Schmidt orthogonalization, that will be one of the methods, I form other vectors; v 2 dot, dot, dot, vn, where this part, this is orthonormal, and this span, this part, it is obviously, say orthogonal. This is an orthogonal basis, orthonormal basis of Cn. You give me v 1; then by Gram-Schmidt orthogonalization, I form the other orthogonal vectors; v 2 to vn. If I consider the v 2 to vn only, they only span this space; that is very crucial to see, this is the case. This v 2 to vn, I form a sub matrix; v 2 dot, dot, dot, vn; that means, what is A. A is not A and then I form one matrix U 1, as v 1 and W. Clearly, this is an unitary matrix, because all the elements of this matrix, all the columns are mutually orthonormal. That means, U 1 Hermitian, if you take U 1 Hermitian u 1, you will get identity. That is obvious, because our columns are mutually orthonormal, this is given to me.

Now, if I consider U 1 Hermitian AU 1. You remember, when we diagonalize the matrix, this is what we get, I mean, on unitary matrix; its Hermitian, times the given matrix, times original one, and that results in a diagonal matrix. Now suppose, I am going in that direction, so I formed an unitary matrix, using this basis and then consider its Hermitian; U 1 Hermitian then multiplied by AU 1; is it diagonal? Answer is no. It will not be diagonal so easily, but we are going one step in that direction,

what happens to this matrix? If you really write this way, this is actually v 1 Hermitian, if you take U 1 Hermitian, first column becomes first row with conjugate transpose, so this is a thing; A and this is v 1 W. So, this is U 1 Hermitian A and then if you do elementary matrix multiplication, you will see these results; v 1 Hermitian Av 1; v 1 Hermitian Aw; W Hermitian Av 1; and W Hermitian Aw. Out of which, very quickly, Av 1 is lambda 1 v 1. So, lambda 1 goes out; v 1 Hermitian v 1; v 1 is unit norm, so that is 1. So, first element is lambda 1. Av 1, that is again, lambda v 1; lambda goes out; lambda 1 here, lambda goes out; W Hermitian v 1 will be 0, because v 1 is orthogonal to all the elements of, all the columns of W. So, this becomes a 0 Hermitian; that is, a column vector. Hermitian are, it can be transpose, because it consists of only 0s.

On this side, Aw, now I told you that any vector, if you pick up from v 2 to vn, when they all belong to this orthogonal complement, and if you pick up any vector from there, say v 2 to vn; apply A for that; that also remains there, which means inner product between that and v 1 will be 0. That is, v 1 is orthogonal to Aw, all the columns of Aw, because Aw is nothing but Av 2, Av 3, Av 4, dot, dot, dot, and Av 2, Av 3, they are all belonging to this orthogonal complement, that you have proved just a while back. Therefore, their inner product v 1 will be 0. So, I get again, here, in fact, I should write just 0 vectors; column vectors by notation. This should be 0 Hermitian, and this one, Aw. Now, the question is; this is again Hermitian. You can call it A prime, you can easily see it is Hermitian. That is A prime, if you take A prime Hermitian, you get A prime. That is because of the structure, you take the Hermitian W Hermitian transpose, A Hermitian, A Hermitian A, and this WH goes to the end and becomes W. You get by the original one, it is n; this is Hermitian.

That means, this become, so but since now we concentrate on this A prime. Whatever you have done so far the same approach, you can apply on this A prime, but this time, it is the vector space, Cn minus 1 because dimension of this matrix is n to n minus 1 plus n minus 1.

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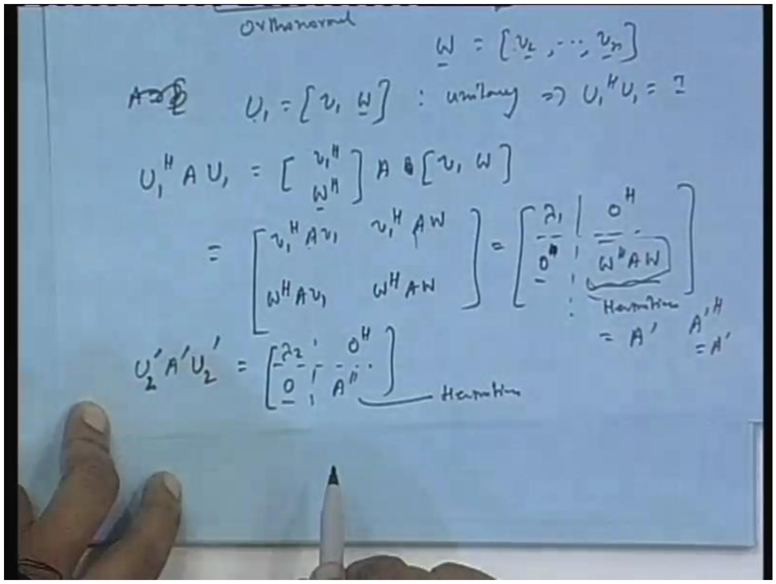

That means, we can construct again, some U 1 prime; we can construct some U 2 prime. In the same way for this Hermitian matrix, we can consider in the same way, but they are all dimension n minus 1. If the vector length is n minus 1, it is a matrix n minus 1 cross n minus 1. So, U 2 prime; that U 2 prime A prime U 2 prime is again, taking this kind of structure; some eigenvalue lambda 2, 0, 0 Hermitian and then another another Hermitian matrix; A double prime Hermitian.

