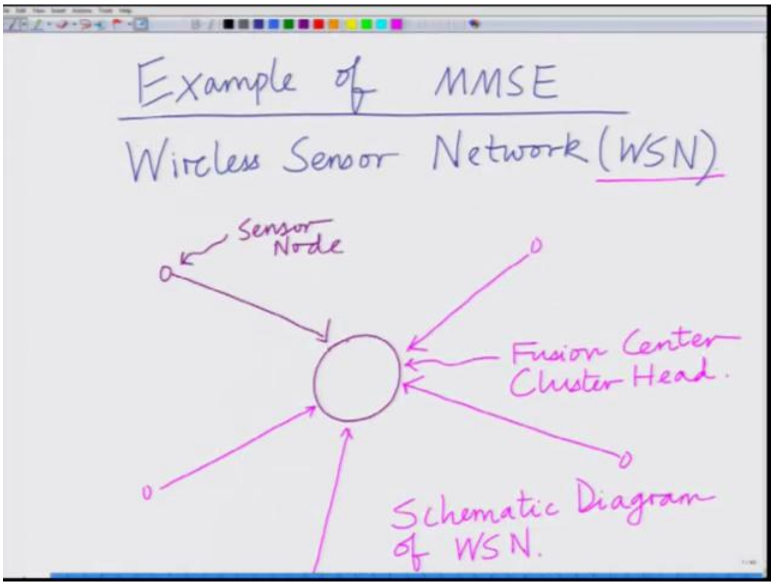
Bayesian/MMSE Estimation for MIMO/OFDM Wireless Communications

Prof. Aditya K. Jagannatham
Department of Electrical Engineering
Indian Institute of Technology, Kanpur

Lecture – 06
Minimum Mean Squared Error (MMSE) Estimation
Application – Wireless Sensor Network

Hello welcome to another module in this massive open online course on Bayesian MMSE estimation for wireless communications. So, far, we have looked at how to compute the MMSE or the minimum mean squared error estimate with the parameter and the observation are jointly Gaussian. So, now, let us look at the application of that principle in the context of a wireless sensor network. So, what we want to do today is we want to look at an application of the MMSE estimation. So, the application we want to look at an application of the MMSE estimation principle in the context of a, in the context of a wireless sensor network.

(Refer Slide Time: 00:59)

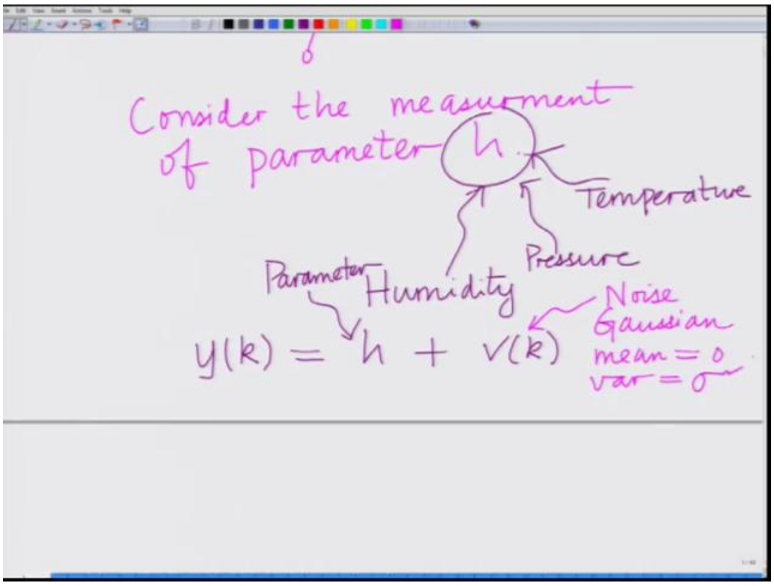


So, we want to look at an example of MMSE in the context of a wireless sensor network or WSN.

Now, what we have in a wireless sensor network, we have already seen a model for this. What we have is we have several sensor nodes which are communicating with the fusion sensor. So, we have a several sensor nodes for instance this is a sensor node, but typically we do not have a single (Refer Time: 1.51) sensor node, but rather we have several sensor nodes which are communicating or sending their. So, all these are small circles are the sensor nodes which are sensing their sending their measurements to a fusion center or what is known as in a wireless center wireless sensor network this also known as a cluster head, which is basically cluster head means it is the head node of this cluster of sensor nodes. Fusion center means it is basically the center which fuses the measurements from the different sensor nodes all right, this is also known as fusion center in the signal processing context in a networking context it is also known as a cluster head of the wireless sensor network.

So, now what is the idea this is the schematic diagram. So, what does this, this is the schematic diagram of a wireless sensor network, wireless sensor network which we are abbreviating by WSN remember whenever OK, this is the abbreviation WSN for a Wireless Sensor Network. We have a schematic diagram of the wireless sensor network, what these different sensor nodes are doing are these are basically trying to measure a parameter h. So, consider the measurement of parameter, this parameter we are going to denote by h.

