Bayesian/MMSE Estimation for MIMO/OFDM Wireless Communications

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Lecture - 27
Linear Minimum Mean Square Error (LMMSE) Channel Equalization

Hello, welcome to another module in this massive online course on Bayesian MMSE estimation for wireless communications systems, alright. So, we have looked at the model of an inter symbol interference channel, model of a wireless channel with inter symbol interference and we also said that removal of inter symbol interference system as equalization alright.

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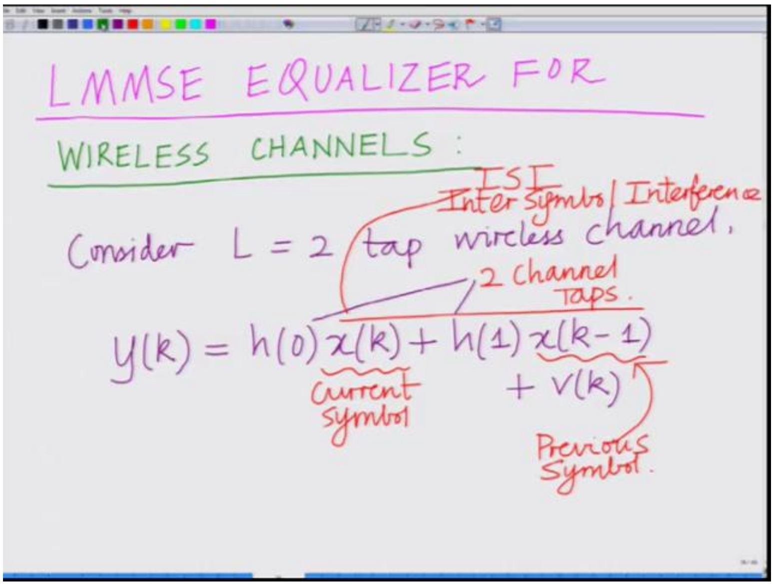

So, now let us look at, in this module let us look at LMMSE or MMSE based equalization for a wireless communication system. So, what you are going to look at starting this module is basically LMMSE equalizer design, basically equalizer or design of a LMMSE equalizer for wireless channels. We are going to look at design of LMMSE equalizer for wireless channels.

So, towards this end for this purpose consider a two tap if the number of taps is L consider an L equal to 2 tap, consider an L equal to 2 tap wireless channel therefore, we can express since there are 2 channel taps we will have y k equals H 0 times x k plus H one times x k minus 1 plus v k correct. So, these are the 2 channel taps H 0 and H 1, these are your two basically coefficients of the channel filter these are also termed as a channel taps x k is the current symbol alright at time instant k, x k minus 1 is the previous symbol at k minus 1 the previous symbol from time instant k minus 1. So, x k minus one is causing interference to x k alright in y k we have contributions on both x k as well as x k minus 1 and this is what we termed as inter symbol interference. So, this interference is what we termed as ISI or inter symbol inter symbol interference and removal of ISI is termed as this. So, this is your noise v k is of course, noise or as a (Refer Time: 03:23) Gaussian noise.

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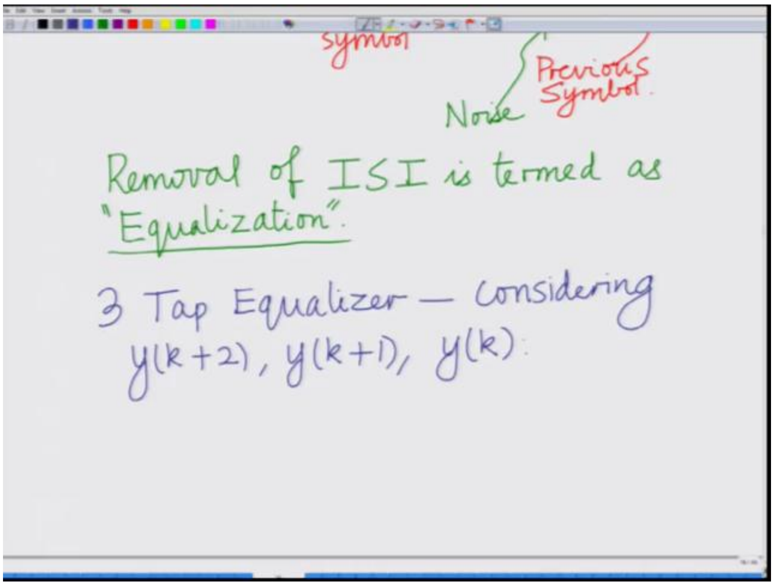