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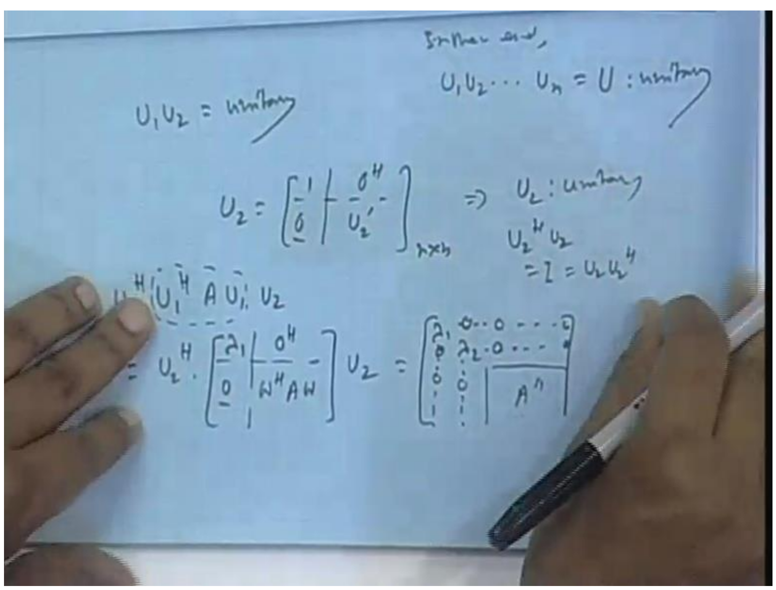

Now, if I define U 2 as simply, 1, 0, 0 Hermitian and this U 2 prime, then again it becomes n cross m. Because, our problem was that one matrix U 1 is n cross m, and this is n minus 1 cross n minus1. So, there is a problem in connecting these matrices; size was there is an incompatibility. So, I just put that

U 2 prime as a sub matrix of individual matrix. Just added 1 here and 0s and 0s; U 2. U 2 is Hermitian, you can see; sorry, U 2 is unitary, you can easily see; this addition of 1 and then 0 and 0; it does not change unitariness, because U 2 prime was unitary. That is how you took U 2 prime for this Hermitian matrix, was unitary, you can easily see that U 2 H u 2 is identity, which is also u 2 u 2 H, this easily you know.

Therefore, if you consider now, this case U 2 Hermitian, U 1 Hermitian, AU 1 U 2, originally, you had these; that give you this matrix. Again, applying this structure with 1 here, and 0 and 0 and U 2 prime, if you put that back here, you will see, I will not do this elementary matrix multiplication results; you will see. First, we will have lambda 1, lambda 2, then 0s, 0s. Here is a 0, 0s, here also, 0s and then there is another matrix. That is that A double. You see by progressive, and this is again, Hermitian. So, by going progressive, doing it again and again and again, the entire matrix gets diagonal, and U 2, remember, I get U 1 U 2 here. This is a unitary matrix, but product of two unitary matrices makes it a unitary matrix. In general, I will get in the end, I will get a product like, I mean, U 1 U 2 dot, dot, dot, up to U n; n number of unitary matrix. This product, you can call it U; this will be unitary.

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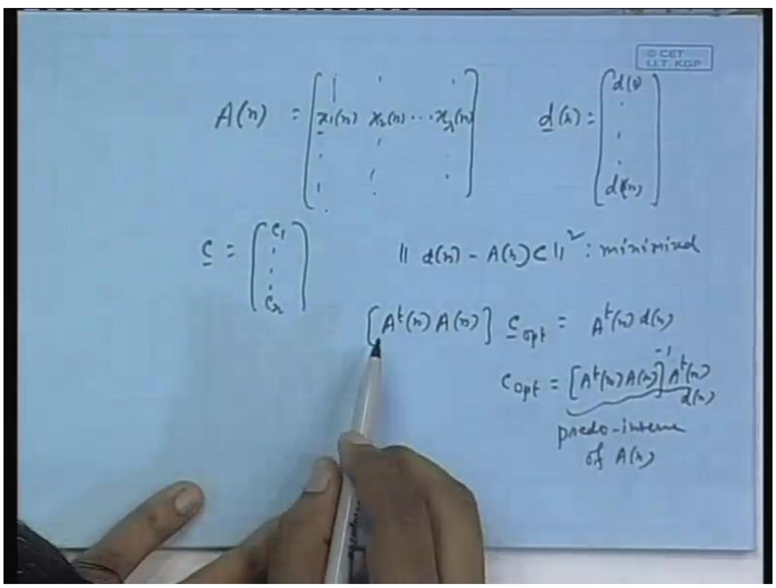

