Bayesian/MMSE Estimation for MIMO/OFDM Wireless Communications
Prof. Aditya K. Jagannatham
Department of Electrical Engineering
Indian Institute of Technology, Kanpur

Lecture - 05
Derivation of Minimum Mean Squared Error (MMSE) Estimate for Gaussian Parameter - Non-0 Mean and Vector Parameter/ Observation

Hello welcome to another module in this massive open online course on Bayesian MMSE estimation for wireless communications. So, in the previous module we have seen the Bayesian MMSE estimate for the parameter h, given the observation y and h and y are both jointly Gaussian, and h and y are 0.

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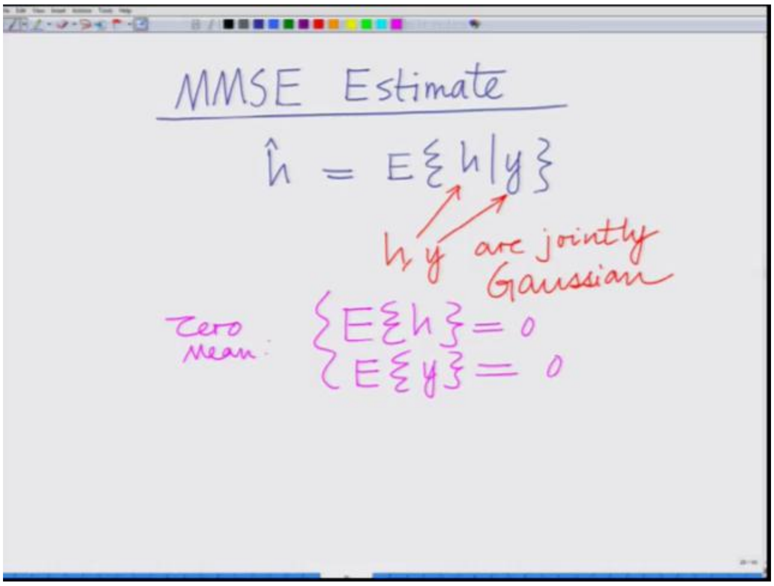

So, now in this module let us considered the scenario, where the means are non zero correct. So, previously we have seen the MMSE estimate. We have already seen the MMSE estimate; that is h hat equals expected value of h given y when these parameters h comma y are jointly Gaussian, this is the first point, and also expected there is 0 mean; that is expected value of h equals 0, expected value of y equals 0 meaning that both of these, these are 0 mean, where the parameter h and the random variable y both of these are 0 mean. Now let us consider. Let us now considered the scenario where these are non zero mean.

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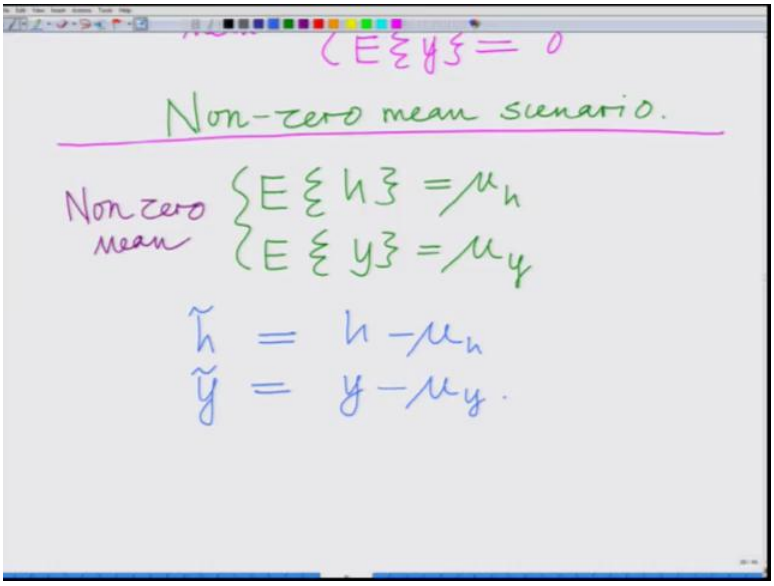

Let us now consider the non zero mean scenario. Let expected value of the parameter h is equal to mu h, and also let expected value of the observation y is mu y. So, we are basically now considering non zero mean, non zero mean, this is the non zero mean random variable . So, they have means which are not 0. Now what we are going to do, is define two new random variables h tilde, which is h minus mu h, and y tilde equals y minus mu y. So, from h we subtracted it is mean mu h from y we subtract its mean mu y.

