Bayesian/MMSE Estimation for MIMO/OFDM Wireless Communications
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Lecture – 08
Minimum Mean squared Error (MMSE) Estimation Application - Wireless Fading Channel Estimation

Hello, welcome to another module, in these massive open online courses Bayesian MMSE Estimation for Wireless Communications. So, now, so far in the previous modules, we have looked at a application, we looked at MMSE estimation, when both the observation, and the parameter as distributed as a Gaussian random variables, and we also we looked at application, of the MMSE estimator for a wireless sensor network let us now, look at an MMSE estimation example for Wireless Channel Estimation. So, we want to look like.

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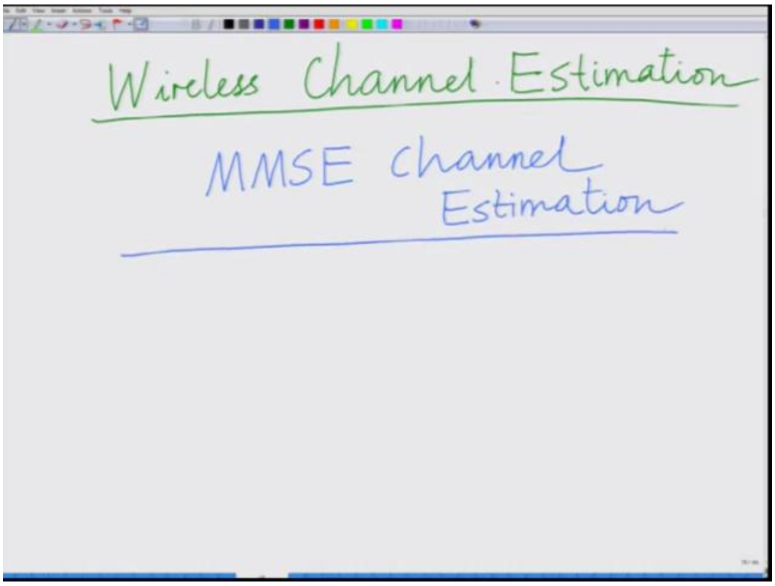

So, today, we would like look at, or I would, we would like to look at, Wireless Channel Estimation, or more specifically, an application of MMSE. So, MMSE, which stands for Minimum Mean Squared Error, Minimum Mean Squared Error Channel Estimation. So, this is what we would like to look at.

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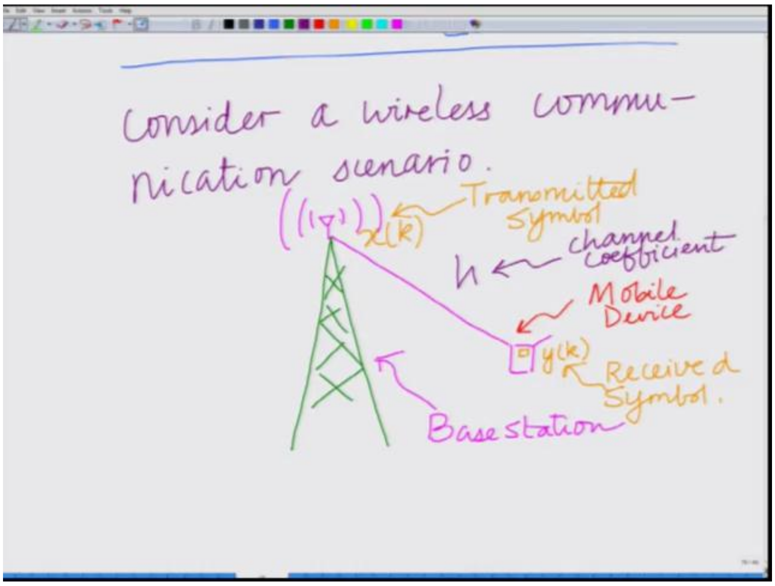

Now, let us start by considering a wireless communication scenario. Consider. So, consider a wireless communication. In a wireless communication scenario, what do we have, we typically have a base station, which is transmitting to a, mobile terminal or a mobile device, basically, all right. So, let us consider this scenario, let us schematically represent this scenario, where I have a base station. I have this base station, which is transmitting to a mobile. So, I have a base station, with an antenna, which is transmitting. So, what is this? This is my base station, which is transmitting to the mobile or wireless device, so your mobile device, over the radio channels.

Now, let say the transmitted symbol is denoted by xk, the received symbol is denoted by yk. So, we have transmitted symbol, which is denote the kth transmitted, k denotes the time instant. So, xk is the transmitted symbol. The received symbol at your mobile device, is yk, this is your, this is your received symbol, and we have a channel, h, this is basically denotes the channel coefficient, the wireless channel coefficient. So, sk is the transmitted symbol that traverses through the channel or that basically traverses this radio channel, characterized by this channel coefficient h, and the corresponding received symbol is given by yk.