And removal of ISI or basically inter symbol interference is termed as, removal of inter symbol interference is termed as equalizer and we want to design an equalizer towards this process of to perform equalization and this wireless communication channel they by removing the effect of inter symbol interference.

So, now let us design for this purpose let us consider the design of a three tap equalizer which means we will consider y k, y k plus 1, y k plus 2. So, let us design a 3 tap equalizer. So, we will consider the design of a 3 tap equalizer considering y k plus 2, y k plus 1 and y k 3 tap equalizer and the system model the model for this can be expressed as; these are the three output observation that we are going to consider for say time instant k, time instant k plus 1, time instant k plus 2.

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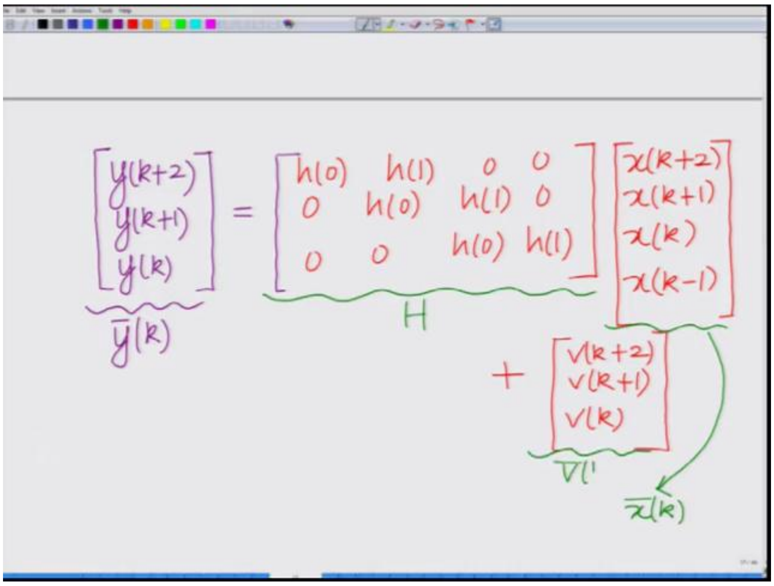

The model for this can be formulated as - the model for this can be formulated as, let us formulate the model for this that will be y k plus 2, y k plus 1 y k these are the three symbols at time instant k, k plus 1, k plus 2 let us call this as y bar k that is the vector y bar k. This you can see, can be expressed as from the model for the inter symbol interference wireless channel this can be expressed as H 0 the matrix H 0 H 1 0 0, 0 H 0 H 1 0, 0 0 H 0 H 1 times x k plus 2, x k plus 1, x k and x k minus 1 plus the noise vector of course, we have the noise vector which is v k plus 2, v k plus 1 and v k. And of course, for instance you can verify this as y k plus 2 equals H 0 times x k plus 2 plus H 1 times x k plus 1 plus v k plus 2 and so on alright. So, we can denote this by this matrix by H this is the effective matrix.