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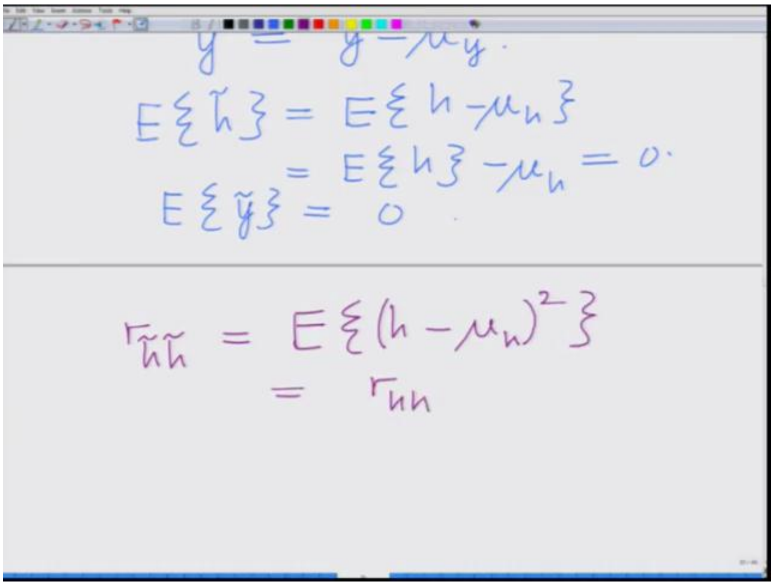

Now you can see h tilde and y tilde have 0 mean. So, that naturally follows, because expected value of h tilde equals expected value of h minus mu h, which is equal to expected value of h minus mu h, this is equal to 0. Similarly, expected value of y tilde equal to 0, now, given non zero mean, parameter h and observation y, we have converted them to 0, the 0 mean equivalent h tilde 0 mean equivalent parameter h tilde, and 0 mean equivalent observation y tilde. Now, I know the estimate of h tilde given y tilde. I am going form that first and from that derive the estimate of h tilde, and that is very simple. So, now, I need r h tilde, h tilde which is expected value of h minus mu h whole square, which is noting, but the variance of the parameter h correct, this is r h comma h.

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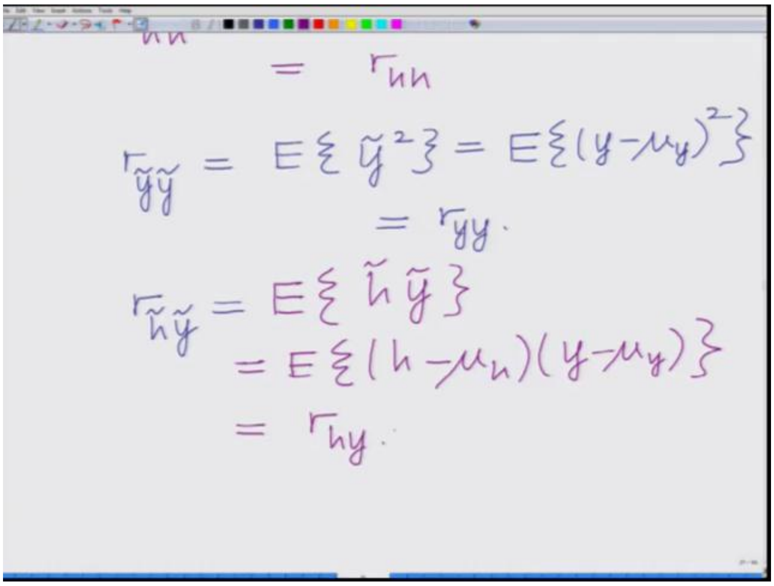

Similarly r y tilde y tilde, this is the variance of y tilde which is r y tilde square, expected value of y tilde square, which is expected value of y minus mu y square which is equal to r y y. Other thing is we need the cross; we need the co variance that is expected value of that is r of h tilde comma y tilde, r of h tilde comma y tilde, which is equal to. this is expected value of h tilde into y tilde, which is equal to, which equals the expected value of h minus mu h into y minus mu y, which equals again, the co variance of h comma which is r h comma y r h comma y. Now what we have, now the estimate of h tilde given y tilde; that is the estimate of the, this is modified parameter tilde given the observation y tilde, we know this is given as.