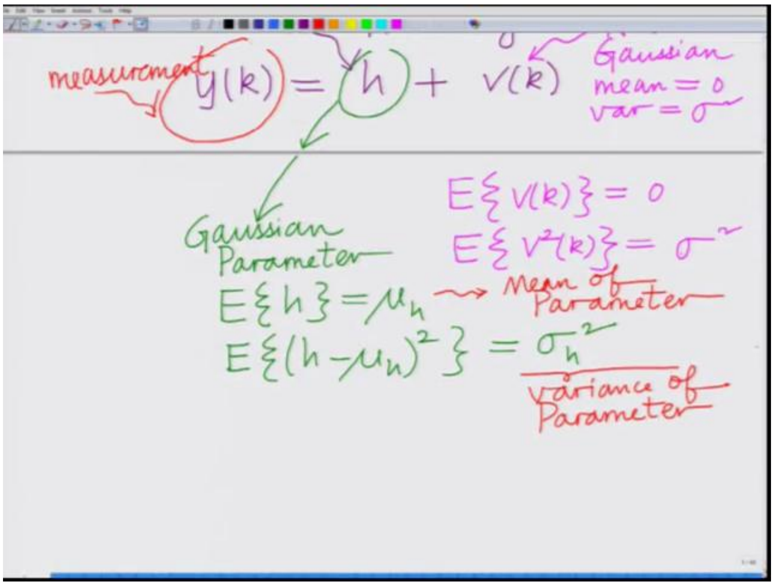
(Refer Slide Time: 03:18)



So, that is our parameter and this parameter can be anything for instance in a wireless sensor network typically you measure parameter such as the temperature or the pressure, can be anything, pressure or for instant for that matter the humidity; the wireless sensor networks have several applications all right. So, the moisture content of the soil and so on. So, there are several applications where wireless sensor network can be used such as (Refer Time: 4.09) agriculture weather monitoring etcetera. So, there are different parameters one can measure the temperature pressure etcetera. So, h denotes the parameter. So, we are denoting by h the parameter that is been measured.

Now, each sensor makes a measurement; obviously, that measurement is going to be noisy all right, that is the point of the estimation because you have noisy measurements or noisy observations all right. So, we have the observation let say at kth sensor, we have y k equals the parameter h plus v k. So, we have the parameter, we have your parameter and then you also have v k which is your noise which is, which you are going to assume as Gaussian with means 0 variance equals sigma square which means basically expected mean 0 means basically your expected value of v k equals 0 variance equals sigma square means basically expected value of v square k since we considering the real parameters the real noise.

(Refer Slide Time: 05:13)

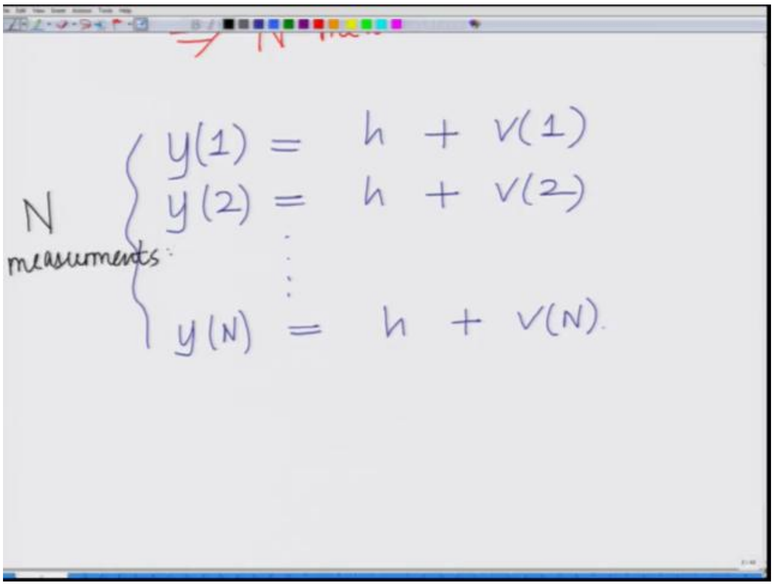


If its complex then we have to write expected value of magnitude v square, sorry that is not 0, that is basically your sigma square. Now h is the parameter, now in this context we are going to assume h is a Bayesian parameter that is a random parameter; remember there are 2 different settings this is the Bayesian MMSE setting therefore the parameter h is random in natures more specifically in this context since we are considering a

Gaussian estimation scenario we can consider the parameter h to be Gaussian with a certain mean and a certain variance which I am going to shortly write down.

So, h is a Gaussian parameter, very simple this is a Gaussian parameter, everything is a Gaussian. So, this is a Gaussian parameter with expected value of h equals mu h, expected value of h minus mu h square equals sigma h square. So, mu h what is this? This is basically the mean of the parameter and what is this? This is the variance of mean of the parameter and this is the variance of the parameter. So, now, what we have done, we have modelled a single measurement this y k you have not written this down this y k, this is your measurement. Now what we have is we have measurement y k equals the h the parameter plus noise v k, but of course, we will not have a single measurement rather will have a group of n measurements that is y from because we have the n sensors. Let us say correct in the sensor network we have many sensors, let say we have n sensor therefore; naturally we will have n measurements.

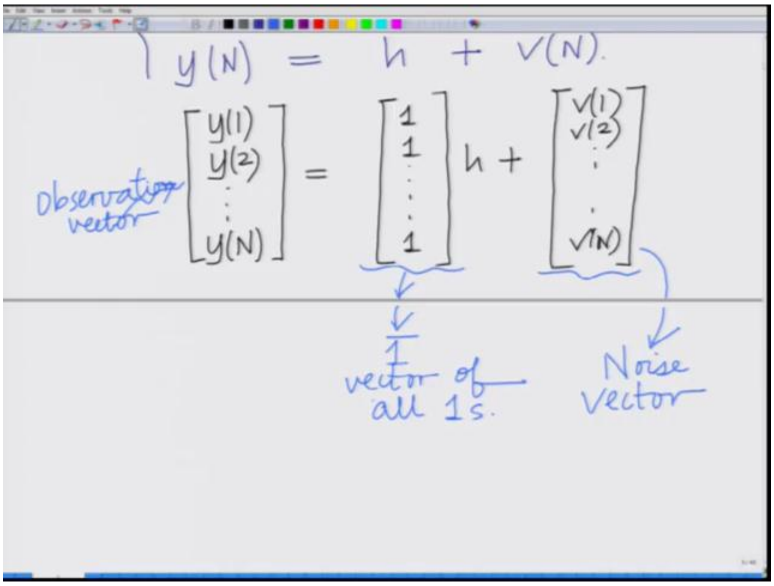
(Refer Slide Time: 07:38)



So, we consider a WSN with capital N sensors which imply that there are n measurements. So, the n measurement can be denoted as y 1 equals h plus v 1, y 2 equals h plus v 2, so on y n equals h plus v n. So, these are your n measurements. So, these are the, these are your n measurements and measurements of the same parameter mind you that is important to keep in mind is n measurements of the same parameter.

