**Adaptive Signal Processing**
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**Lecture - 18**
**Vector Space Treatment to Random Variables (Contd.)**

So, last time I gave you this axioms of vector space, I will not go into all that I was just, I also told linearly independence dependence span of a finite set of span by a finite set of vectors, over vector space. What is meant by subspace of a vector space, linear dependence independence utility of this independence dependence and all those things right. I have not given any examples that I will do and then I was in the midst of an exercise. You know that I will continue again that is suppose you are given it a vector space V from which you pick up…

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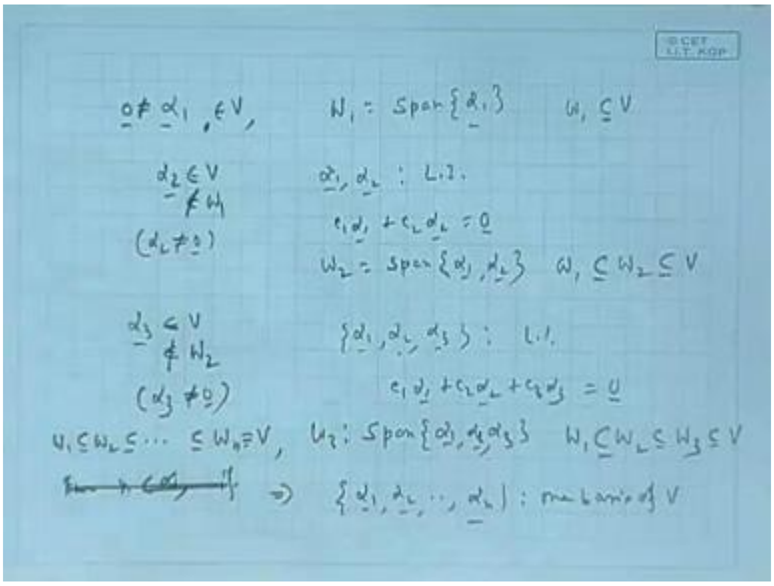

Again I tell you that it is my way of arriving at some concept of dimension without diluting any mathematics. Suppose you find out one alpha 1 and again mind you this alpha 1 could be anything. So, again there is plenty of choice, alpha 1 where alpha 1 is non 0, this is not the 0 vector of the vector space. Pick up and alpha 1 element of V then you define W 1 as the span of alpha 1. That means typical element of W 1 will be what, a constant times alpha 1. Span of a few element means all possible linear combinations, here only we have only one element. So, all possible terms like C into alpha 1 for all C, W 1. Next you consider alpha 2 element of V, and not element of W, that is from outside W, but from inside V because after all here you find W 1 is a subset of V.

So, if W 1 is less than V that way size, you go outside W and remain in V pick up somebody from there and call it alpha 2. So, obviously, alpha 2 cannot be 0 vector because alpha 2 is outside W 1, 0 is unique for the vector space. So, the 0 of V, the 0 of W 1 they are same only one 0, and that is why this cannot be 0 because I am outside W 1. So, I cannot be, it cannot be 0, then it means alpha 1 alpha, 2 is LI, linearly independent. This I proved last time again we can see because if you form a summation like this equated to 0. Is it that only solution is C 1 equal to 0, C 2 equal to 0.

There is a question because suppose it is not, suppose both C 1 and C 2 are non 0. Obviously, you can write alpha 1, alpha 2 in terms of alpha 1, some constant times alpha 1 which means alpha 2 is a part of W 1 which is contradiction. So, that is not possible So, is it that one of them non 0 and other 0 will that be ok? So, this is 0, C 1 alpha 1. So, C 1 non 0, C 2 0. Suppose is this fine? C 1 non 0, alpha 1 0 is C 1 non 0 C 1 alpha 1 0 means from our previous class we know in such case alpha 1 has to be 0.

But alpha 1 was started with non 0, so that is not possible. On the other hand, is it that C 1 non 0, C 1 0, C 2 non 0, then C 2 times alpha 2 equal to 0 means alpha 2 has to be 0 which is the contradiction, that is not possible. So, only possibility is 0 0. If that be then you consider W 2 a span of. So, that easily you can see W 1 after all they are considering all linear combination alpha 1 and alpha 2.

So, the combiner coefficient with alpha 2, if you hold that 0, then you will get only constant multiple of alpha 1. So, that means, all elements of W 1 are present in W 2, is not it. Suppose W 2 is still not V, you understand, but from W 2 I am going to a bigger subspace, which contains W 1, that subspace W 2. If it is still not V, then again go outside W 2 from within V, but outside W 2, that is you pick up another guy say alpha 3.

This W 1, W 2, if it is not in W 2, it is not in W 1, that means it is outside W 2. You find out. Obviously, alpha 3 cannot be 0, like here I should say alpha 2 cannot be 0 vector, alpha 3 cannot be 0 vector because W 2 consists of 0, 0 is unique that has same 0 of V, I am outside W 2 alpha 3 cannot be 0 vector. Again I append this. This is again LI what else happening is whenever I go outside the space and take somebody from there append these two the basis. Mind you, this is alpha 1, alpha 2 is a basis of W 2 because it is spanning these and they are linearly dependent, is not it.

Alpha 1, alpha 2 is a basis of W 2, alpha 1 is a basis of W 1. What is happening is, if you go outside the space, subspace bring somebody from there, then my claim is that somebody and the previous basis if you form a bigger set, that is a linearly independent set. That is why I am trying to say, this is the one line conclusion. Then you have been given a basis and you are finding the subspace fine, span by that they are linearly independent because of these basis. You go outside the space bring somebody from outside that and this basis forms a linearly independent set.

From that you simply you leave that fellow cannot be written as a linear combination of this basis because that means, that fellow could, third fellow new fellow could is part of W 2, but that is not possible. Most formally if you want to see that the LI, you form an equation like this. I go up to this and then I can apply dot, dot, dot because they logically will be very clear. I do not have to go for alpha, I mean you know W 4, W 5 and all that. From this we can conclude. Alpha 3 we picked up, Is it that all solutions are, I mean some non 0 solution possible?