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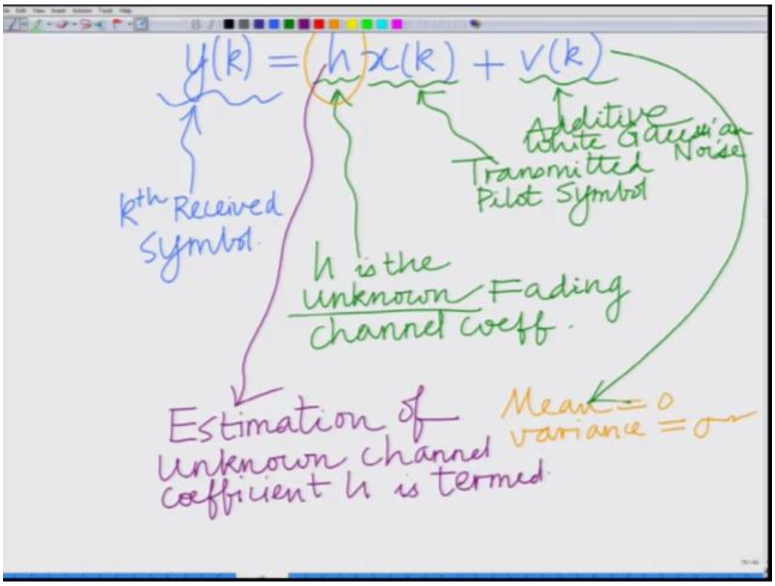

Therefore, this system this input output model, where the input is the transmitted symbol, and the output is a received symbol can be modeled as.

So, wireless MMSE, for wireless MMSE Channel Estimation, we have the model given as yk, equals h times xk, plus vk, where yk is kth received symbol, correct? This is the kth received symbol, at the mobile. We are considering a simple downlink scenario, although this can also be an uplink scenario. So, this is kth received symbol, h is the unknown fading channel coefficient, which has to be estimated, this is the unknown, that is the important point, this channel coefficient is unknown, is the unknown fading channel coefficient, xk is the transmitted pilot symbol. This is the transmitted pilot symbol; I am going to talk more about this, shortly.

So, this is kth transmitted pilot symbol, and vk is the additive, white Gaussian Noise. So, vk this is additive white Gaussian Noise, this has mean equal to 0, the mean of the Gaussian Noise is 0, variance equal to sigma. So, mean of this vk, which is additive white Gaussian Noise, the mean is 0 and variance is sigma square. And key aspect here, the important part here, is that this channel coefficient h, this is the channel coefficient h, which is unknown, and estimation of h, estimation of h is termed as Channel Estimation. Estimation of this unknown channel coefficient h, estimation of this unknown channel coefficient h, is termed as; this is termed as Channel Estimation.

This is termed as Channel Estimation, and for this purpose, for this purpose of Channel Estimation; the base station transmits a set, or a sequence of pilot symbols. There are 2 kinds of transmitted symbol, 1 is, the information symbols, these are unknown to the mobile all right, because these carry the information; However, the transmitted pilot symbols, which do not carry any, any information, and a simply used for Channel Estimation. These are a fixed number, of or a fixed sequence of pilot symbols, which are transmitted by the base station, to aid, the mobile or to help the mobile perform Channel Estimation.

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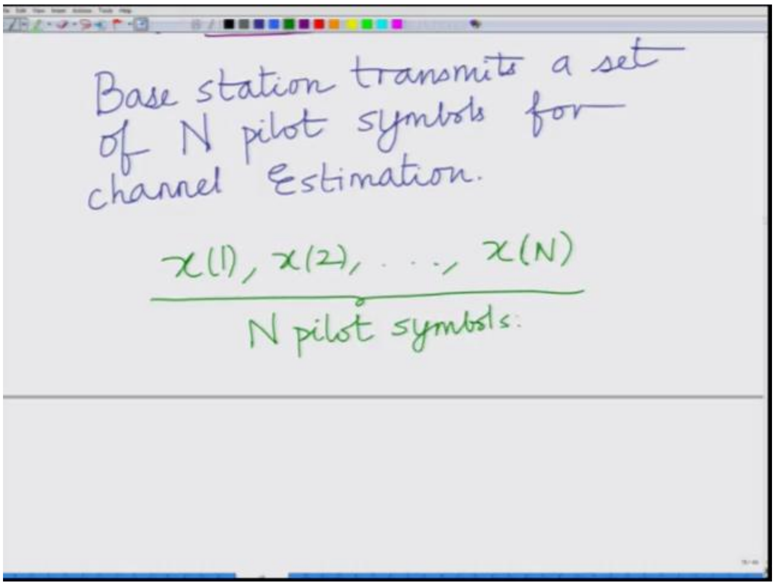

So, base station transmits, a set of N pilot symbols, for Channel Estimation. It transmits a set of N pilot symbols, which transmits an extra for a Channel Estimation. And these channel pilot symbols, we are denoting by x 1 x 2 so on up to xn. These are the N pilot symbols, transmitted for the purpose of Channel Estimation.

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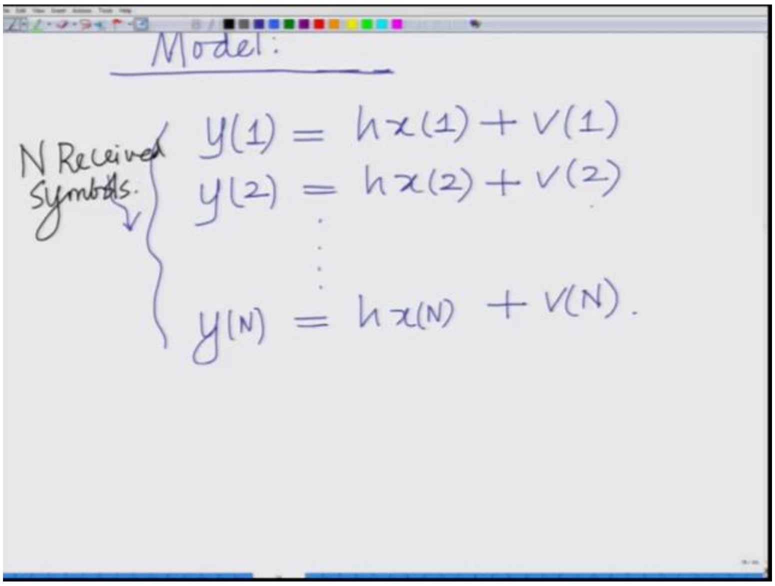