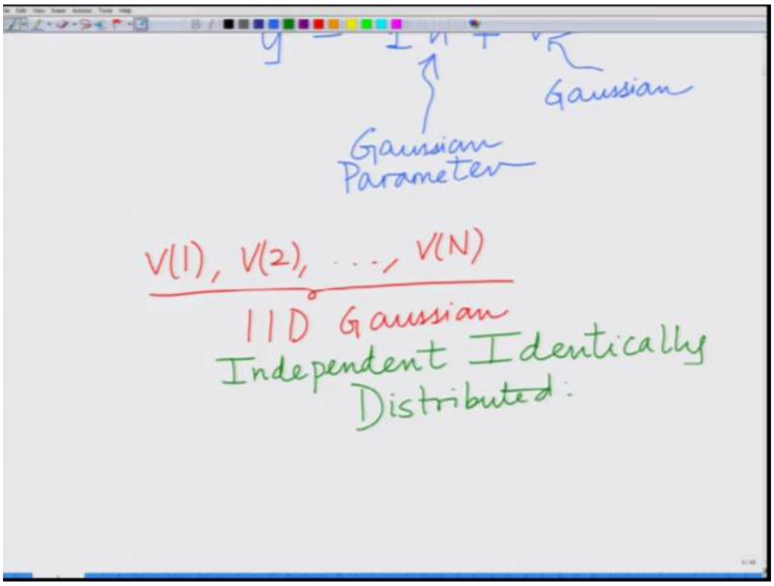
So, your multiple measurements of the same parameter from which you are trying to estimate this parameter h; obviously, as the (Refer Time: 8.48) number of measurements increases we expect the estimate of the parameter to improve correct if you take more and more measurement all right; that means, we expect by using this more by using this larger number measurements we expect to improve the estimate accuracy. Naturally that is the point behind taking more measurements or that is the point. In fact, behind having a large number of sensors in the wireless sensor network, so one has to keep that in mind.

(Refer Slide Time: 09:17)



So, now we have, I can write this in the form of a vector, your observation vector y 1 y 2 so on up to y n. This is your observation vector, this is equal to the vector of all once, this is equal to the vector of all once times h plus the noise vector, v 1 the vectors are n dimensional naturally because you have n sensors and n measurement. So, this is now your observation vector. This is your vector of all once which you are going to denote by 1 bar, this is the vector of all, this is the vector of all once and this is your, this is basically your noise vector. This is v bar which is your, this is v bar which is your noise vector. This is the noise vector that is fine. Now let us look at this now v bar is Gaussian because the noise samples are Gaussian. So, v bar let us denote this by this is let say v bar.

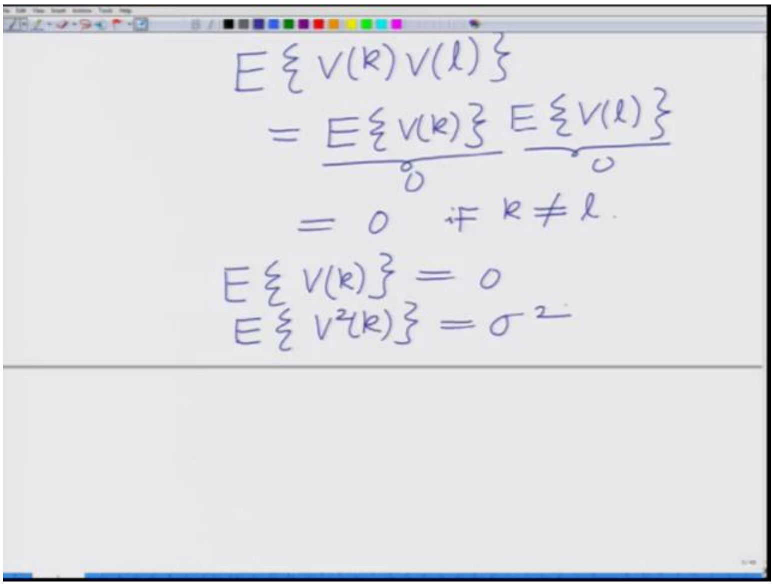
(Refer Slide Time: 11:16)



So, if we can write this model, we can write this now naturally this is your observation vector that is y bar. So, I can write y bar equals 1 bar times h plus v bar. Now look at this this is a Gaussian vector, this is a Gaussian parameter all right, that is what you have assumed and therefore now if you look at this this is for parameter h is Gaussian, vector v bar is Gaussian, you are adding two Gaussian quantities therefore, the output y bar that is also Gaussian all right. Because of the property of the Gaussian random variable when you add two Gaussian random variables you get another Gaussian random variable. So, the output observation vector y bar that is also a Gaussian vector.