Firstly suppose C 1, C 2, C 3 are non 0, then obviously, alpha 3 can be written in terms of alpha 1, and alpha 2 as linear combination which is a contradiction not possible. That means all non 0 not possible, then two non 0. So, suppose these two are non 0 since, alpha 1 and alpha 2 has been proved to be linearly independent all ready, they cannot be non 0. So, is these two, these two means again alpha 3 in terms of alpha 2, alpha 3 can be written as in terms of alpha 2 or in these 2, alpha 3 can be written in terms of alpha 1 in either case alpha is part of W 1 and therefore, W 2 or part of W 2 that is not possible.

Student- Sir, alpha 3, alpha 3 is not a sign of alpha 1 or alpha 2. So, it means alpha 3 can be jointly expressed in terms of alpha 1, alpha 2?

No, no, no, no, no, that is you did not understand anything, however I am saying W 2, W 2 is a span of this means set of all possible linear combinations. You have forgotten. So, this is span of, it includes only multiples of alpha 1, multiples of alpha 2, and not only that, like X axis Y axis, but any linear combination of x vector y vector. So, in that x y plane and I am going in the third direction that cannot be written in terms of either some combination of x y vectors, is not it?

So, understand two also not possible because this two, then they have to be 0 they cannot be non 0 because LI, and these and any of the two not possible because alpha 3 is again becomes part of W 2, not possible. So, two not possible and then one, one means suppose this is, only this is non 0, that means, alpha 3 is 0, which is not possible and suppose this is not, either this or this again that contradiction. Obviously, alpha 1, alpha 2, none of them is 0. One at a time is also not possible, is not it?

So, they are linearly independent only solution is non zero. This way you can proceed because when you, whenever you add extra vector that extra term, if the coefficients that is non 0 and some others are non 0 that extra term will be linearly expressively in terms of rest that is not possible. If that term has 0 coefficient others have to be 0 because others have to be already been proved to be linearly independent. Okay rests follows linearly, you understand that this way what I can do from W 3.

So, I consider now W 3 as a span of alpha 2, alpha 3 right. Now, if W 1 content in W 2, content in W 3, content in V. Again I go out of W 3 bringing alpha 4, that cannot be 0 vector append to, append that to these can be proved to be linearly independent in same way form a linear combination. The C 4, alpha 4, if C 4 is non 0 and either other three coefficients are at least one of the three coefficients non 0, immediately alpha 4 will be written in terms of either alpha 1, alpha 2, alpha 3 or part of them, which is not possible because then alpha 4 will be part of W 3, which is contradiction, because I am going out of alpha 3. If C 4, alpha 4, C 4 is 0, that means, the other coefficients have to be 0 because the LI. C 1, C 2, C 3, they have to be 0 this way.

So, suppose this an algorithm of constructing bigger and bigger subspaces within V. If it so happens that, after a finite number of steps, I arrive at W n, and then W n coincides with V, that is there is nothing outside W n that is for n less than infinity. If I mean you are getting a sequence of subspace is not it. W 1 if it so happens W n you arrive and that is equivalent to V. Then, that means you have alpha 1, alpha 2, dot, dot, dot, alpha n you have constructed one basis, fine. But remember there is no in it nothing unique because alpha 1 I started with a purely arbitrarily, there are infinite sets of alpha 1, just do not think that 0 1.

Then again I found a W 1, outside W 1 I went, I put any alpha 2 that term any. Again alpha 1, alpha 2, I found out the span W 2, outside that I went I took any alpha 3. So, this choice is not unique, the number n, however. So, what it means is you can start with another say element beta 1 not 0 or say it is span, W 1 prime go outside that, but inside V take another vector say beta 2, beta 1, beta 2 linearly independent take its span call W 2 prime dot, dot, dot, dot, this way you go.

So, you get another sequence of vector spaces, but total number will be same, the total number is n, and that n is called dimension, dimension does not depend on the choice of the basis we will prove it. Have you, have you understood what I am saying, that there are there is nothing unique about basis, through examples it will be clear later, there is nothing unique about the basis. I can start at alpha 1, then alpha 2, alpha 3 purely and I can start at some beta 1 find the space span by that, go outside that, pick up a beta 2.

So, beta 1, beta 2 again find the space span by that go outside that pick up another fellow beta 3. So, this way also you can get beta 1, beta 2, beta 3, and subspaces which are content in next step subspaces one content in other that also content in another. Finally as it will coincide with V, after some finite number of steps and my claim is that number of steps will be, will same as n.

That is either it is alpha 1 to alpha n, or beta 1 to beta n or could be gamma 1 to gamma n, number is same and that number is called dimension, it is a unique number independent on the choice of basis. But basis does not have to be unique as you can see, you can construct a basis in your way, in your own way, and that is infinite choice right. So, this dimension thing we can prove easily. Suppose I start with beta 1 and beta 2 and all that.

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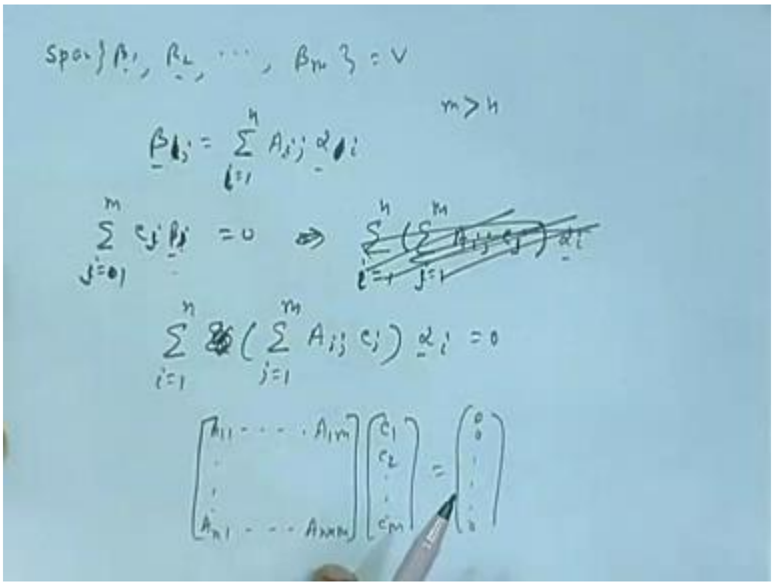


See we go like this, beta 1, beta 2 dot, dot, dot, dot, and suppose if I have got our terms up to beta m, and m and are not same I will prove that m is equal to n. I will prove m and n are same. You understand, but in general I am starting with m, that means this span also span of this also V suppose, then I have to prove that m is nothing but n. That is what I have to prove I started with beta 1, then got a beta 2, then got a beta 3, that process went on suppose after n th step the corresponding subspace span by these coincides with V, that is I stopped there then my claim is that m will be nothing but our good only, it cannot be a new number, I have to prove it.