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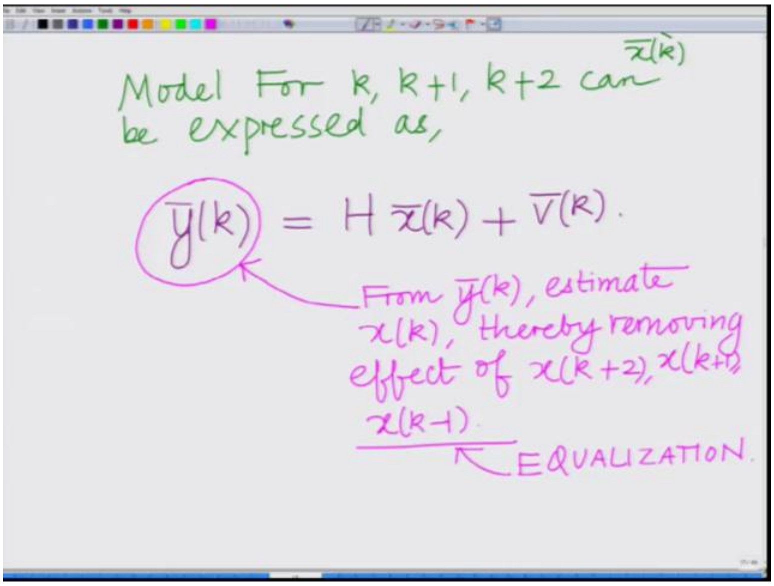

This matrix by x bar k and this matrix and this vector rather by v bar k. So, we have this model considering, model for k, k plus 1, k plus 2 can be can be expressed as we have y bar k equals this is the model that we have developed for a 3 tap equalizer equals H times x bar k plus v bar k.

So, this is the model that we can x derive for this that is your y bar k y bar k consists is a vector, consisting of observations of y k, y k plus 1, y k plus 2 or rather it is a vector which y k plus 2, y k plus 1, y k in that order equals the matrix H times the vector x k plus 2, x k plus 1, x k, x k minus 1 plus the noise vector v k plus 2, v k plus 1 v k. And from this basically now from this vector y k plus 2, y k plus 1 y k basically based on this observation vector we have to estimate x k that is the process of equalization. So, that basically you are removing the effect of x k plus 1 and x k, from y k - from y bar k estimate x k thereby removing effect of x k plus 2, x k plus 1 comma x k minus 1 and this is basically your equalization. By using y k plus 2, y k plus 1 and y k basically we are trying to recover x k and remove the effects of the other symbols, other interfering symbols which is in this case are x k minus 1 x k plus 1 and x k plus 2 this is the concept of equalization and we will use the LMMSE principle to deriving the equalizer.

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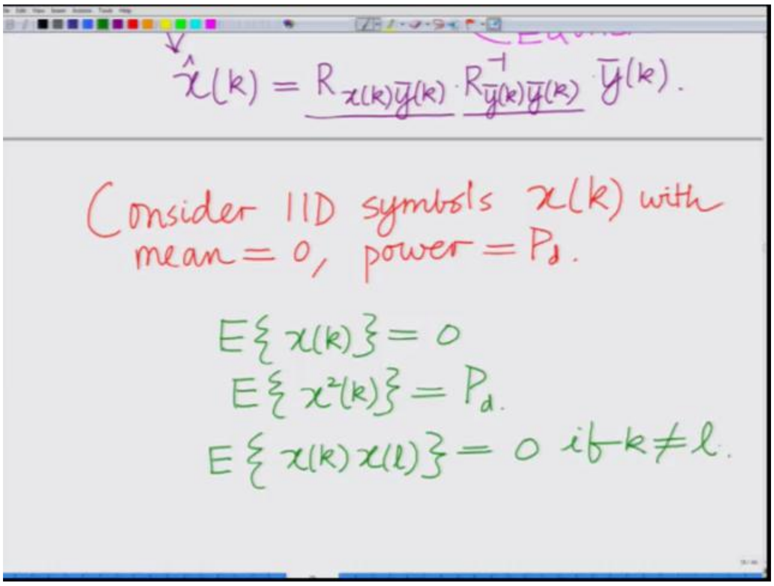

And therefore, naturally what we have is let us start. So, naturally LMMSE estimate or the LMMSE equalizer is given as we already know this x hat of k will be R x k y bar k the cross covariance between x k and y bar of k times r. The covariance matrix of the observation vector R y bar k y bar k inverse times y bar of k, this is the LMMSE estimate of x of k. And towards the side basically we have to derive these two quantities R x k y bar k the cross covariance R y bar k y bar k the covariance matrix of the output vector y bar k. Now for this let us consider the standard assumption that is IID symbols (Refer Time: 11:08). So, consider IID that is independent identical symbols of x k with mean zero, mean equal to 0 comma power equal to P d, consider IID symbols we are considering the transmitted symbols x k to be IID that is independent identical distributed with mean 0 and power P.