So, the model that we have, for Channel Estimation, is the following, all right? We have by y1, that is the first received symbol, corresponding to k equal to 1, equals h times x 1 plus, v1, y2 equals h times x2 plus v2 so on and so forth, yn equals h times xn plus vn. So, this is the model. So, we have N received symbols, and this is basically, the model. So, we have N, N received symbols, corresponding to the n transmitted pilot symbols, and v1, v2 up to vn, are the N noise samples. And what we are going to now is, basically we are going to vectorize this model, that is, we represent this model using Vector Notation. And that will be convenient for manipulation. So, that will be convenient, to later manipulate it to a form, that is, and get the final expression for the Channel Estimation at h.

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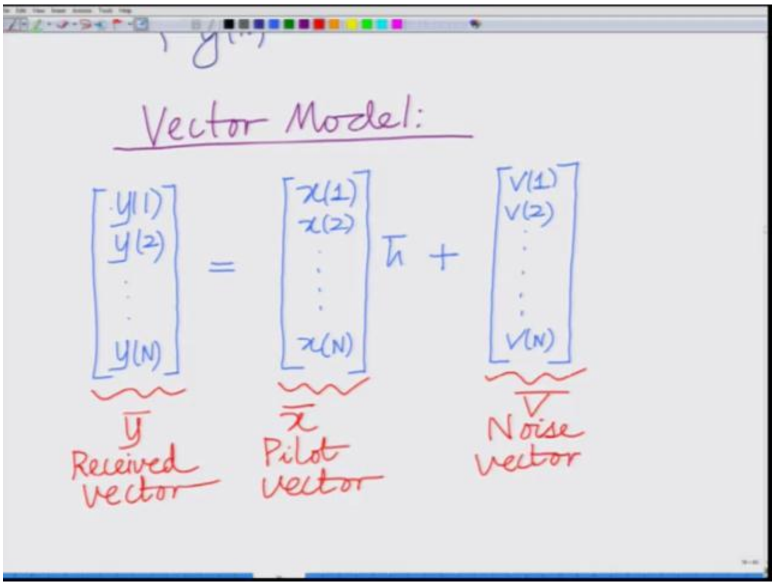

So, now, what we are going to do is we are going to recast this, as the Vector Model. Going to recast this as the Vector Model - First, we can write the received Vector y bar, that will simply, that will simply consist of, the received symbols, y1 y2 up to. So, we have the received Vector y bar, which will consist of, the received symbols y1, y2, yn which is basically, the channel Vector, which is the channel, the transmitted symbol of Vector of pilot symbols, x1 x2 up to xn times h bar, plus v1 v2 up to vn, this is the noise Vector. So, what we have? We have y bar, which is the received Vector, or observation Vector. We have x bar, which is the pilot Vector, and we have v bar, which is the simply the noise Vector.

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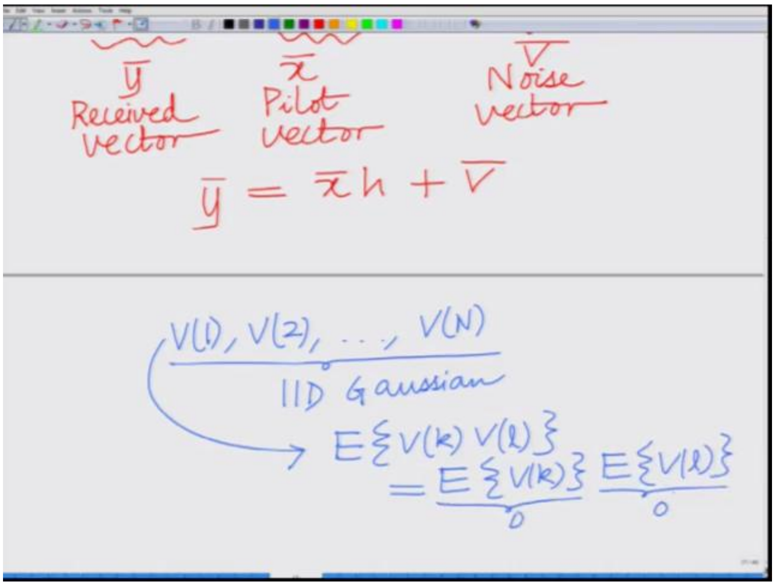

And therefore, what we have is, we have v bar, equals x bar times h, plus v bar, and h is the unknown channel coefficient, and each Vector, y bar, h bar, and v bar, are of size n cross bar, so these are n crossed 1 Vectors. h is the unknown channel coefficient, and this is our model for Channel Estimation.

And one more assumption, I remember we are already made this, we are going assume, that these noise samples, v1, v2, vn, are IID Gaussian. IID Gaussian means, independent identically distributed Gaussian, similar to the wireless sensor network, these each has mean 0 variance sigma square, and they are independent, independent means. So, v bar v1, v2, vn, these are independent, just to briefly repeat this v1, v2 up to vn, these are IID Gaussian, means each as means 0 variance sigma square, and they are independent, means that, expected value of vk into vl, 2 different noise samples, vk into vl is equal to 0, that is this is equal to, this is can be simplified that, simplified as, if k is not equal to l, this is expected value of vk, into expected value of vl, and since both these expected values are 0, this can be simplified as, equal to 0, if k is not equal to l.