But one thing we know this will be linearly independent by the construction. You just take the space span by beta 1, take beta 1 go outside that space find out any beta 2. So, beta 1, beta 2 linearly independent and then I take the space span by beta 1, beta 2 go outside that, take a beta 3, beta 3, beta 1, beta 2 that proof general proof, we have done is not it? So, this LI there is no doubt about it only thing is m should be equal to n that we have to prove fine. That means.

So, suppose start with we do not assume m equal to n, that means either m is greater n or m is less than n. You take the case m greater than n and proof that this is not possible. So, that means, m could be less than n, then in the same proof you reverse the role of n and m and immediately you get that proof that m cannot be less than n also. You understood my logic. I will start with m greater than n I will show it is not possible. So, you can ask the question, that means, m should be less than n. But by the same proof replacing interchange the role of n and m, I will say that n also cannot be greater than m.

So, only possibility that will appear is possible is m equal to n. It is not very difficult it is a easy proof. Since, beta 1, beta 2 they are all part of V, and V has another basis alpha 1 to alpha n, but each of these can be written as a unique linear combination of alpha 1 to alpha n. Any vector of V can be written as a linear combination of any basis vector. I am taking that to be the basis alpha 1 to alpha n set. That means beta j, this notation I have to check A whether i j or j j i. Let we start with i j alpha j why i because, why j because I am taking sorry, it is beta j because beta i that particular vector is a linear combination of those alphas.

So, j because of the corresponding basis vector alpha j and i because starting vector is beta i. So, i comes here, j comes here, and j will be 1 to n, but this betas are linearly independent. See if I form a summation C 1 beta 1 plus C 2 beta 2 plus dot, dot, dot, equate to 0, only solution should be C 1 to C n, C n equal to 0, is not it. So, suppose I form an equation like this C i, beta i, i equal to sorry 1 to m after all m elements. I am considering the case m greater than n. Suppose this is 0 since, that proven to be, proof to be linearly independent about that there is no doubt.

So, there is betas or alphas they are mutually linearly independent, that we have, that is true from the way we have constructed them, we have already given that proof. LI is never in question, that number is in question whether m and n will be same or not. So, since they are LI only solution for this would be C 1 equal to C 2 equal to dot, dot, C m equal to 0 that we know. But suppose here I replace beta i by this expression and interchange the summation because, yes interchange the summation, what will you get. See better you have been A ji here, anyway you will understand what I am tell to mean. This beta is replaced by this here, then the two summations are interchanged.

So, j equal to 1 to n goes outside and i summation, i equal to 1 to m, A ij C i. Let me do one thing just, there is no I mean with this also is possible, but to get this in either form let this be beta j and this will be alpha i, C j beta j. So, beta j is given by an expression like this, beta j is a linear combination of alpha. So, I am just interchanging I mean changing this, in this area actually, beta j is given by a linear combination of alphas like this. These are the combiner coefficients.

Student- Sir, i equal to 1 to n.

So, i this will be summation over i, i equal to 1 to n, A ij, alpha i, i is coming from alpha and j is coming from j that is better fine. So, now I am linearly combining them equating them to be 0, they are proven to be linearly independence, only solution should be that C 1 equal to C 2 equal to W equal to C m equal to 0. And you replace and interchange the two summations. So, that means, i equal to 1 to n will be here j equal to 1 to m is that right? Cj

Student: C j, sir that will be j equal to 1 to m.

What is the problem? This I am replacing here, this summation is here, then I am interchanging the two summations here, here, okay let me correcting it let me in briefly again once again beta ji am replacing this beta j this one, this way you can correct this. I did not bother about this, I thought it is implied already I was taking that in mind and proceeding.

This is where you went right. C j beta j you replace beta j by these interchange the two summations. J equal to 1 to m your A ij c j right times alpha i equal to 0. But alphas also have been proven to be linearly independent, that means, this a this a scalar remember that this are scalar summation times summation of something times something, this scalar.

So, scalar number dependent on i is a function of some constant i subscript i that times alpha i equal to 0. Alphas are linearly independent, that means, each of this term should be 0 for each i, is not it. For each i this would be 0, but this term you can write as like this. You get this, but if m is greater than n, that means, what you have got more unknown less equation.

So, obviously I can find, I can say take some coefficient to be non zero and subject to that I can find a solution for that. So, that this is satisfied. It is a basic thing I have all there if you have more unknown and less equation I can all 0 is not the only solution you can find a non 0 solutions easily, is not it. Where there is a contradiction because whatever the Cs after all here they are linearly combining another linearly independent set, that is betas. There are two linearly independent summation set, set of betas, set of alphas, they are combining the betas. Betas are also linearly independent.

So, therefore, the only solution for this equation should have been that all coefficients are 0. But if you have m greater than n, you can otherwise construct solutions for C 1 to C m, where not all are 0 were still this is satisfied. So, that is a contradiction is not it. So, that is not possible, that means, m has to be greater uh either equal to n or less than n. If m is less than n then either interchange the role of n and m, interchange the role of n and m. I have just proved the property m cannot be greater than n interchange the role of alpha and

Student: Alpha and beta.

Alpha and beta. Betas were expressed in terms of alpha and expressed alphas in terms of beta. Same thing you go ahead. You understood this. So, only possibility is m equal to n and if m equal to n you had a square matrix times this and that is possible this square matrix in that case will be invertible. Any invertible matrix times of vector equal to 0 means the solution is 0. So, this shows that two, these two integers cannot be different they have to be same and that is called dimension of the space. We will take some examples.