This means therefore, again, this is also straight forward that is expected value of x of k each symbol x of k equal 0. Expected value of x square of k which is the power this is equal to P d and expected (Refer Time: 12:10) symbols are independent expected value of x k into x l equal to if k is not equal to l. Now what we wanted to do is, basically now, let us start by looking at expected value of x bar k into x bar k transpose

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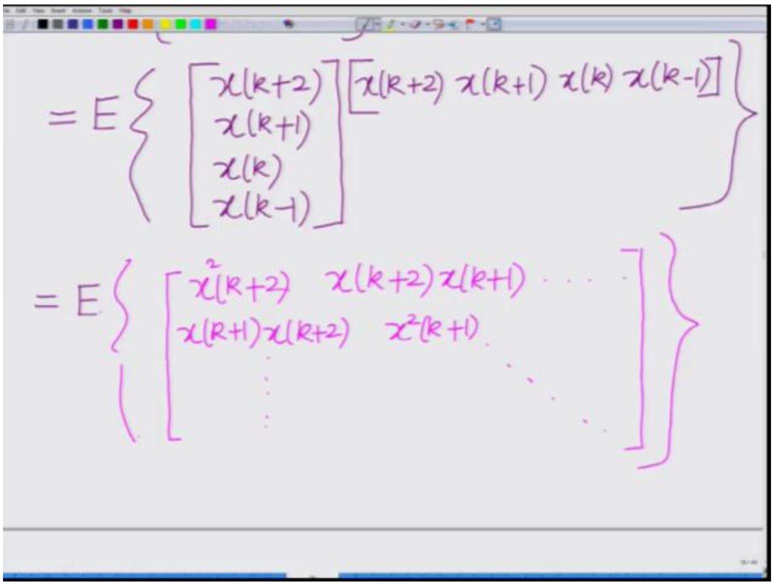

Remember, x bar k this is the vector, x bar k is the vector that is different from x k x bar k is this vector which contains x k plus 2 x k plus 1 x k and x k minus 1. So, something to keep in mind that x bar k and x k are slightly different, x bar k is the vector which contains x k plus 2 x k plus 1, so this x bar k contains the vector x k plus 2 x k and x k minus 1 that is your vector x bar k into its transpose and the transpose will be x k plus 2, the row vector same vector, but the row vector - x k minus 1. Now this will be equal to expected value of x square of course, the diagonal element will be x square k plus 2, x k plus 2, x k plus 1, x k plus 1, x k plus 2, x square k plus 1 so on so forth.

And now if you can see the diagonal elements are going to be x square k plus 2, x square k plus 1, x square k, x square k minus 1. Expected values of diagonal elements are each P d, but if you look at the off diagonal elements they will be the expected value of x k expected for instance expected value of x k plus 2 into x k plus 1 and expected value of x k plus 2 into x k plus 1 is 0 because the different symbols are independent. So, the off diagonal elements, these elements will go to zero one once you take the expectation that is important once you take the expectation operator inside at this will be equal to each equal to P d. So, this will be a diagonal matrix with P d on the principle diagonal so this is basically going to be P d times identity matrix.