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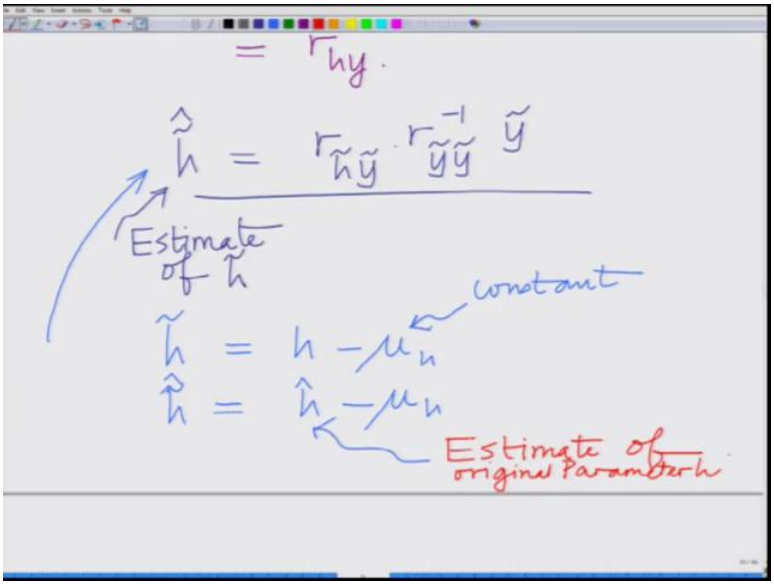

Let us write this down, this is h tilde hat, estimate of h tilde. This is your estimate of h tilde, this is equal to r we have already know the relation, r h tilde y tilde r y tilde y tilde inverse times y tilde. So, this is your MMSE estimate for h tilde given y tilde. They are the same expression h tilde hat equals r h tilde y tilde times r y tilde y tilde inverse into y tilde. Previously we had h hat equals r h y into r y y inverse into y.

Now, what I am doing is in the same relation I am replacing h by h tilde and y by y tilde. And now you can see clearly, look at this we have h tilde equals h minus mu h. This is the constant, this is the mean. Therefore, the estimate h tilde hat equals simply the estimate h hat minus mu h correct h hat minus mu h, or in other words, because mu h is simply a constant. So, h tilde equals h minus mu h which implies the estimate of h tilde; that is h tilde hat equals estimate of h h hat is. What is this h hat? h hat is estimate of h estimate of the original parameter; that is the simple relation. Original parameter h, estimate of original parameter h is h tilde hat is h hat minus mu h.

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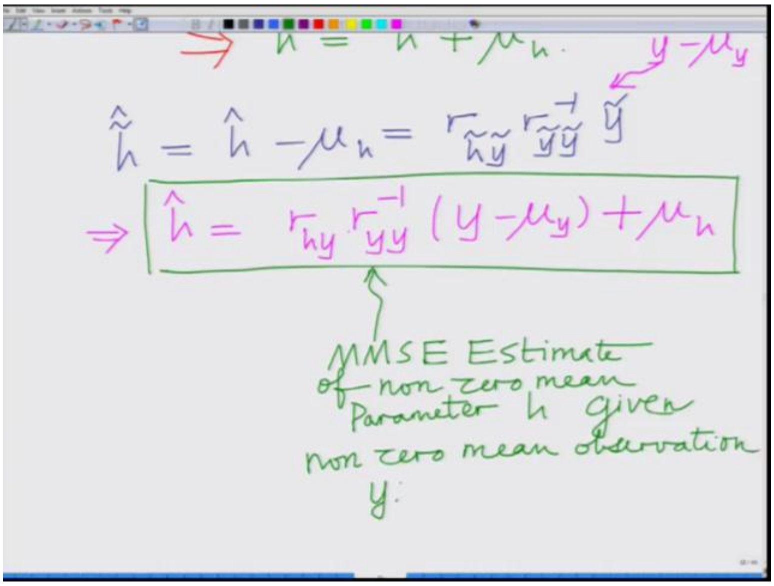