That means, you will have a situation like this; U Hermitian like here, this is U; this is Hermitian of that; U Hermitian original AU will be nothing, but all the eigenvalues; lambda 1, lambda 2, up to say lambda something, lambda n. They may be distinct or they may be repeated; this I am not bothered. Therefore, you can take this UH on this side; you can make it even A U, as if you multiply by U on this side, U UH is 1, so here also, U and the same matrix. Then, you can easily see that it is the columns of, you take the columns of U; they are mutually orthonormal, and A times each column is 1 lambda times the corresponding column. So, they correspond to eigenvectors. These eigenvectors, you can easily see, they and how many? As many as that dimension of n, that is n. So, the matrix is diagonalizable, which also means here, that I get totally n mutually orthogonal basis vectors. Some of them can correspond to the eigen subspace of lambda 1, if it is repeated, some could correspond to eigenvectors of lambda 2, so on, dot, dot, dot, but they are mutually orthogonal and is diagonalizable. This shows that a Hermitian matrix is diagonalizable. See, it completes the proof, some SVD applications, you know there are many. Unfortunately, I cannot get into them, but one thing you see, we have done this least squares minimization thing.

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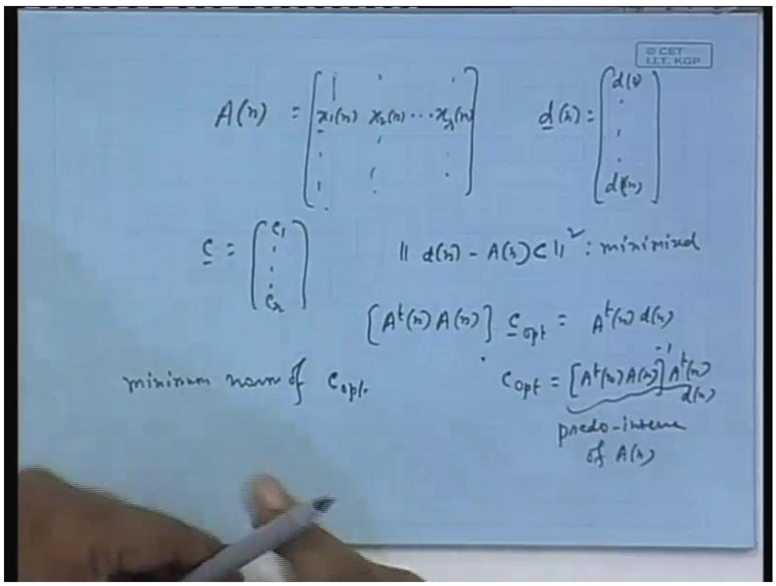

That is, suppose, there is a data matrix given, say A, say data matrix given; I do not know what notation was used; An. It consists of the data vectors; x 1 n starting from 0 to n, x 2 n, dot, dot, dot, say xrn, and there is a dr response vector; d 0 dot, dot, dot, dn. We have done the least squares minimization and that time, we tried to say that find out coefficients of vector c as c 1, dot, dot, dot, cr; that is as many as the number of columns here, so that dn minus An C; its norm square is minimized. We got the, I mean, we connected this to orthogonal projection problem and all that, and see the orthogonal projection is unique. I mean, we get one such unique projection exists, and we got that equation as A transpose n, An times C; C opt was A transpose n dn. Then if the columns, that time we assumed, the columns are linearly independent, its full rank, and therefore, we could easily, I mean, if it is full rank, then A transpose An also will have will be strictly positive definite, if the columns are positive definite, then therefore, invertible. So, you take it on this side. That time, we said that A transpose An inverse A transpose n dn; and this part I call as pseudo inverse of An, but suppose, A transpose An, An is such the columns are not linearly independent, then you cannot invert it, which means you got a matrix here, which is rank efficient and therefore, this equation has more than one solution. Projection will be same, because whichever solution you put here, Anc will be same; An times c will be same. But solutions are many.

In this case, SVD can be used to find out the solution, which has the minimum norm of C opt; that is, there are many solutions possible. If you apply SVD here, then you get the solution, which is a minimum norm. This is a very vital application of SVD. Another is that suppose, given a matrix of rank efficient square matrix, and you want to approximate it to; I will give you a square matrix of some rank, may be, full rank. But you want to approximate it to a lower rank matrix. Then, the best such approximation can be also obtained by SVD. There are many algorithms to compute SVD and all. There are tremendous applications in signal processing control and communication. My purpose was to introduce this topic as an appendix to you people. Also, through that process, do this bit of lineariser, which I did not do in this course. So, at least lineariser wise, this course is, kind of complete, you can say; we tried to cover a lot. So, that is all for this course and if you have any feedback on it, you can communicate to me later.

Thank you very much.