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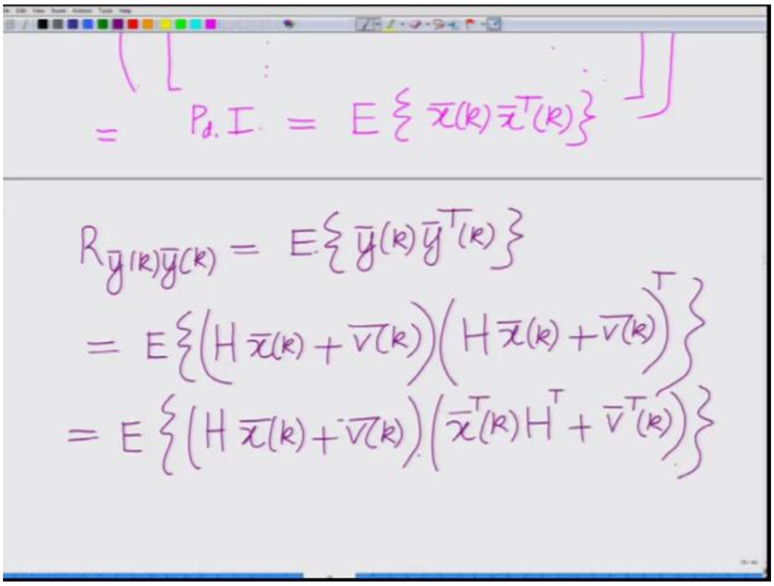

That is your basically expected value of x bar k into x bar transpose of k that is considering IID independent, identical distributed elements we have expected value of x bar k transpose k is P d times identity matrix where P d is the power of each symbol.

Now, let us now look at come to R like we have computed so many times before the covariance matrix of the output vector y bar is expected value of y bar k, y bar transpose of k this will be your expected value of H x bar k plus v bar k times H x bar k plus v bar k transpose which is equal to expected value of H x bar k plus v bar k into x bar transpose k H transpose plus v bar transpose k. Now what we are going to do is also assume the standard assumption, assume symbols each symbols x l comma noise v l are un correlated.

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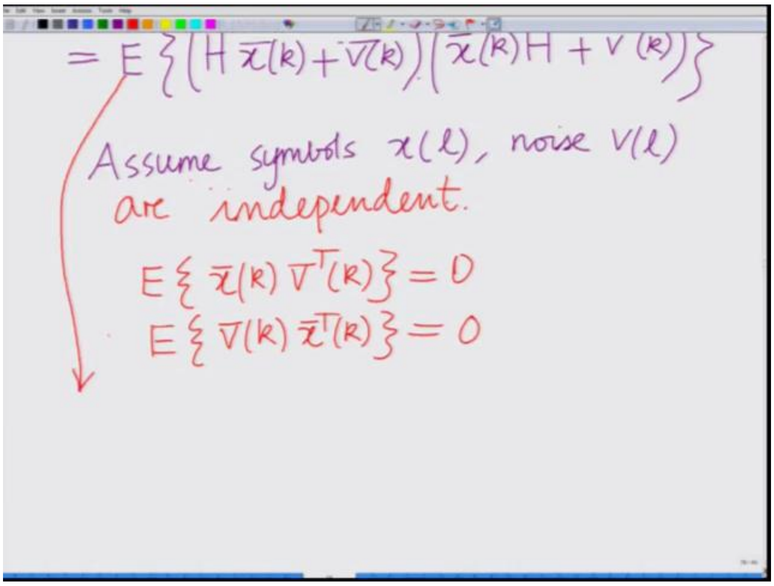

In fact, they are independent so let us assume they are independent. For us we need only the, so in realties the symbols from the transmitter the symbols are transmitted by the transmitter and the noise is at the receiver. So, these two process are different process and therefore, these two are independent let me write this again clearly.

So, let us assume. So, these two are independent and therefore, naturally you we are going to have if you assume zero mean noise expected value of x bar or even otherwise expected value of x bar k v bar transpose k equals 0 expected value of v bar trans v bar k x bar transpose k both of this things are 0. And therefore, now what we are going to have is if you expand this covariance are y bar y bar k what we are going to have is expected value of, we are going to have expected value of H x bar k x bar transpose, so we are going to have expected value of. So, if you expand this H x bar k x bar transpose k into H transpose plus the next term will be H x bar k v bar transpose k H k v bar transpose k plus v bar k x bar transpose k H transpose plus v bar k v bar transpose k and now splitting this expectation operator we will have x H expected x bar k, x bar transpose k, H transpose plus H expected x bar k v bar transpose k plus expected v bar k x bar transpose k, H transpose plus expected v bar k v bar transpose of k.