Which also implies is now, if have to write it this way. Now rearranging this with this implies that h hat equals h tilde hat plus mu h, we have that simple relation. Now substituting this we have h tilde hat equals h hat minus mu h which is equal to r h tilde y tilde r y tilde y tilde inverse into y tilde, but remember y tilde, this is equal to y minus mu y, which implies now h hat equals. Now look at this r h tilde y tilde is nothing, but r h y. This is nothing, but r y y inverse; because r y tilde y tilde is r y y y tilde is y minus mu y plus mu h from the left. I bring it to the right I have mu h, and that is the expression for your MMSE estimate; that is your expression for the MMSE that is your expression for the l MMSE; that is your expression for the, what is this. This is the MMSE estimate of your non zero mean parameter h, given non zero mean observations, given a non zero mean observation y. So, what we had previously we considered in the parameter h and y to be both Gaussian and also 0 mean.

Now what we have done we have relaxed the 0 mean assumption and we said h and y are still jointly Gaussian; however, h has a non zero mean mu h y has a non zero mean mu y, and in this scenario what is the estimate of h given y. And you have a simple expression what it is, basically are basically simply subtract the mean mu y from y compute the estimate, and then add the mean mu h separately. And that is what we have over here if look at this expression this is simply r h y into r y y inverse into y minus mu h and then followed by addition. So, initially we are subtracting the mean mu y from y, computing the estimate, and then separately adding the mean mu h. So, that is a very simple relation. Now the variance the m s e or the mean squared error, this will be.

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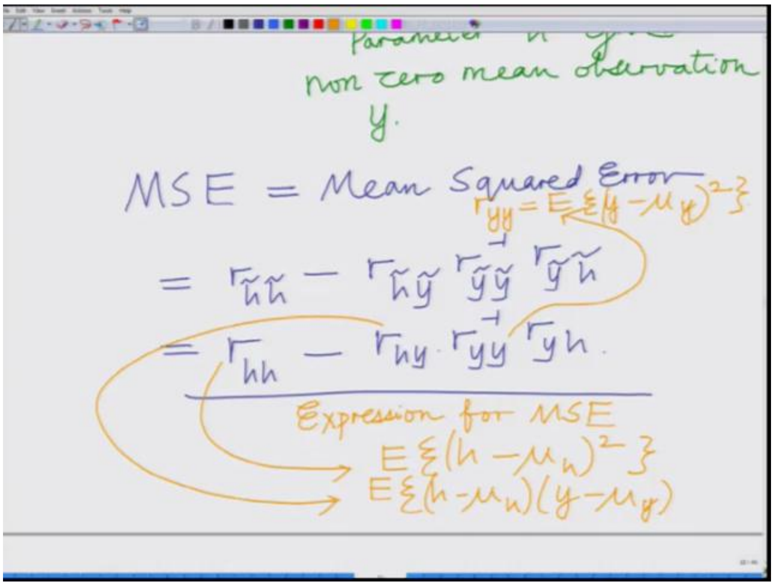

Remember previously we had the expression r h h minus r h tilde y tilde, wherever I had y I am substituting y tilde, wherever I had h I am substituting h tilde, now, r h tilde h tilde. Remember this equal to r h h. So, this remains the same expression r h y r y y inverse r y h. So, this expression is valid again for, this is the expression for the mean squared error; however, now remember r h h r h y these quantities are defines slightly differently. remember r h h is not expected value of h square, but rather expected value of h minus mu h whole square, r h y is the co variance that is expected value of h minus mu h into y minus mu y, and similarly r y y equals expected value of. Let me write this separately r y y equals expected value of y minus mu y the square. So, just remember this changed definition for the non-zero mean scenario.