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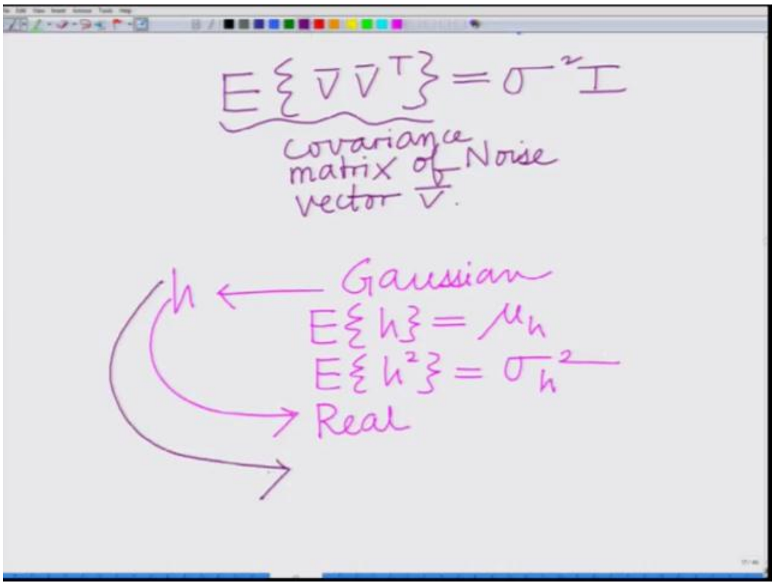

Therefore, the covariance, and this is also something that we have again derived in the context of the previous example, for the wireless sensor network, that is expected v bar, v bar transpose, is sigma square times identity matrix, this is the covariance matrix, covariance matrix, of your noise Vector, so variance matrix of the noise Vector, v bar. Further we are going to assume, similar to the previous MMSE estimation framework, because remember for the MMSE, in the MMSE estimation scenario, the parameter h, is consider to be a random, random parameter, that parameter h is considered to be a random in nature. And similar to the MMSE, since this is we are considering the Gaussian MMSE estimation framework, we are going to consider, assume the channel coefficient h, to be Gaussian in nature, all right, we are going to assume the channel coefficient h, to be Gaussian in nature.

In fact, this is a very valid assumption, because, for a practical wireless scenario, the channel coefficient h is considered to be a complex Gaussian channel coefficient. This is also known as, the Relay Fading Channel Coefficient, that is a symmetric complex Gaussian channel coefficient h, which also known as a relay fading channel coefficient, because it is amplitude, follows the relay probability density function. For the purpose of illustration, however, we are going to simply consider real channel coefficient h in this discussion; however, it can be very easily extended to a scenario, where the channel coefficient h is, complex in nature. So, our assumption here, is basically that this channel coefficient h, is Gaussian, with mean that is expected value of h, equal to mu h, and variance, and variance expected value of h square, equal to sigma h square. We are going to consider this at present, to be real in nature, but we will also note that, it can be extended to a scenario with complex channel coefficient h.

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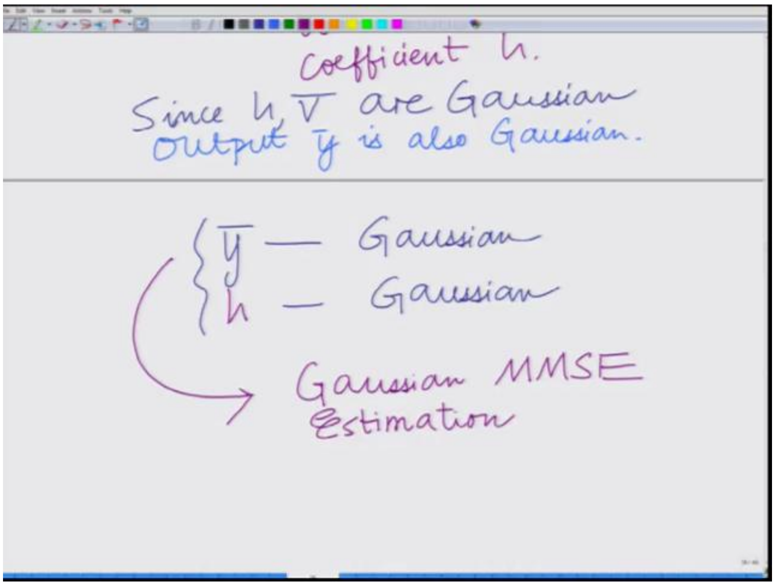

This can be easily extended, to a scenario with complex channel coefficient h, and therefore, we have assumed something about the channel coefficient. We assumed that the noise Vector to be, the noise samples to be, IID independent identically distributed Gaussian, with mean 0 variance sigma square. These are the 2 assumptions that we have, fine?

Now, let us start with the mean. Now let us make also one, one other observation. That is, look at this, if you look at this, we have y bar, which is a linear combination. So, y bar is a linear combination. It is a linear combination, of h coma v bar. And therefore, since h coma v bar are Gaussian in nature, y bar is also Gaussian in nature. Since h coma v bar are Gaussian, output, output y bar is also Gaussian. So, basically we have an output Vector y bar, which is Gaussian, parameter h, which is Gaussian. So, we have Gaussian, Gaussian scenario. So, now, we can use the principle of Gaussian, MMSE parameter estimation. So, you can use the principle of MMSE estimation, for Gaussian parameter. So, y bar is Gaussian, this is an important aspect, h bar is Gaussian.