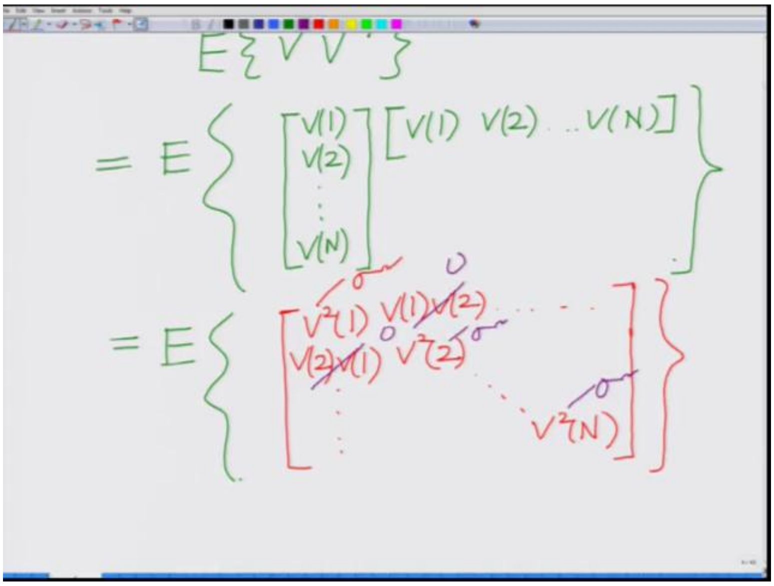
So, this is a Gaussian vector. So, now, we have a scenario in which the observation is Gaussian that is y bar the observation vector is Gaussian the parameter h is also Gaussian therefore, now I can use the MMSE theory develop so far of estimating Gaussian parameter h from the Gaussian observation vector y bar. So, that is what I am going to do. Before that we are going to characterize the noise and this is important. I am going to assume the noise is basically, we already said the noise is 0 mean. We are going to assume the noise samples v 1 v 2 v n and this is going to be an assumption. We are going to often use v 1 v 2 v n. We are going to assume that these are IID Gaussian, where IID stands for Independent Identically Distributed. This is independent identically distributed Gaussian, what do we mean by that we mean that each v 1 v 2 v n are Gaussian, is Gaussian.

(Refer Slide Time: 13:37)



Now, also they are identical which means the expected value of each v l or v k is 0, expected value of v k square is sigma square that is what we already seen. And further this different v s are independent which means expected value that is if I look at expected value of v k into v l since they are independent, this will be equal to expected value of v k times expected value of v l. So, this is 0 this is 0, therefore, this is equal to 0, but mind you if k is not equal to l. So, expected value of v k in into v l is 0 if k is not equal to l of course, if k is equal to l this become the expected value of v k square which is sigma square that we already know, that is expected value of each v k the mean is 0 and the expected value of v square k equals sigma square.

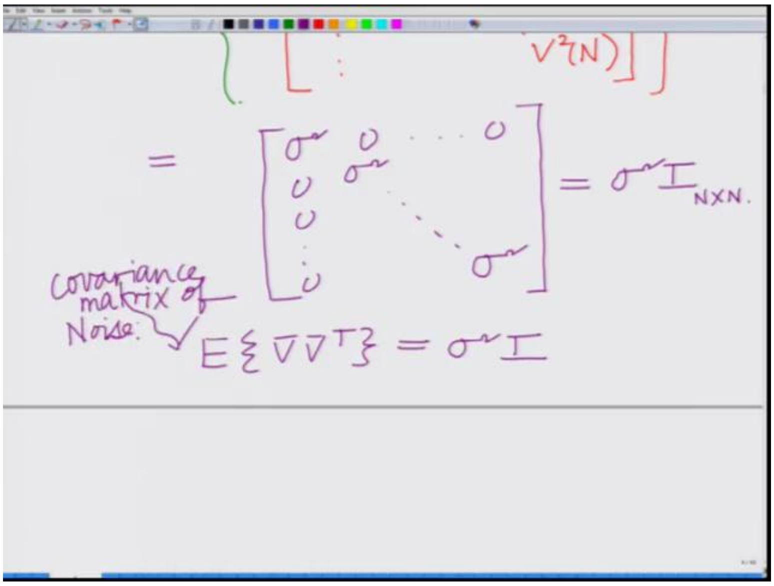
(Refer Slide Time: 14:36)



Therefore, if we look at the covariance matrix of this noise, now, what we want to look at is the noise, what we want to look at is the noise covariance matrix and what is this? This is simply expected value of v bar v bar transpose which is equal to your expected value of. Now I am going to write v bar, v bar is your vector v 1 v 2 v n times the vector v transpose which is column vector transpose is row vector, v n.

Now I can simplify this if, you can see if you now multiply this column vector by row vector what you will get is you will get expected value of v square 1. On the diagonal, you will get all the square elements v square 1 v square 2 and so on up to v square. On the off diagonal, you have v 1, you have v 1 v 2, and you have v 2 v 1. Now you can see if I take the expectation operator inside all, these elements are the diagonal elements expected value of v square 1 v square 2 so on. These are sigma square, all the off diagonal v 1 v 2 these are 0 all right if you take the expected addition operator inside.