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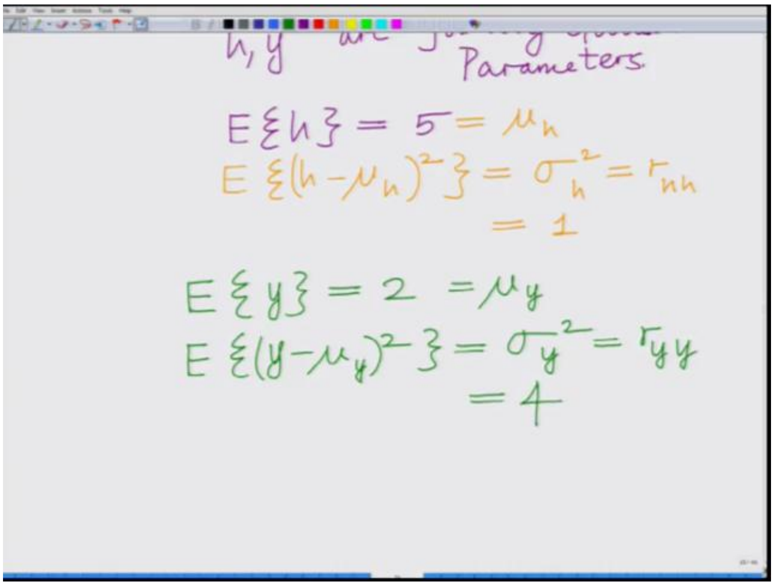

So, let us do a simple example to understand, this MMSE to understand this m s e estimation process better. Let us do a simple example to understand this. So, let considered is simple example. we have two jointly Gaussian parameters, Let us say h comma y are these are jointly Gaussian parameters, with expected h is equal to 5 that is this is the non zero mean, and expected the variance of h is expected h minus mu h.

So, expected h is 5 which is basically your mu h, h minus mu h square expected value is nothing, but the variance sigma h square, which is also r h h this let us say is equal to 1. So, sigma h square is equal to 1, which means sigma h is also equal to 1. Now, consider expected y equals, let us say this equals to two which is equal to mu of y expected y minus mu y whole square equals, let us say this is equals to which is basically your sigma y square, which is basically r y y. This is let us say this is equal to 4 sigma y square equals 4 which means sigma y equals 2.

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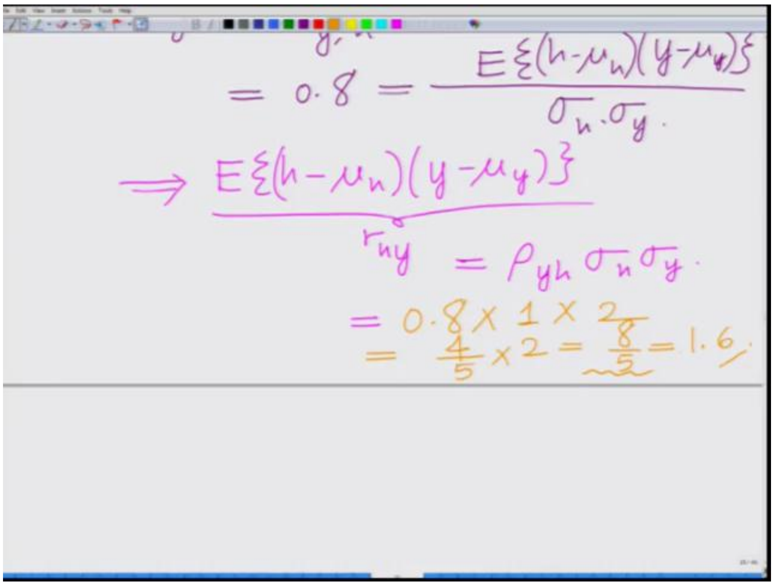

Now, let us say the correlation coefficient rho, which is basically your correlation coefficient of y comma h which is equal to, let us say this is 0.8 which definition is basically. Remember the definition of the correlation coefficient is expected value of. I will write it properly this is basically nothing, but expected value of y minus mu y or this is the scalar scenario h minus mu h times y minus mu y divided by sigma h into sigma y, which basically implies, now from this correlation coefficient. from this correlation coefficient I cannot derive the co variance, and I can derive the co variance, now you can see this co variance is nothing, but expected value of h minus mu h into y minus mu y. this is basically your r of h comma y.

You can see this is nothing, but rho y h times sigma h times sigma y. And what is this, this rho of y h, this is given as 0.8 sigma h square equals 1 which means sigma h equals 1 sigma y square equal to 2 which means sigma y equals 2. So, this is 0.8 into 2, this is 4 by 5 into 2, this is basically your. Let us put it this way, this is basically your 8 by 5 which is equal to point. This is equal to 8 by 5 which is basically 1.6. So, this is basically your r h y. So, r h y is basically 1.6 or basically 8 by 5. So, this is basically your 8 by 5, so now we have r h. So, basically now the MMSE estimate. So, now, we have all the required quantities, and now I can compute the MMSE estimate of h given y, given that h and y are jointly Gaussian, and we know the relation for that.