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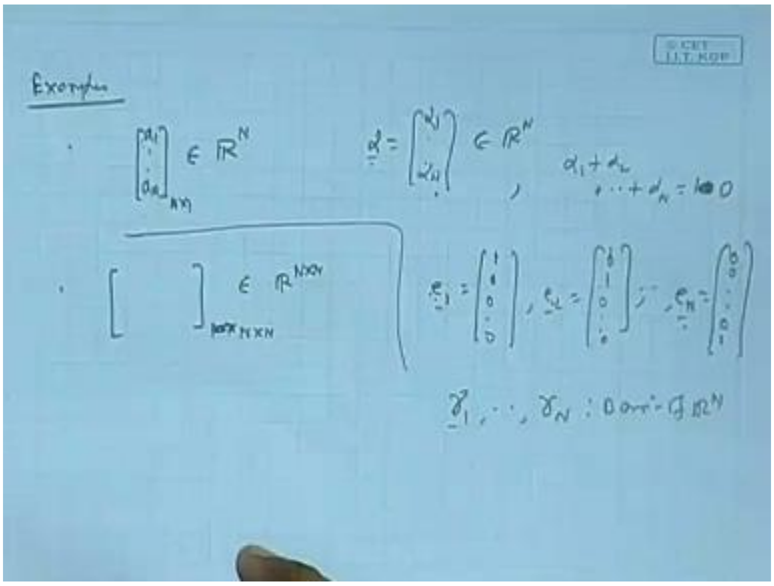

Consider examples of vector space. One. Consider just column vectors say n cross one column vectors, a 1 to say a n element of what is called R N, you know this R N means. R is real number, R into R into R. Basically all this means all the elements are from real. Now, set of all these that is R N actually, set of all these vector space under usual notion of vector addition and all that because if you take one vector another vector if you add, you get a vector of the same dimension, is not it. You get a vector of same the dimension it does not become n plus 1.

So, remain in the same space if you take one vector multiplied by the scalar you still get another vector of the same dimension. So, you remain there then there is a 0 vector means all 0s like that associativity commutativity they are, there are you know they are valid very trivially. Two vectors are added means just terms wise they are added. Normal addition of column vectors.

So, that is of course, commutative if that associativity all those things exists. There is a 0 vector if you put all 0 here that is a 0 vector in this vector space because with that if you add any other column vector you get back the same column vector. Then giving any vector you can find it is so called negative because you just reverse the sign with that if you add the original vector you will get that 0 vector, so on and so forth. So, it is this is a vector space. Now, suppose in this vector space I consider set of all vectors, whose sum if you sum all the element you say equal to 10.

So, those vectors will form a subset of this vector space, is not it. Because after all I will consider only those vector where all the terms added will gives rise to say 10. So, it will not be any vector for R N, but only selected vectors. So, that set will be a subset of this vector space. But is that a subspace also? You understood my question? Not understood. So, this is the R N, now I am considering vectors like this say alpha, element of this R N is the space here, vector space, element of this.

So, that summation of alpha 1 plus alpha 2 plus dot, dot, plus alpha n equal to say 10. It could be any number 10. So, I collect only those vectors from R N form a set, that is a subset of R N. Subset fine, but is it also a subspace? Answer is no. You take one vector alpha 1 from that set, another alpha 2 in both the cases if you add that two terms you get 10, but when you add alpha 1 plus alpha 2 and then you add the terms you get 20.

So, immediately you go outside the set, is not it, or take a vector all that terms added equal to 10 and now multiply the vector elements by a scalar immediately the sum of the elements after the multiplication will be different. So, you were not remaining in the set. So, that set is not, we say that set is not closed under addition, not closed under scalar multiplication, you go outside the set. And if is not closed it cannot be a subspace.

Now, suppose instead of 10, I make it 0. I say I make it 0, that all the terms added equal to 0, then it will be a subspace because you take any vector whose terms are 0 added 0, take another one, again terms added equal to 0. Now, if you add the two vectors, resulting vector also if you add the terms you will get 0 only or you take anybody any vector like that multiplied by scalar and then resulting vector if you take at take all these terms and add still it will be 0, the scalar will be common. Are you following me? Suppose take this and multiply by C, C into within bracket alpha 1 plus alpha 2 plus dot, dot, dot, to alpha n.

So, alpha 1 to alpha n, if added equal to 0 after multiplying by C also is 0, is not it. So, that will be not only a subset, but that will be a subspace of R N. Consider matrices, say M cross, not M cross, it is a square matrices to start with. This is we write as the R N cross N know, I think how it is written R N cross N. This set of matrices is also a vectors, obviously same logic add two matrix of this side you get the matrix of the same size. Take a matrix multiplied by the scalar you get matrix of the same size. All 0 elements put 0 elements in all places you get all 0 matrix, reverse the sign of each element of a given matrix you get the negative of the matrix. Those are trivially obtained.

So, there is a space, vector space. There if you consider say all matrices which are diagonals, then there is a subset of this. But it is not only a subset, it will be a subspace because if you consider the set of all diagonal matrices, that set is a subset of this I agree. Because diagonal matrix of this size also is a matrix of size N cross N is not it. A diagonal matrix of this size also is a matrix of this size that is part of this vector space. But if you only pickup this diagonal matrices from a, from a set that is a subset of these because only we are picking up a special kind of N cross N matrices.

So, that is a subset. But my claim is that that is not only the subset that is also a subspace. Suppose, if you take one diagonal matrix, take another diagonal matrix, if you add them it still remains the diagonal matrix only. If you take a diagonal matrix, multiplied by a scalar, it still remains a diagonal matrix only, closed under scalar multiplication addition. Other properties follow easily and diagonal matrix with all 0 diagonal entries and 0 matrix. Diagonal entries you find, you reverse the sign it becomes a negative arithmetic these are trivially done.

Coming back to this, this first example, what is the dimension of that original vector space? but how do you come to that? Dimension of these column vectors, you know its N capital N, how to show that? To show that suppose I consider a set of basis vectors trivial, we call it a trivial basis, then how many you can get you can easily. Suppose, I consider e 1 where I put 1 and 0s, then e 2 and finally dot, dot e n. How many you can get such N. Are they firstly linearly independent?