So, basically since both of these are Gaussian, we can use Gaussian, MMSE result for Gaussian MMSE estimator, with a parameter y bar, where the observation Vector y bar and the parameter h are both, Gaussian.

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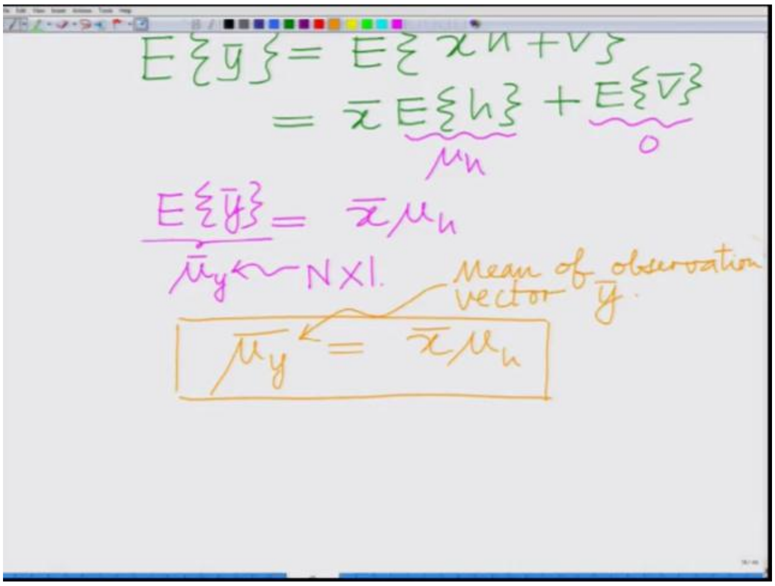

So, towards this end, first let us find the mean, of the observation Vector y bar. Well what is this? This is the mean, of y bar equals, x bar h, plus v bar, which is equal to x bar, into, expected value of h, plus expected value of v bar, well expected value of v bar, this is equal to 0, since the noise is 0 mean, we have said that expected value of h, the unknown channel coefficient, is mu h. So, therefore, this will be equal to x bar, times mu of h.

So, this is expected value of y bar, we can also represent this as mu bar y. Which is basically, by the way, since y is an n cross 1 Vector, mu bar y is also an n cross 1 Vector, therefore, let us rewrite this, clearly, mu y bar equals, x bar times, mu of h. So, this is an important result, what is this? This is mean of the observation Vector y. Mean of observation, mean of the observation Vector y bar. So, we have an expression for the mean of the observation Vector y bar, that is mu y bar, and we know that the expression of the Gaussian MMSE estimate, we know that the expression of the Gaussian MMSE estimate, is given as, h hat is equal to Rhy into Ryy inverse, times, y minus mu bar y, plus, mu h where Ryy is the covariance matrix of y, Rhy is the cross covariance between the parameter h, and the observation Vector y.

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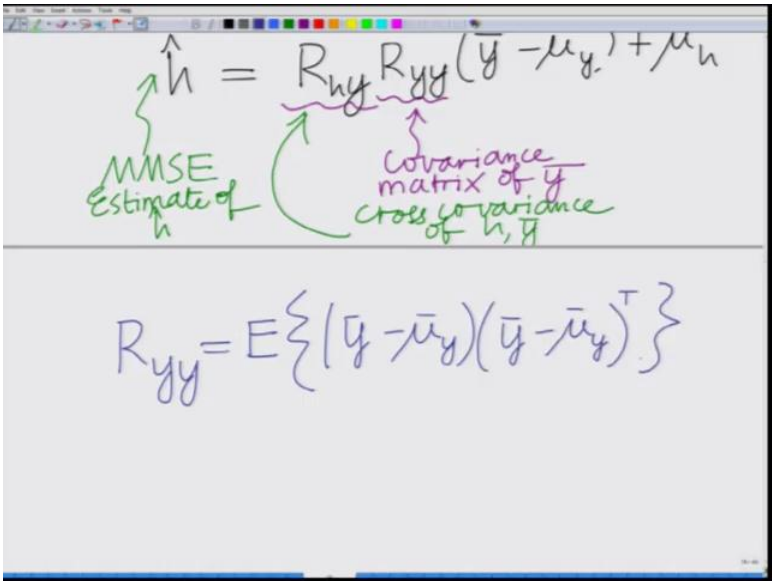

So, let us also write that down, we know that the MMSE estimate, h hat, equals, Rhy into Ryy into y bar, minus mu y, plus mu h, Ryy, this is the covariance of y, or covariance matrix rather. This is the covariance matrix of y, this is the cross covariance, cross covariance, of h coma y bar, this is the MMSE; this is the MMSE estimate, of the channel coefficient h.

Now, what is Ryy? remember Ryy, the definition of Ryy equals the expected value of y bar, expected value of y bar, minus, mu bar y, times y bar, minus mu bar y, transpose this is the definition of Ryy, which is the covariance matrix of y.