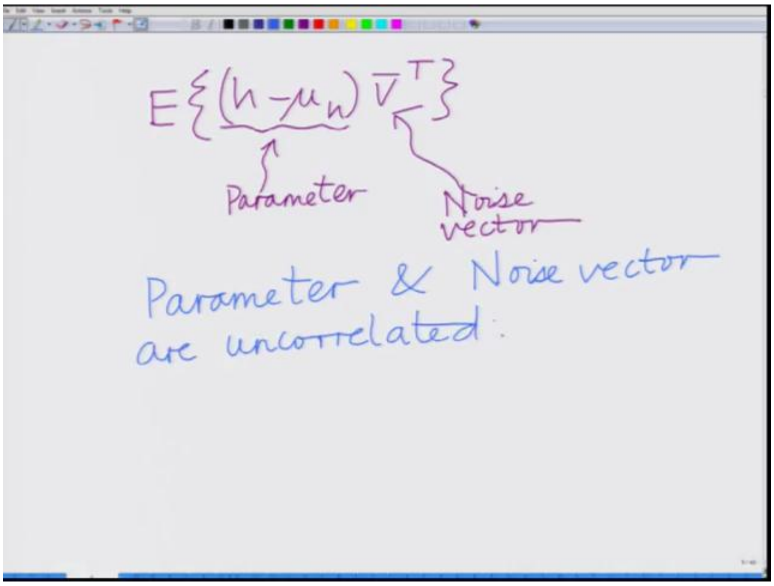
(Refer Slide Time: 16:32)



And therefore, what I have is the covariance matrix, if I look at this that is sigma square, sigma square sigma square and the off diagonal elements are 0. So, basically this becomes sigma square times the identity and most specifically times the n cross n identity matrix. This is what is this, this is the covariance matrix of the noise vector v bar, that is expected value of v b v bar transpose this is equal to sigma square times identity this we already said is the covariance matrix, this is the covariance matrix of the noise vector.

Now, further we are going to make another interesting and important observation which is also very natural and logical that is we are going to assume that the noise vector v bar and the parameter are independent or basically uncorrelated. Now of course, since they are both Gaussian uncorrelated also implies independent, one implies the other that is the un correlated implies independence and independence implies uncorrelated because it is Gaussian. So, at this also logical because the parameter comes from the environment all right this is based on for instance if you are trying to sense the pressure or temperature or so on and the noise all right, the noise is basically the noise the measurement noise which arise arises because of the, the thermal noise in the sensor equipment, the sensor nodes and so on.

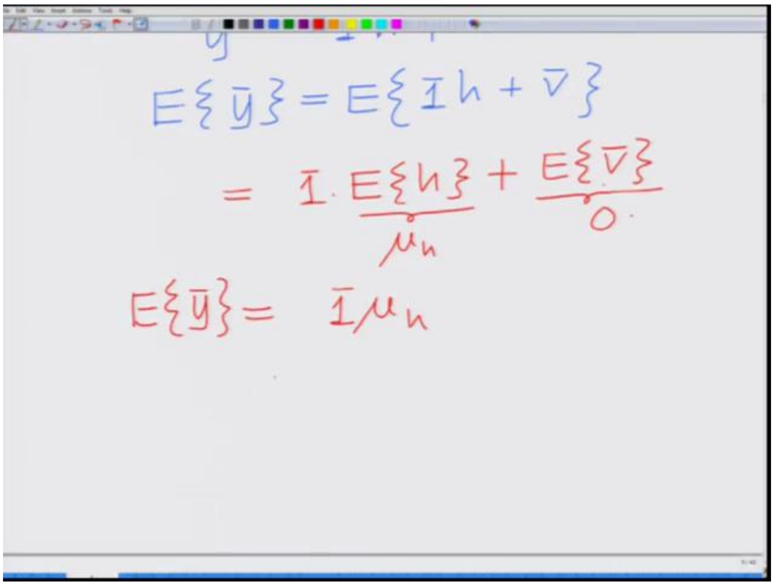
(Refer Slide Time: 18:32)



So, it is logical to assume that these two quantities are noise. The noise and the, the parameter that will make sense are independent, because these arises from totally different effects one can say, so the parameter h. So, we have the parameter h. So, we have expected remember h minus mu h times v bar transpose this is the parameter minus of course, you are subtracting the mean, this is your noise vector. So, the parameter and noise vector are uncorrelated that is the important assumption that we are making this is also as we said logical parameter.

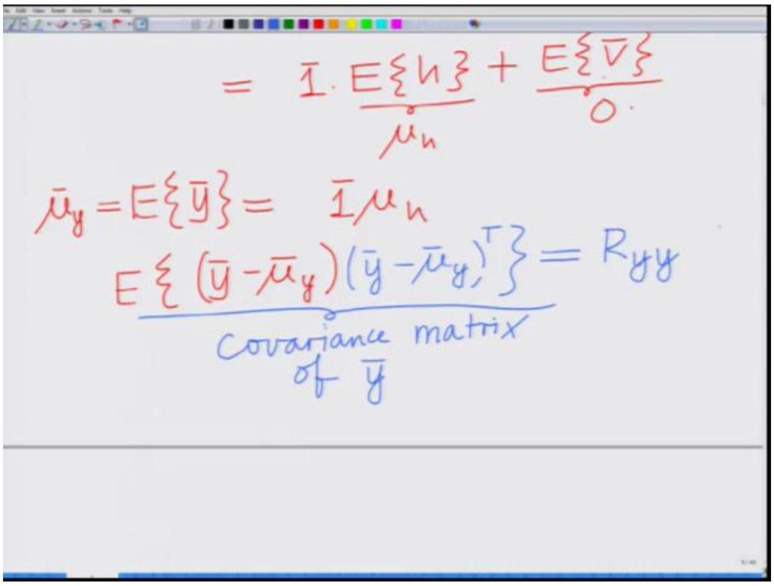
And noise vector are actually they are independent, but here uncorrelated (Refer Time: 19.24) because they both Gaussian. So, naturally if they are uncorrelated if follows that they are independent, they are one and the same. So, for that we are just going to say that they are uncorrelated which means the expected value that, that is the covariance expected value of h minus mu h times v bar transpose that is equal to 0. So, this is basically the frame work of estimation parameter estimation in wireless sensor network all right. So, now, let us proceed towards estimation of the parameter h from the observation y 1 y 2 up to y n that is our ultimate aim. So, now, what we have let us first compute we have y bar (Refer Time: 20:00) equals 1 bar h plus v bar.