Answer is yes, physically you can see any vector cannot be written as a linear combination of the rest because where this guy is non 0 that is equal to 1, others are all 0. If it you combine the rest by some linear combiner coefficients and all, you cannot produce 1 out of the 0s with others. If you to have to do it purely mathematically you multiply these by C 1, multiply these by C 2 and dot, dot, dot, and add. What will you get C 1 times, C 1 means only C 1 here 0s here. C 2 times, C 2 means 0, C 2, all 0s so on and so forth.

So, you can get a vector where all elements will be C 1, then C 2, then C 3, dot, dot, dot, C n. Equate that to 0 means C 1 equal to 0, C 2 equal to 0, C n equal to 0. So, linearly independent. So, fine they are linearly independent. Then the next question is, do they span the entire R N. That means if I pick up, they span a subspace that a set of all possible linear combinations of this. That is a subspace span by them, a space span by them. So, they are linearly independent.

So, basis of that space also all that is fine. But is it that the space span by these fellows is nothing but R N is not subset of R N, but it is a whole of R N? Answer is yes because if I give you, if you give me any vector like these from R N, I can simply write alpha as a linear combination like alpha 1 times e 1 plus alpha 2 times e 2 plus dot, dot, alpha n times e n, is not it. That means they are not only linear independent, they also span the inter space R N.

So, that means they are a basis, they are one particular basis of R N. Then from this I find out what is the number capital N, but that is all I want. What is the number how many basis elements N. So, I know dimension of the space that is N. Now, these basis you know if you really expand alpha as a linear combination of this what are the combiner coefficients you get? Alpha 1 alpha 2, do you get any new information?

No. Alpha 1 alpha 2 that alpha N they already available within a vector. But maybe you can form another set of basis vectors. How many total N, the moment I find another set of basis vectors and total in capital N. Another set of linearly independent vectors total in capital N, obviously, that will span R N because dimension of R N is N. Then may be using them if I try to express alpha as a linear combination there will be new combiner coefficients c 1, c 2, maybe they carry some extra information which are useful to me.

This is the basis of all transforms which you study. In the case of all, like I am writing as a vector you just you stop vertically you make it horizontal that will be a sequence of length N. DSP is all about sequences, 0 th point to n minus 1 th point, and this will become trivial sequence basis 1 0 0 0. That is delta n, delta n sequence 0 1 0 0 0 that is delta n minus 1.

So, this way we have n sequences, they form a trivial basis of the sequence space, sequence space means sequence of all length. I mean space of all sequence of length capital N. I only want to know the dimension of the space, by this trivial basis I found out dimension is n, then forgot about these basis. They give me nothing but the original samples of the given sequence alpha as they are linear.

So, I want to find out new basis. So, that using that those basis vectors I can get new linear combiner coefficients to express current sequence, you understand. That is maybe I want to find out some new basis vectors say gamma, gamma 1 to say gamma N basis of say R N, another set of basis, but total number must be N that is important. See if you can find out linearly independent vectors, you cannot find more than capital N obviously. But if you find up to capital N which are, they obviously span R N. What is the logic for that? Suppose they do not span R N, then you consider the span of this, that is content in R N because on gamma one to gamma N they are all content in R N.

So, their span also content in R N. If the span is not full of R N, but content in R N, that means outside the span, but within R N, there is another vector, there is subspace from that I can pick up a vector append to this. This becomes a linearly independent set of N plus 1 element which is contradiction. Because in a vector space of dimension N cannot have more than N linearly independent vectors, is not it. So, you start constructing gamma 1, gamma 2 dot, dot, and moment you arrive at N you cannot get any more, but this gamma 1 to gamma N will be one basis, that was there is infinite choice of such basis.

So, here I will pick up some basis. So, that the linear combiner coefficients that is given any, say alpha I will write it as C 1 gamma 1 plus c 2 gamma 2 plus dot, dot, dot. So, c 1, c 2 may give this me some extra information some useful information not visible directly in alpha. So, depending on that various basis vectors have been proposed one is will, I will come to this little later again. One is DFT, another is DCT, another is ((Refer Time: 36:20)) transform, another is (Refer Time: 36:21)) transform.

So, much and all these, I will consider this little bit more, litter later actually. In some cases the basis could be such, then so happens that when you try to find out linear combiner coefficients C 1, C 2, C 3 up to C n, you may find then only few are turning out to be big sizeable, others are very close to 0, because of the statistical properties of these vectors and all that. See that cases I can achieve a lot of compression, is not it. I do not need all that to store or transmit all the n that is the basis of DCT, in DCT it so happens.

You have a compression in the coefficient domain, these coefficients when you find out linear combiner coefficient, those are the transform coefficients. And when you write using the coefficients as a linear combination original vector that is the inverse transform relation. May be little more I will consider after I cover inner product. So, you got this example I will not, I am not covering any further example. Now, because sequences are things over primary importance.

Similarly, instead of sequence if you are dealing with continuous function, 0 to capital T some period. Take all functions within these, they form a vector space because you add two functions you get a function within the same duration only. Take a function multiplied by scalar you get a function of the same duration only. A function that takes 0 values all along these a 0 vectors, that is a 0 function. You take any function reverse the amplitude at any point, you get another function over the same size, same duration where there is so called negative of the original function, so on and so forth.

But they are I will not going to that is not algebra they are actually the dimension is infinity. Dimension is infinity and this summation becomes integral, you get Fourier transform, analog Fourier transform, inverse transform. That is not difficult to study. In algebra, you do not have the problem. This algebra all finite addition summation, there is no integral, there is nothing continuity discrete this plus this or this plus this. And finite dimension also, I am saying that process will of finding a basis will converge after a finite number of terms that I put cleverly, that is where everything after all just finite number of terms things converge.

That means any vector will be a, what linear combination on the basis vector. What kind of linear combinations finite, and finite linear combinations are already defined in my definition because in a vector space definition I give you definition of addition, vector addition where you take only two. So, two means if you have to add two 10 vectors two at a time, then another one, then another one like that. This is possible as far as those axioms, but if I say infinite, there is no notion of infinite addition there.