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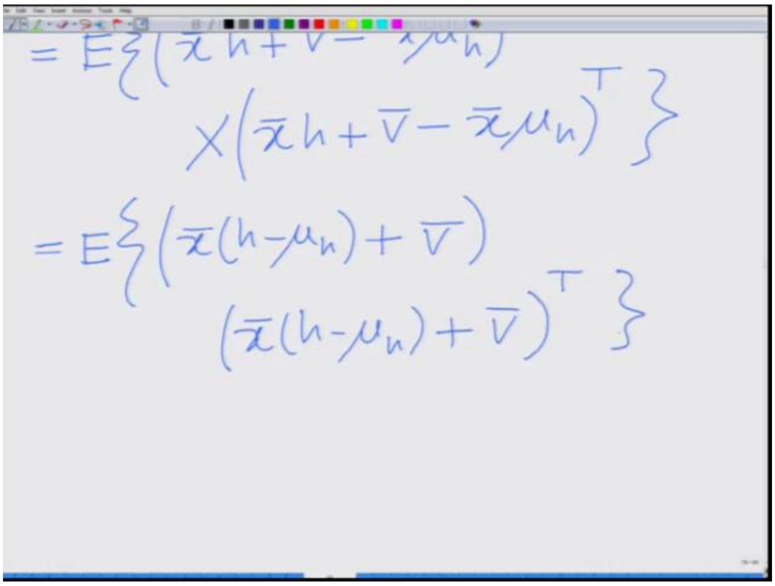

This is equal to, now let us substitute for both y bar, of course, we know y bar equals, x bar times h, plus v bar, minus mu y bar, equals x bar times mu h, times, the transpose of this quantity, x bar times h, plus v bar, minus x bar, into mu h, transpose, and this is equal to, now this can be again, just write it in this fashion, which will help us to simplify x bar into h, minus mu h. Just remove the mean of h that is mu h from h, plus v bar.

Now, we can see, times x bar, into the same thing, mu h plus v bar transpose. Now we are going to make another assumption, which is also, a very intuitive assumption. We are going to assume that the noise v bar and the parameter h are uncorrelated. Because a parameter has the channel we trying to estimate, the noise is the noise at the receiver.

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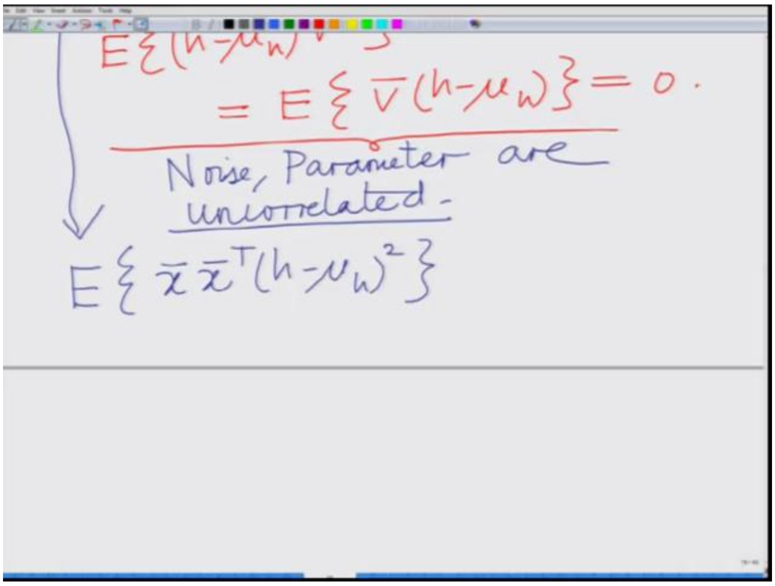

So, naturally these 2 will be independent in fact, so we are going to assume that, these 2 quantities are uncorrelated, and in since h is Gaussian, v bar is also Gaussian, uncorrelated will also mean that they are independent anyway. Because h and v bar are Gaussian in this particular scenario. So, we are going to make the assume, assumption, that, expected value of h, minus mu h, times v bar transpose, equal to expected value of v bar, into h minus mu h, is equal to 0. This basically implies that, noise coma, noise, and parameter, in this case, are independent. Or let us simply write that they are uncorrelated, because this simply implies, that they are uncorrelated.

But since they are Gaussian that is also translates into independence, of h coma v bar. Now, therefore, I can simplify this expression. I can simplify this expression for Ryy as, I can write it as a product of the terms, expected value of, well, expected value of, the first term will be, x bar, x bar transposes times h minus mu h square, since h is the scalar quantity.

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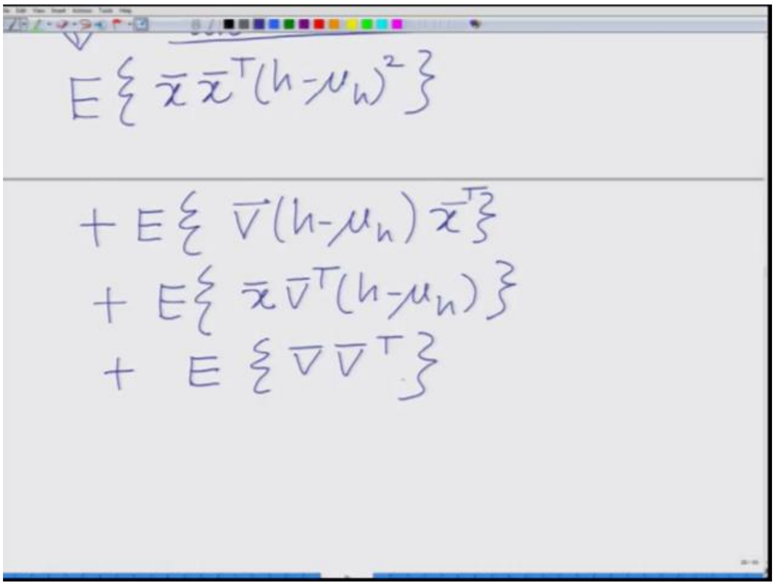

Plus of course, we can take look at the other terms, these are going to be expected value of v bar, times, h minus mu h, into x bar. We are going to simplify this later, since x bar is a constant quantity, constant pilot Vector, which will therefore, and come out of the brackets. So, this will be expected value of, well, this is x bar transpose, v bar h bar, and this is expected value of x v bar transpose, into h, minus mu h, just write each term carefully x bar plus expected value of well v bar, v bar transpose.