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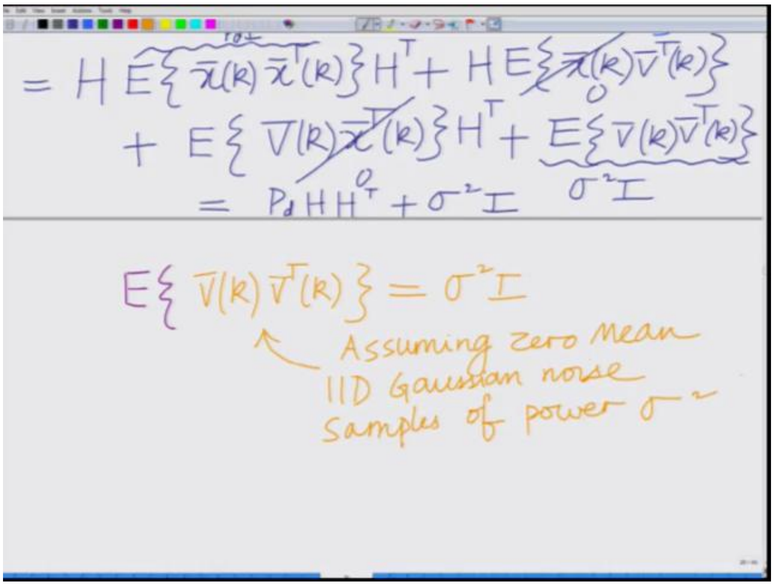

So, this is what we have, once we take expand it into the expectation of the separate terms and take the expectation operator inside. Now let us simplify each term and now we know expected x bar k v bar transpose k this is 0, these two terms are zero of course, assuming IID Gaussian noise samples of mean zero variance sigma square, we already seen this is sigma square times identity, this assuming IID noise IID symbols of power P d this is P d times i. So, this net matrix becomes P d H H transpose plus sigma square i. So, let me just clarify that once more regarding the noise covariance I am not repeating since we are seen this many times before expected value of v bar k v bar transpose k equals sigma square i, this is assuming zero mean IID Gaussian noise samples of power sigma square or variance sigma square.

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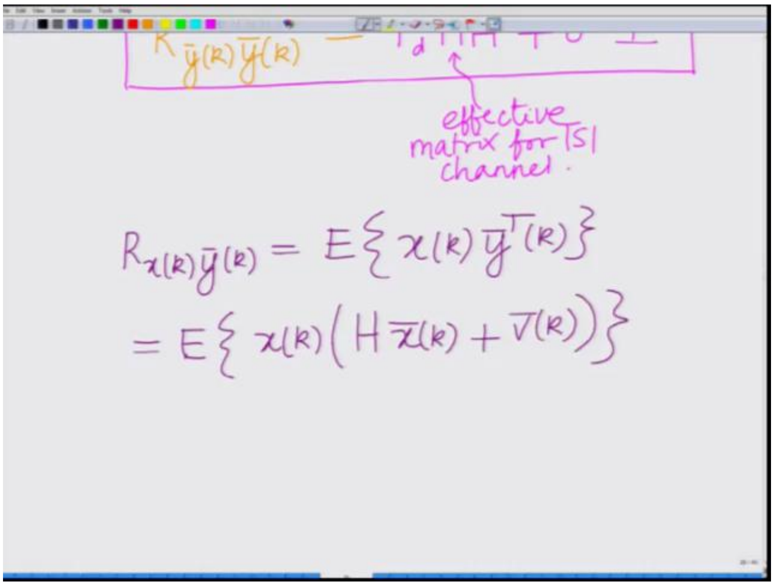

So, R y bar R k y bar k the covariance matrix of y bar that is basically P d H H transpose plus sigma square times identity, we have computed the covariance of the output covariance of the output of course, H is the effective matrix for the ISI channel - effective matrix for the ISI channel and this is the structure of H alright if you do not you can recall that this is H which is the effective matrix of the ISI channel.