What is meant by infinite addition how many times you never know. Those that axioms give only addition of two, it takes two vector give you another vector, not infinite vector. That infinity brings in all hell a lot of problems actually, luckily for all our purpose, we are not dealing with those cases. Finite dimension of vector space all accurate no problem in that. Now, I come to another notion, that is of inner product which is actually equivalent of your dot product. In that 3 dimensional, another example was that 3 d vector space or 2 d vector in that position vector. Suppose this is the way x y axis, if you take this axis, this axis, any two vectors.

The space that span by those two positions vectors is the entire plane. Because one vector, another vector will linearly combine you get another vector. So, each point if you draw a line from the origin it is a vector. So, this space is nothing but a set of vectors starting from origin. Connection of all such possible points is this plane and any vector or any point in this plane is a linear combination of the two directions you have chosen. And they those two directions are linearly independent because one is not in the same direction of the other.

So, one cannot be written as a linear combination of the, or multiple of the rest. And then if you go to the upward direction not in the sense vertically it an angularly upward, that new vector will be outside that space. Because that is not in that space and obviously, that cannot be written as a linear combination of the previous two basis vectors or excess. So, these three will form a basis of this 3 d world, but their dimension is 3. From the physical limitation only you can see, you cannot go beyond. 3 means, but you can choose any set of vectors, not in the sense one of them is that vertical thing x y z.

But it is not necessary that they have to be vertical. So, you can choose there any arbitrary orientation theta 5 and those kind of angles you know. So, the exclamation, one is not in the plane spans by the other two, they are linearly independent and by using them as a, as a linear combination you can express any vector in the space. Dimension remains three choice of basis is infinite either three vertical axis or three points at any angle with each other. Making sure that one does not lie in that direction of another or in the plane span by the other two, then the dependence will come up. So, you cannot span the entire 3 d world. Orthogonal is a special case when they are at 90 degree, but does not have to be 90 degree see this.

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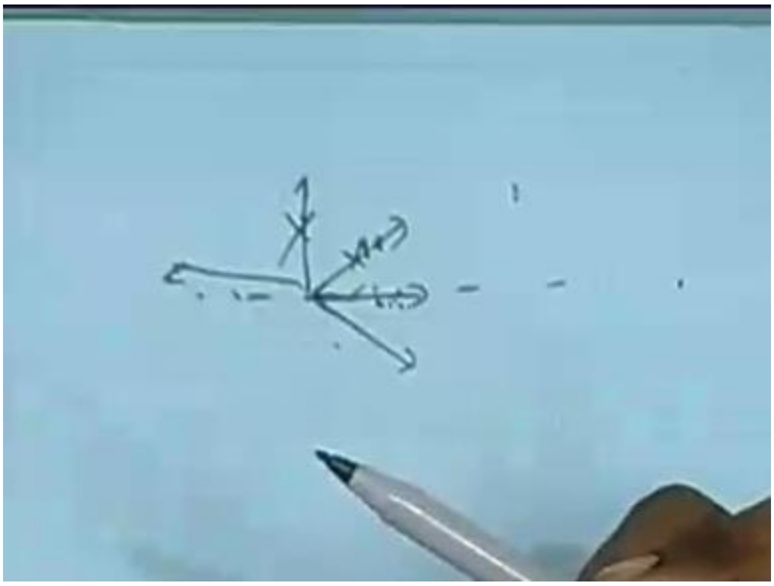

Consider the 2 d plane, this is orthogonal obviously, this two are linearly independent. But suppose I do not take this, I take these two they are not orthogonal, but still they are linearly independent. This is not something times these and vice versa. This not, the space span by this is, this axis any point in this line is a scalar multiple of this vector. The space span by this vector is this line I go outside this.

So, I bring not especially this guy which is at 90 degree, this fellow. This is obviously, if these and these are linearly independent because this lies outside this place this cannot be a multiple of this. Together these two will be spanning one space which is this 2 d plane.

You take any vector, you draw a line parallel to these and all you know you can, you can write any vector is a linear combination of these two, is not it. So, that means these two span the entire 2 d plane, then you go one step ahead, you go vertically up. Not necessarily 90 degree, but just point out by any vector, that vector does not lie this 2 d plane. So, it is not a linear combination of these two.

So, again consider the span of that new guy, this new guy and those previous two. They will span from our original knowledge you know they will span the entire 3 d world, any vector in the space can be written as a linear combination of those three, which means this 3 d world has dimension 3. But basis is not unique, I could have taken, instead of these I could have taken these direction, these I could have taken in these direction these two-third one, I could have in other direction. But total is 3. Now, I go to that dot product thing, which is called inner product. There you are familiar with this notion something dot something. Something dot something, but that dot will be gone, we will be using these notion

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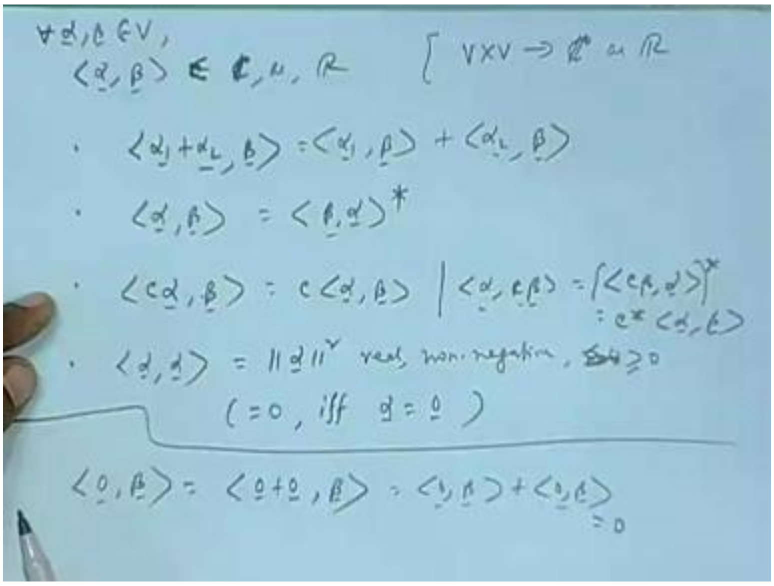


Alpha beta, instead of alpha dot beta alpha beta. I am not writing the full mathematical sentence that there is a finite dimensional vector space V and alpha beta or any two vectors. Just for all alpha beta element of V, this is the inner called the inner product which is equivalent to a dot product. The more generalized notion of the dot product. In dot product, since it is a 3 d world specific vector space, there is a notion of length, there is a notion of angle, and all that you give a specific definition there. Length of p times, length of q times, cos of theta, theta is the angle there is an.