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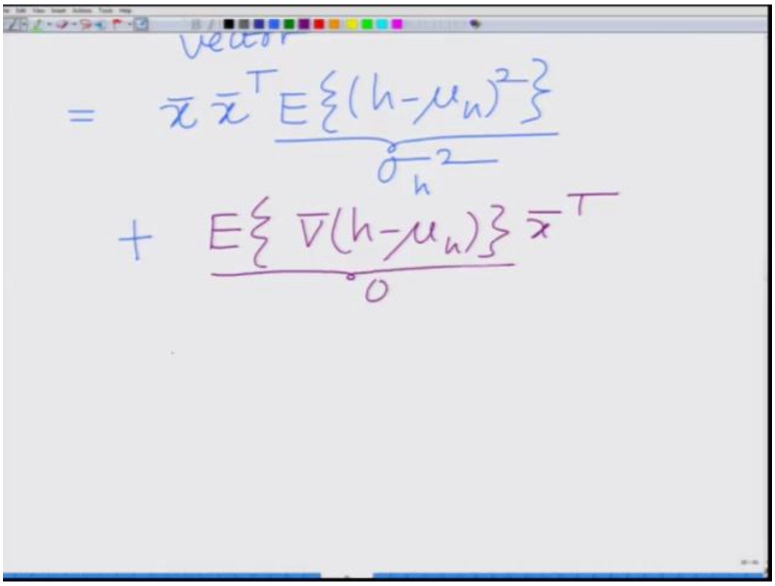

Now again since x bar is a constant Vector, I am going to simplify this, since x bar, x bar is a constant pilot, we have x bar is a constant pilot Vector, therefore, this is equal to expected x bar, x bar transpose, expected value of h, minus mu h square, we know this is equal to, sigma h square, plus, expected value of, v bar into h minus mu h, into x bar transpose, we know this is equal to 0.

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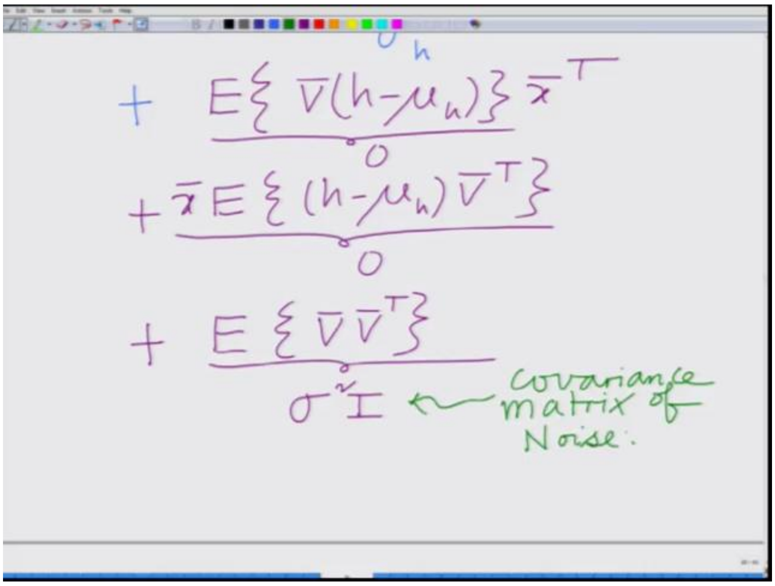

We just said that noise and the parameter are uncorrelated plus expected value of, x bar, v bar transpose, into, or expected value of x bar, into h, minus mu h, into v bar transpose, we know this is also 0, once again, because the noise and parameter are uncorrelated, and the last term is v bar, v bar transpose. We know this is sigma square times identity, this is the covariance matrix of noise. This is the covariance matrix of noise; this is the covariance matrix of noise.

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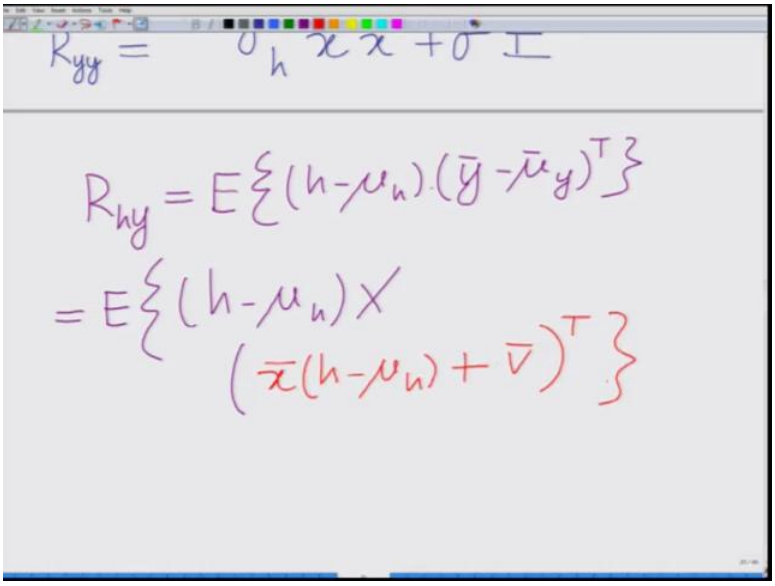

Therefore, the net, this expression can be simplified as, sigma square times x bar, x bar transpose, x bar transpose plus, or sigma x square times x bar, x bar transpose, plus sigma square times, identity. This is the expression for the covariance matrix of the observation matrix y bar. That is r y bar, y bar, Ryy equals sigma x square, times x bar, x bar transpose, plus sigma square times the identity matrix.