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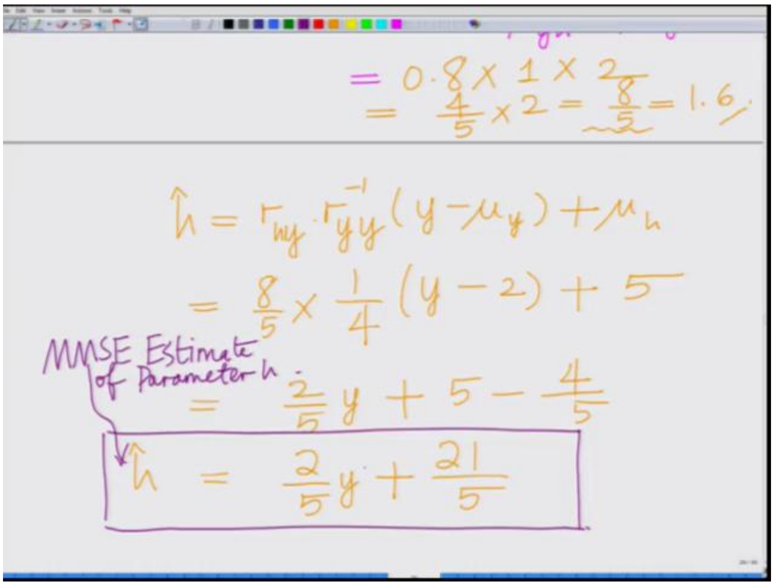

That is basically your h hat equals r h y into r y y inverse into y minus mu of y plus mu of h. we know r h y that is equal to 8 by 5 times r y y inverse. We know r y y that is sigma y square equals 4. So, this is 1 by 4 y minus mu y. We know mu y that is given as 2 plus mu h that is given as 5. So, this is basically what this comes out to, is this basically your 2 by 5 into y minus plus 5 minus 16 or rather minus 4 by 5, which is basically your 2 by 5 y plus 21 by 5. This is basically your MMSE estimate of the parameter h.

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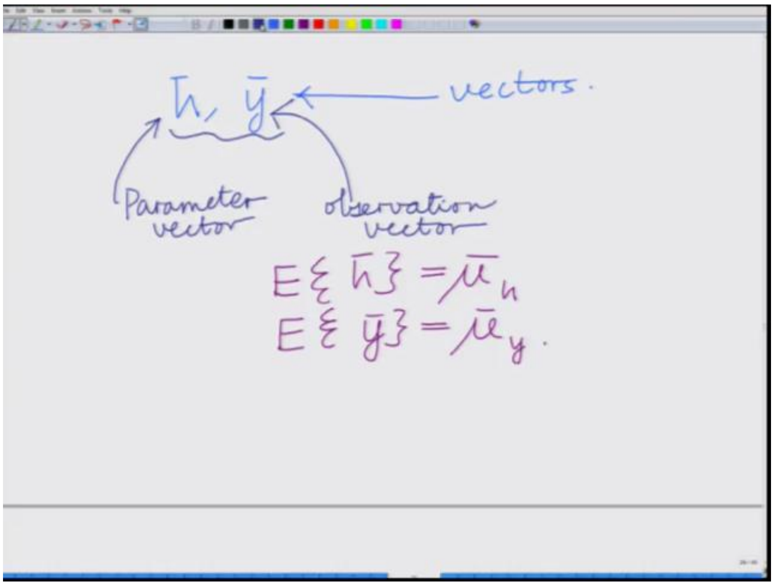

And remember we are also given the, this MMSE estimate given that; of course, this is assuming h comma y are; that is an important assumption, is an important assumption that h comma y are jointly Gaussian. In fact, we will see later that even when h comma y are not jointly Gaussian, this still an important estimate this is known as the l MMSE estimate; that is the linear minimum mean squared error estimate. This is also valid when h and y are not jointly Gaussian, except with the slight change of the framework, or with the slight change of basically the estimation (Refer Time: 20:41), but anyway will come to that later.

Right now given h and y are jointly Gaussian non zero mean this is optimal MMSE estimate of h given y, and this simple example clarifies. So, in today's example what we have seen, is basically we have computed the MMSE estimate or the joint, the MMSE estimate of h given y, when h and y are jointly Gaussian and not necessarily 0 that is basically non zero mean.