(Refer Slide Time: 20:09)



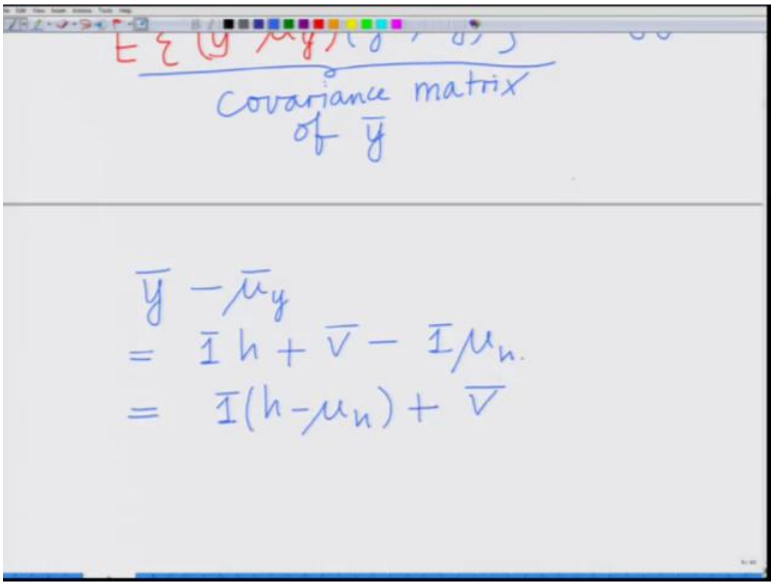
So, let us now proceed towards estimation. So, I can calculate the expected value of y bar as the expected value of 1 bar h, sorry not h bar, 1 bar h plus v bar which is equals to of course 1 bar is a constant, 1 bar times expected value of the parameter h plus expected value of v bar. Now this we know expected value of h is mu h now this we know is 0 expected value of. So, we have expected value of y bar is expected value of 1 bar h plus v bar which is equal to 1 bar into expected value of h that is mu h plus expected value v bar which is 0, because each expected value of the mean of each noise element is 0. So, it follows that the expected value of y bar equals 1 bar times mu of h, the mean of the observation vector equals 1 bar times mu of h.

(Refer Slide Time: 21:39)



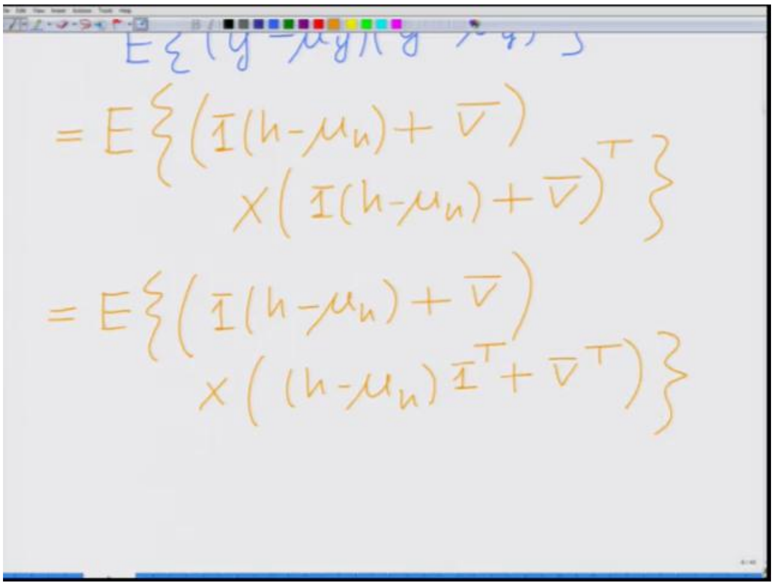
Now, we want to find RYY, remember because that is the another quantity we need in the MMSE estimation that is expected value of y bar minus mu y bar times expected value of y bar minus mu y bar transpose that is r y, that is the covariance matrix of the vector y bar. So, we need expected value of y bar minus mu bar y, this is you can say as the mean of y bar times, y bar minus mu bar y transpose. What is this? this is the covariance, this is let me right it properly this is the covariance matrix of the observation vector y bar this is also denoted by if you remember our notation also denoted by capital RYY which is equal to now let us substitute this.

(Refer Slide Time; 22:30)



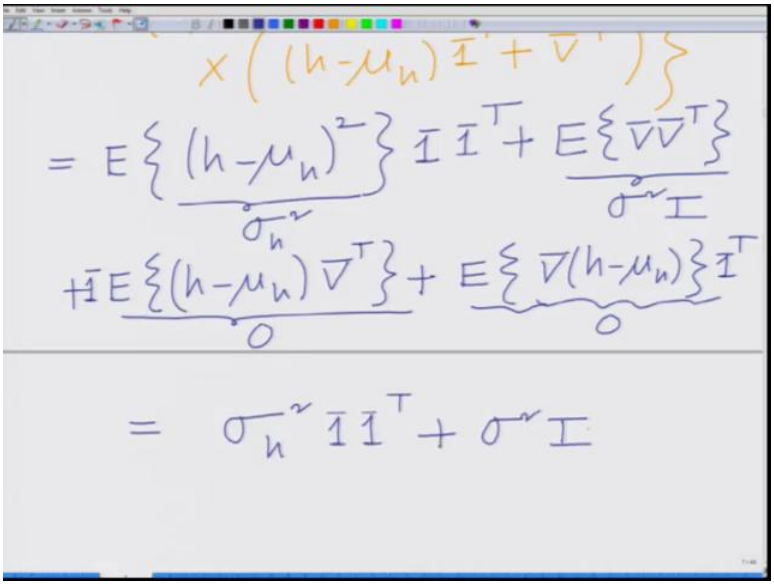
First let us look at this quantity y bar minus mu bar y that is equal to y bar is 1 bar h plus v bar minus mu bar y, the mean of vector which is 1 bar mu of h correct that what we have derived, mu bar y is 1 bar mu of h. So, I have 1 bar in to h minus mu of h plus v bar. So, this is your quantity y bar minus mu bar y.