Similarly, now I can also simplify the expression for Rhy, the cross covariance between h and y bar, that will be given as, Rhy equals expected value of, well, h minus mu h, into y bar, minus mu y bar transpose, which is basically, I can simplify this as h, minus mu h, times, writing this below, this will be, we already simplify this, x bar into h, minus mu h, plus v bar transpose, which will be equal to, well x bar, once again taking x bar out of the bracket, x bar times, expected value of h, minus mu h, whole square, plus expected value of h, minus mu h, into v bar transpose.

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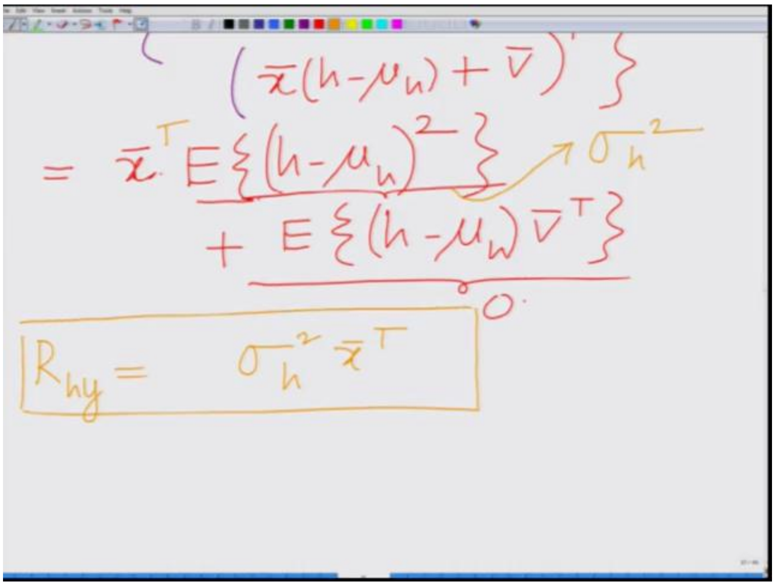

Of course, we know this is equal to 0, because a noise and parameter are uncorrelated. And this remember, once again, is simply expected value of mu h, h, minus mu h square, this is equal to sigma h square. So, therefore, this is equal to basically simply, a well, this is simply equal to well, there will be a transpose here, x bar transpose, sigma h square, x bar transpose, which is basically your Rhy. So, this is basically your, expression for, Rhy in the, above this, we have derived the expression for, Ryy. Now all we need to do, is substituted in this expression here, for the MMSE estimate, substitute in this expression for the MMSE estimate, and we are going to derive the final MMSE estimates.

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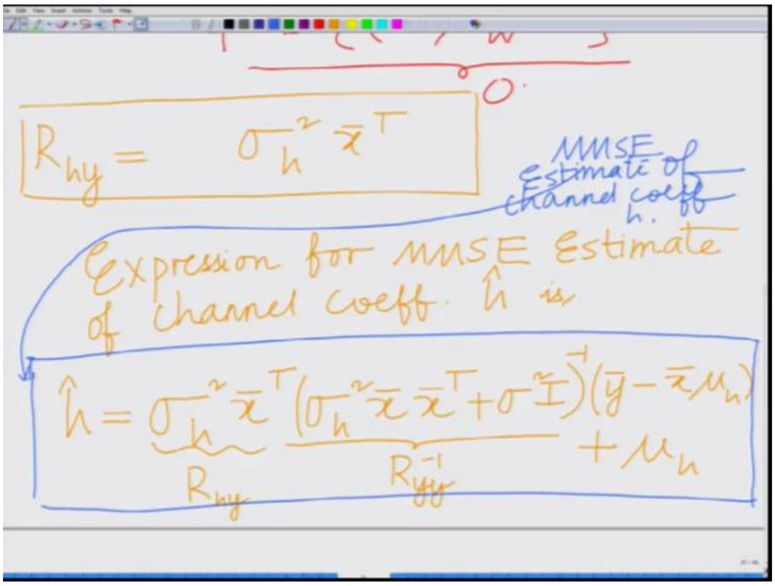

So, basically the final expression for the MMSE estimate, is given as, expression for the MMSE estimate h hat is, expression for MMSE estimate of your channel coefficient h hat is, h hat equals Rhy, which is basically, we derived Rhy, the sigma h square, x bar transpose, times, Ryy inverse, which is sigma h square, x bar, x bar transpose, is sigma square times identity, this inverse into y bar, minus x bar, into mu h, plus, mu h. And what is this? This is your Ryy inverse, and this is your Rhy, and this is, therefore, your final expression for the MMSE estimate of, this is the final expression for MMSE estimate of, this is the final expression for MMSE estimate of, channel coefficient h. This is the final expression for the MMSE estimate of the channel coefficient h.

So, what we have done, in this module is, basically we have derived succinct expression, for the MMSE estimate of the channel coefficient h, in terms of the observation Vector y bar, and the pilot Vector x bar. Now what we are going to do? Of course, this expression is complicated, similarly to the wireless sensor network scenario. Again, similar to the wireless sensor network scenario, we are going to simplify this expression, and give a nice, nice expression, a much more simpler and nicer expression, which has a lot of, which make sense intuitively, or which, which one can explain, and which yields a lot of interesting insight, into the nature of this MMSE estimate.

So, we will stop this module here, and continue with the simplification, in the next module.

Thank you very much.