Now let us also look at what happens when h comma y are vectors. So, that will basically comprehensively we considered the scenario, when h comma y; these are vectors, and that is also very simple is the straight forward extension now; that is your observation. This is your parameter vector h bar, and this is your observation vector y bar. And now you can relatively easily extend this by observing the following, the first let us say that the mean of h bar equals mu h bar, the mean of course, is a vectors, the mean of both the parameter h bar and the observation y bar are also going to be vectors, this is mean mu bar of y. Now, instead of co variance, instead of the variances in co variance what we need are the co variance matrix, the co variance matrix of the observation y bar, co variance matrix of the parameter h bar, and also the cross co variance between these two vectors that is what we need.

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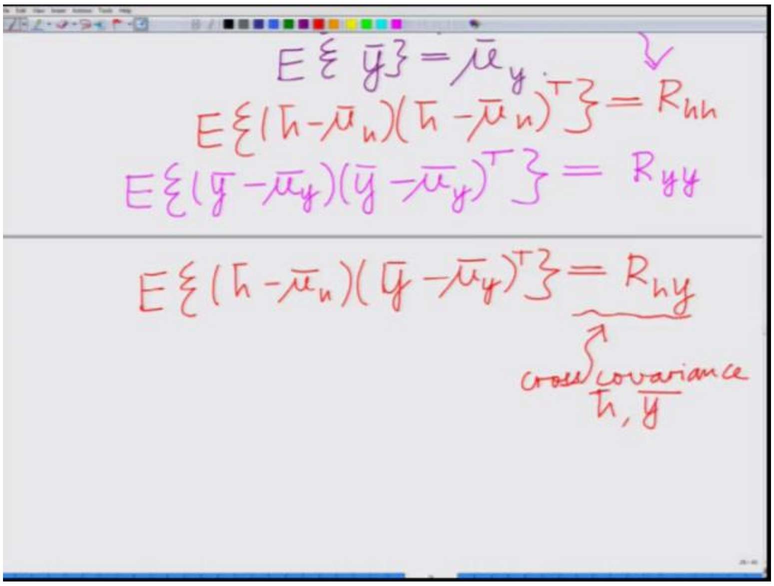

So, what we need for estimation is, again expected value of h bar minus mu h bar or mu bar h into h bar minus mu bar h transpose, this is r h h. Since their vectors I cannot simply consider the variance anymore, this is the co variance matrix of h. Or rather the parameters h bar. similarly expected value of y bar minus mu bar y times y bar minus mu bar y transpose, this is the co variance matrix of y. And the cross co variance can be defined as expected value of y bar, or cross co variance of h bar comma y bar; that is h bar minus mu bar h into y bar minus mu bar y transpose, this is equal to r h comma y. So, that is what we have, this is basically your cross co variance of h comma y. And now again it goes without saying that we are assuming h bar and y bar, when this scenario we are assuming h bar comma y bar are jointly.

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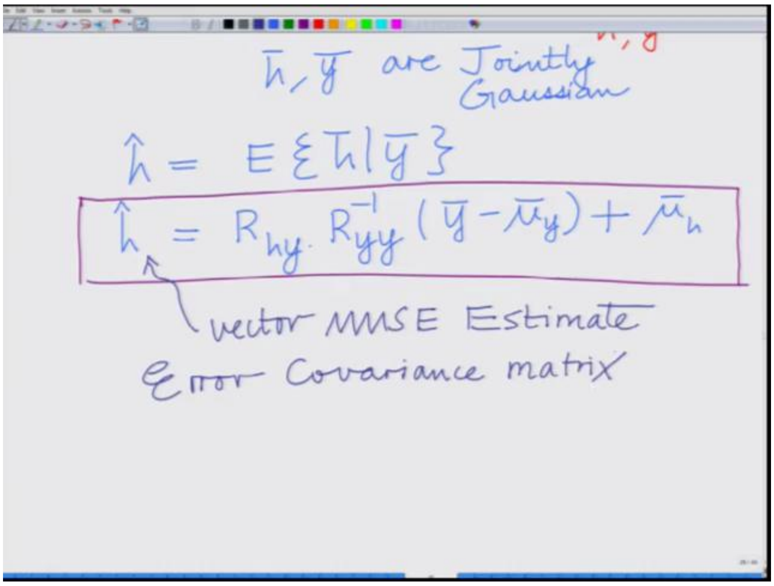

We are assuming that h bar comma y bar are jointly Gaussian, now in this scenario expect the estimate, the vector estimate. Now h hat which is also the conditional mean of the parameter vector h bar given the observation vector y bar, this is given as. Now instead of the small r h y and y r y, I am going to replace them by the capital that is the co variance instead of the variances and co variances, I am going to replace them by the cross co variance matrix and the co variance matrix respectively.