But here in the abstract vector space there is notion of angle. But people mathematicians are also intelligent, they found out that angle and all they are not important actually, they give only the specific value, but there are certain basic properties that dot product satisfies there which can be written in a very general way. And those, what is that enough to get all beautiful results that you get there also. The presence of constant only gives to a specific value of that dot product exact value. But properties do not depend on that, what is the more general.

So, they found out and wrote some axioms, that alpha and beta are the inner product, it is nothing but it is this means these are element of the field C or R. Our vector space is defined over C or R mathematically, this means alpha beta that is V cross V inner product means V cross V map to C or R. You understand this, V cross V take one element of V call it alpha, take another element of V call it beta. For any pair alpha beta, on every pair alpha beta you define a scalar number, you map it to a scalar number, which is complex or real depending on the field you have chosen.

If you do not understand this, it means this inner product will give you a number scalar number which could be a complex number or real number depending on whether dealing with complex field or real field. But then it should be satisfy some basic properties axioms, one is alpha 1 plus alpha 2 comma beta, is separable it is. These are axioms this must be satisfied. If instead of one vector, you write as a summation of two vectors and then inner product is same as inner product with alpha 1 inner product with alpha 2 addition of the two scalar numbers, then alpha beta is the complex conjugate of beta alpha. Alpha beta or beta alpha, one is the conjugate of the other.

In the case of real field, there is no meaning of this because conjugation has no meaning there. Then if you take C alpha, C for any scalar, this C can be pulled out that is you take C alpha, then inner product is same as first do the inner product between alpha beta, multiply the scalar by C we will get the same result. And last one is very important all are important actually. That if you take the inner product of the vector with itself, like in the case of dot product p dot p, what does it give square of the length, that is more generally called norm square, it is denoted as.

These number whether you are dealing with irrespective of whether you are dealing with complex field or real field. That is inner product is in general a complex number or real number this must be real. This is real and non negative always, that is less than 0, less than equal to 0 sorry, greater than equal to 0. And equal to 0 that is greater than equal to 0, in that it will be equal to 0, if and only alpha is the 0 the vector. Remember this alpha with itself will be denoted as norm square, that is greater than equal to 0 always real non negative, but it can be equal to 0 only if alpha is 0. Now, certain things we have to see what happens.

Student: sir, second was a conjugation?

Conjugation, yes. What happens if I take a 0 vector and with that I do any inner product. Out of these axioms it will follow easily, 0 means I can write 0, 0 plus 0 that is the definition of 0 with beta and I apply this first axiom. It is again 0 comma beta plus 0 comma beta obviously, 0 comma beta is 0 scalar 0. See inner product with any 0 vector is 0. This says C alpha beta equal to these if it is, if I generalize alpha and C beta, then it is, you can write C beta comma alpha conjugate from this relation, then from this you pick up that is conjugate of the entire thing, then take C out then you again conjugate. So, you get C star alpha beta.

So, C in the first coordinator, first thing I call it is a coordinate loosely, that can come out as it is, but C if present here it will come out with a conjugation. Now, if two vectors are such that their inner products are 0, they are called orthogonal. If two vectors are such that there inner product is 0, then they are called orthogonal. If on top of it their inner product their individual norm is 1, then there they are mutually orthogonal and their individual norm is 1, they are called orthonormal. And any vector can be normalize easily forgot our orthogonal.

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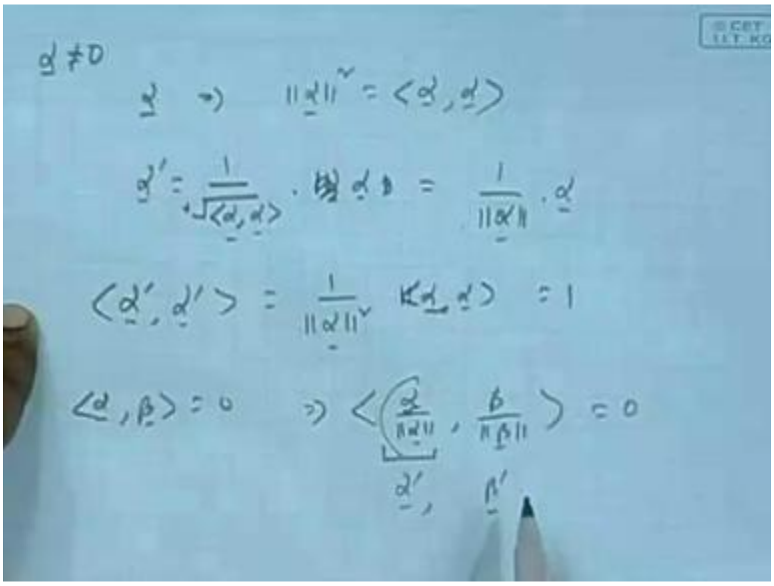

Suppose I use a vector alpha, its norm is, norm square is alpha square, which is these with itself, is not it? I think we have just five more minutes, alpha with itself. Now, suppose I take, I define a vector which is nothing but and alpha is non 0 given to be alpha is non 0. So, this is a positive real number, this a positive real number division by that is possible no division by 0. They are positive number non 0 in that case if I form alpha and if you find a new vector which is alpha only just a scalar multiple there is a scaling of alpha. I will take alpha and use this square root of positive square root times sorry, alpha.

So, it remains otherwise alpha, but just scaled or you can write just norm of alpha times alpha that I call alpha prime, there some dividing by the norm of the original one. Then alpha prime, alpha prime will be unique norm. If you take alpha prime with itself, if you take alpha prime with itself, then what will you get put that here, put that here, put this thing here alpha prime, put the same thing here. Then by the axiom 1 by norm alpha will go out and from the second coordinate again 1 by norm alpha with a conjugate should go out, but this a real number, is not it.