(Refer Slide Time: 23:05)



Now, therefore, expected value of y bar minus mu bar y into y bar it is slightly messy, but it is fairly easy actually to derive this thing, that is equal to I just have to substitute this terms expected value of y bar minus mu bar y is 1 bar h minus mu of h plus v bar into the product 1 bar h minus mu h plus v bar transpose. Now this is going to be equal to, this is equal to what we have over here this is now if I explain this let me write one more step 1 bar into h minus mu of h plus v bar times transpose is h minus, of course this is a scalar quantities. So, it does not matter h minus mu of h into 1 bar transpose plus v bar transpose.

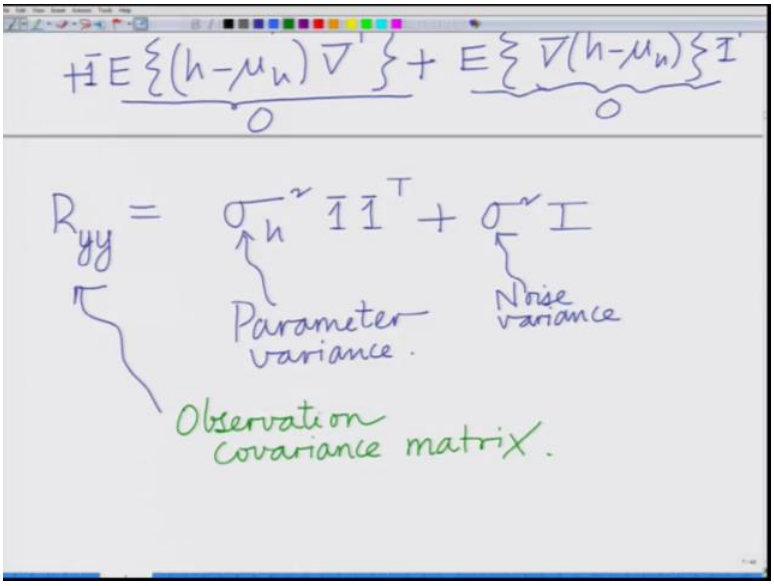
(Refer Slide Time: 24:22)



Now, if you multiply it term by term what you are going to get is expected value of; let us look at the first term h minus mu h square, h minus mu h square times 1 bar 1 bar transpose of course we know this is sigma h square plus if you take the last two terms then you have expected value of v bar v bar transpose. This we know is sigma square identity now it remains to look at the cross terms that is the terms in the middle that is plus expected value of what we have h minus mu h into v bar into or we have 1 bar into h minus mu h into v bar transpose.

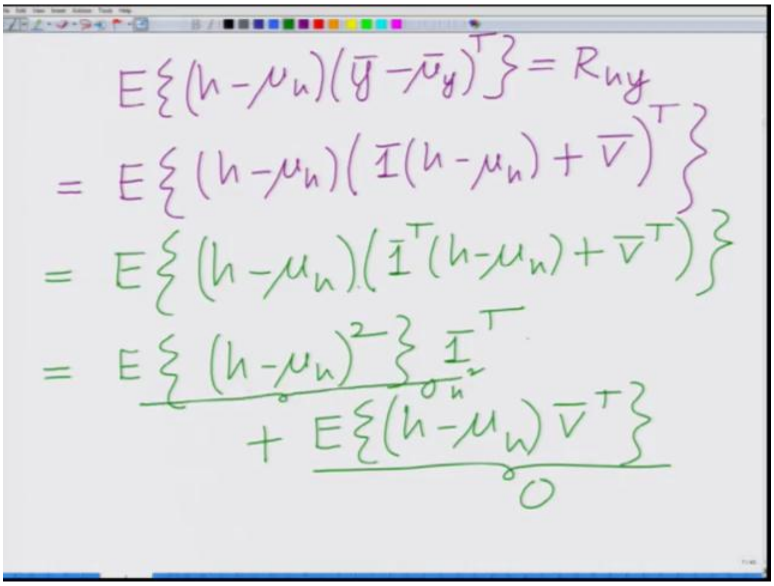
Of course, we know that the noise and the parameter are uncorrelated that is equal to 0 plus expected value of v bar into h minus mu h in to 1 bar transpose. Of course, we know that the noise and the parameter are uncorrelated. So, this is again the expectation is again 0. So, what we are left with is basically finally, we are left with only the first two terms that is the sigma h square times, sorry 1 bar into 1 bar transpose plus sigma square identity where sigma square is a noise variance, sigma square remember is the variance of each noise sample this is the noise variance.

(Refer Slide Time: 26:07)



This is the parameter, parameter prior variance, variance of the Gaussian parameter sigma h square. So, this is what this is my RYY which is observation covariance. In fact, it is a covariance matrix of the observation vector RY by which is expected value of y bar minus mu bar y into y bar minus mu bar y transpose which is equal to sigma x square times 1 bar 1 bar transpose plus sigma square times the identity matrix.