So, that basically that is the relatively simple and straight forward extension of the scalar scenario to the vectors scenario. So, now, I have r h y r y y inverse times y bar minus mu bar y plus mu bar h that is add the mean. So, now, instead to the variances and the co variance I am using the cross co variance matrix and this is the, co variance matrix of y r y y is the co variance matrix of y. So, this is also another, this is another important aspects, and of course, the most general this is the vector MMSE estimate. This is the vector MMSE estimate. And also the error co variance, now we can talk about, because we have a parameter vector.

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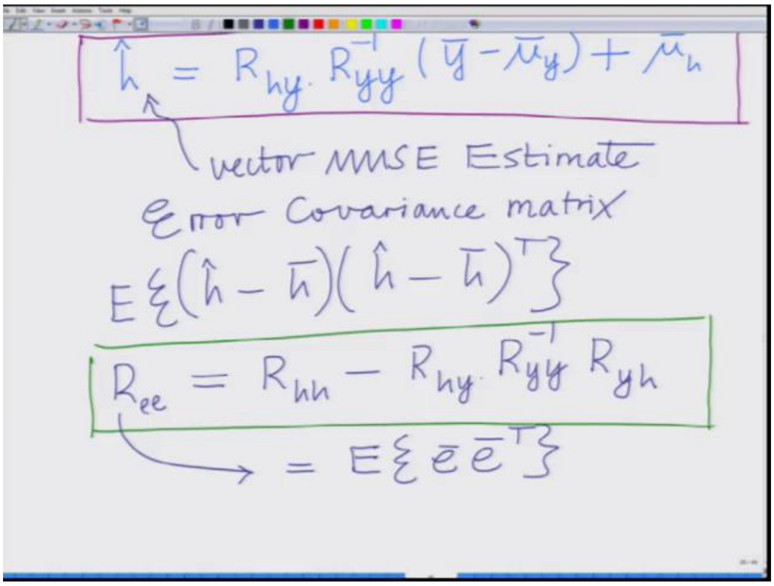

We have not simply the error the mean squared error, rather the error co variance matrix which expected value of h hat minus h bar into h hat minus h bar transpose, and this is again instead of the variances of the co variances, I am going to replace them by the co variance matrix and the cross co variances, I can replace write this as r h h minus r h y, replacing small r y y inverse by capital r y y that is the co variance matrix of y inverse times r y h.

So, this is your this is your error co variance matrix, you can also called this as r e e which is basically equal to expected value of e bar times e bar transpose, where e bar is the estimation error. And that is it; that is basically your corresponding error co variance matrix for the vector scenario. This is the vector MMSE estimate, remember this is the vector MMSE estimate; this is the error co variance matrix for the vector scenario. So, we have the expression for both of them, and we have derived them as extension. So, of their scalar corresponding, scalar counter parts their corresponding expressions from this scalar parameter and observation scenario.

So, this basically comprehensively concludes this MMSE estimate when the parameter h and y are jointly Gaussian. So, what we did is, we started with the scenario when h and y are Gaussian, but have 0 mean. extended to a scenario when h and y are Gaussian non zero mean also, as simple example, and again later that is in the last part derive the comprehensive expression when h bar and y bar are now general; that is they need not be scalars, but vectors h bar is a vector parameter vector y bar is a observation vector and h bar and y bar have non zero means, we mean of h bar is mu y mu bar y, mu bar h mean of y bar is mu bar y, and now we computed the vector MMSE minimum mean squared error estimate and also derived the error co variance matrix. So, using this basically we are going to see how these can be applied in wireless communication scenario; that is something that we have to see yet right.

So, we are going to next see how this can be this framework can be applied for instance in wireless sensor network, to compute the estimate of the parameter at the fusion centre. We will stop here.

Thank you.