So, no conjugation so that means, 1 by norm alpha square goes out. Because it goes out twice from the first coordinate and second coordinate, what are left with alpha with alpha, so this alpha with alpha which is norm square of alpha. So, they cancel and you get 1. So, very simple you take the norm square find norm square of a vector, take the positive square root call it norm just divide the vector by the norm real thing vector is nothing but the original vector just scaled down.

So, that I mean your norm is 1. Now, if there are two vectors alpha and beta they are mutually orthogonal, but not each not having normal norm I mean unique norm. I can normalize alpha and beta by this factor, but orthogonality it will prevail, is not it. Because after all alpha prime will be some scalar times alpha, that is suppose alpha and beta 0. And I find that I what I do, I make alpha by alpha prime sorry, alpha norm beta by beta norm. My claim is these also should be 0, you take 1 by alpha norm outside 1 by beta norm outside. You here you still have alpha comma beta.

That is in the space suppose two vectors are at 90 degree, if you scale up those vectors 90 degree does not change, they remain orthogonal. Same thing, I am scaling. So, that this total thing has unique norm total thing has unique norm, but at the after this scaling also are they remain orthogonal these total vectors? Yes, because take this constant outside 1 by norm alpha 1 by norm beta, that times inner product alpha comma beta which is already 0 given to be 0.

So, that means they remain orthogonal this is still 0, that means they remain orthogonal you call this vector alpha prime, call this vector beta prime. So, alpha prime beta prime also are orthogonal, but with added advantage that their norms are 1. So, unique norm, then they are called orthonormal mutually orthogonal and each having unique norm orthonormal. So, normalization is not a problem you understand.

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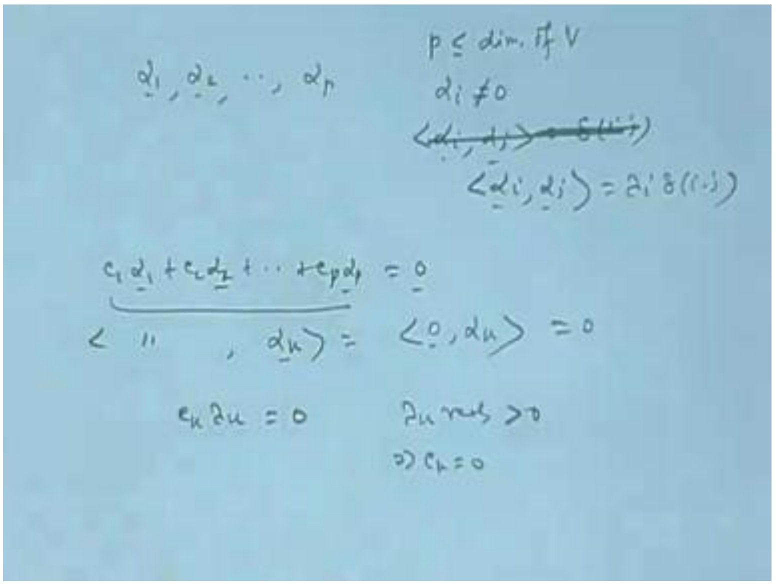

Very quickly you see this result. If suppose you have been given some number alpha 2 alpha dot, dot alpha p in a vector space. P of course, has to be less than equal to dimension of the space, dimension of V that let we will see why. You are taking vectors and no alpha, no alpha i is 0 vector and given alpha i, alpha j is say delta ij what does it mean quickly because the time is running out. This we will see why I am taking p little later

Alpha i non 0 means I am taking up set of vectors alpha 1 to alpha p none of them is a 0 vector and this a orthonormal set, alpha 1 is orthonormal with alpha 2, alpha 3, alpha p, that is why is the inner product the moment in and j are different. 0, i and j are same norm equal to one norm square that is equal to 1. But then you could say that if I just alpha i, if I had alpha i equal to 0 then it is orthogonality would have been present, is not it. If you have to prove this I do not need this orthogonal, I mean, I can have this you know does not have to be orthogonal or normal, we can take some lambda i, delta i minus j.

What I am going to prove, it does not require normality because orthogonality is enough. That is a set of vectors which are mutually orthogonal, mutually orthogonal, that is only when i and j are same that is lambda i inner product. Lambda i is means norm, alpha i with itself is norm. So, lambda is a real positive number, alpha i not equal to 0, because if you append 0, it is orthogonal to everybody because 0 with inner product with anybody and 0 inner product with anybody is 0.

So, I am appending running out. Now, if I consider I will say that this set is a linearly independent set, if they mutually orthogonal they are linearly independent, not the other way. Very easily you can see C 1 alpha 1 plus C 2 alpha 2 plus dot, dot, C p alpha p if you equate it to 0. Is it that only solution is C 1 equal to C 2 equal to C p equal to 0? To see this, suppose I find, I want to find out C k is C k equal to 0. So, what I do suppose I take the inner product of this entire quantity, we say alpha k, then here also inner product with alpha k.

But I have already proved that inner product of any vector with 0 vector is scalar 0. And in this summation, you apply that axiom, you can separate out that inner product of C 1 alpha 1 with alpha k, C 1 moves outside alpha 1 with alpha k, orthogonality 0. C 2 alpha 2, C 2 goes out alpha 2 with alpha k orthogonality 0. So, which term only survives only when C k alpha k, C k goes out alpha k with itself alpha k with itself is lambda k. And since I am not having 0 vector here, lambda k cannot be 0, norm square is not 0, lambda k is a positive number strictly positive number.

So, you get C k lambda k equal to scalar 0, lambda k is real greater than 0, that means C k equal to 0. So, a set of non 0 vectors and a set does not content, when the set does not consists of 0 vector a set of non 0 vectors and non consisting of 0 vector is a is a linearly independent vector. You understand then why I took p 2 be less than the because in a dimension in a vector space V of finite dimension, you cannot have linearly a set of linearly vector independent vectors where number is greater than equal to greater than dimension.

That is why to be on the safe side I took these. This is a very vital result, all the transforms that you study they are basically orthogonal vectors with respect to some inner product. I will take that example may be with some inner product the orthogonal vectors. That is all for today.

Thank you.