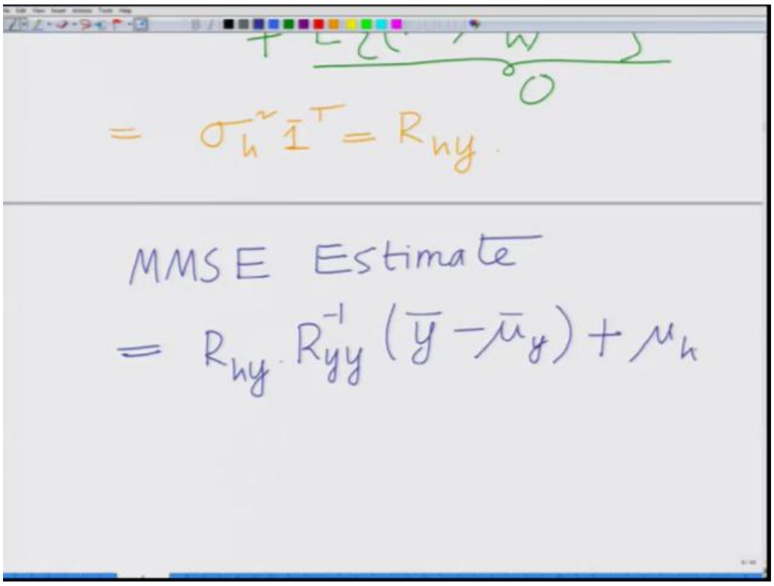
(Refer Slide Time: 27:06)



Now, we want to derive the next quantity that we need is the cross covariance that is expected value of h minus mu h times right y bar minus mu bar y transpose. This is equal to r h, y this is the cross covariance between the parameter and the observation. Now what we can do here is we have h minus mu h that we have anyway y bar minus mu y bar this is 1 bar into h minus mu h, remember y bar minus mu y bar is 1 bar into h minus mu h plus v bar transpose of this quantity that is equal to expected value of h minus mu h times of course.

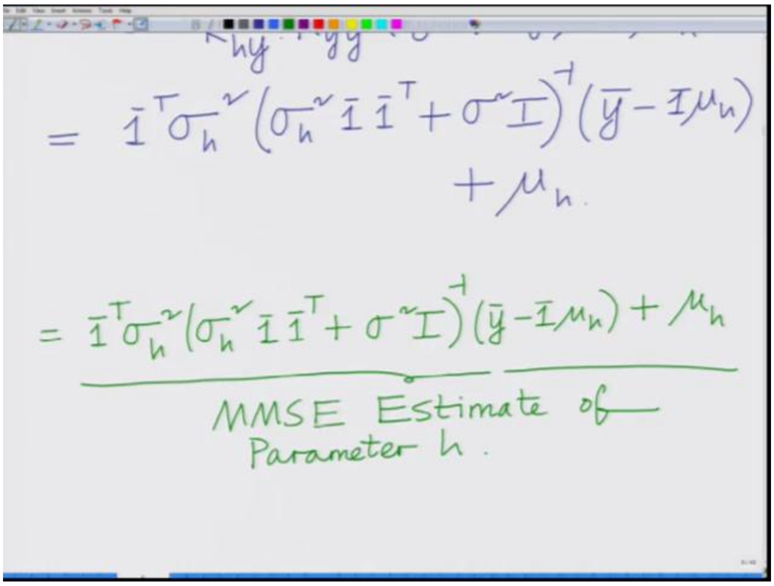
Now I have to take the transpose of this quantity 1 bar transpose into h minus mu h plus v bar transpose h minus mu h is scalar, that is why I have not I can just let it remain where it is equal to expected value of now split it into the components expected value of h minus mu h square into 1 bar transpose plus expected value of h minus mu h into v bar transpose. We know this is equal to 0 because the noise and the parameter are uncorrelated, while this we know not this were this quantity expected value of h minus mu h square this is sigma h square. So, therefore, this net quantity if you can now look at this which is also, very simple again nothing very complicated this is sigma h square into 1 bar transpose which is RHY.

(Refer Slide Time: 29:09)



And therefore, now the MMSE estimate, remember MMSE estimate, now we have RHY we have RHA RYY. The MMSE estimate is given as recall that the MMSE estimate equals RHY cross covariance times RYY inverse times y bar minus mu bar y plus mu h and we know each of these quantities RHY.

(Refer Slide Time: 30:00)



We have calculated is one bar transpose time sigma h square. RYY is basically we have calculated that as sigma h square into 1 bar 1 bar transpose plus sigma square identity inverse into y bar minus mu y bar is 1 bar into mu h plus mu h. Let me just write it a little bit more clearly, this is equal to 1 bar transpose sigma h square sigma h square 1 bar 1 bar transpose plus sigma square identity inverse into y bar minus 1 bar into mu h plus mu of h and what is this? This is therefore the MMSE now what we have is we have a neat expression for the MMSE estimate of we have an expression for the minimum mean squared error estimate of the parameter h.

So, what we have done in this module is in something interesting. We have looked at how to use this principle of MMSE estimation in the context of wireless sensor network. We have considered a wireless sensor network with n sensors making n measurements y 1 y 2 up to y n, the parameter h which is let us say any parameter that is either the temperature of pressure so on is assumed to be Gaussian in nature in the presence of Gaussian noise therefore, the observations are also Gaussian.

Now from this Gaussian vector, we estimate this Gaussian with Gaussian parameter h we calculated the covariance matrix of the observation vector, we have calculated the cross covariance between the observation and the parameter and now we have developed the MMSE that is the minimum means squared error estimate of this Gaussian parameter h in the wireless sensor network. Now, if you notice this expression is slightly unwieldly all right, it slightly messy it is a long expression what we are going to do in the next module is to develop a more is to simplify this further and development nice in intuitive a nice more simplistic expression and also which conveys more meaning also which yields a lot of intuition.

Thank you, thanks very much.