And realize this effective matrix for this ISI channel depends on a couple of things - one it depends on number of channel taps alright and two, it also depends on the order of the equalizer that is the size of the this effective channel matrix it depends on the total number of taps in the channel it also depends on the total number of taps in your equalizer. So, basically this is the effective channel matrix H. And now let us look at the cross covariance between x k not x bar k, but the x k the desired symbol to be estimated. So, now, let us look at R x k y bar k that is expected value of the symbol x k times transpose of the vector y bar k which is expected value of x k times H k bar k, this is different k bar k plus v bar of k.

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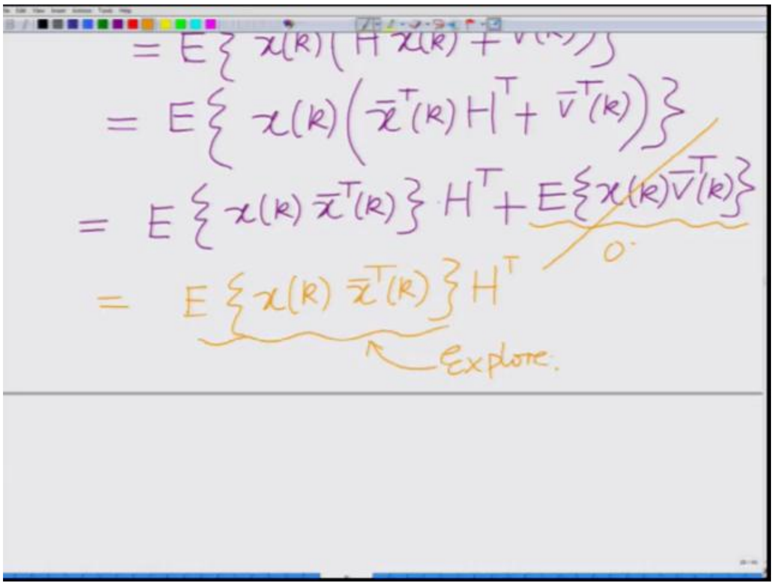

And this is now equal to expected value of let us write this down at x k x bar transpose k H transpose plus v bar transpose k which is equal to expected value of well x k x bar transpose k into H transpose plus expected value of x k v bar transpose k.

Now, basically again we know the noise and symbols are un correlated or independent therefore, this is equal to 0. So, what we are left with is expected value of x k, x bar transpose k times H transpose.

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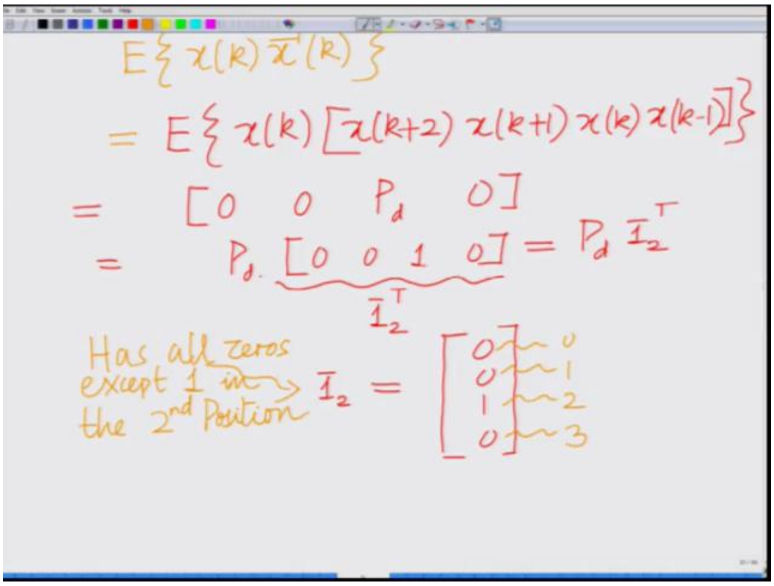

Now, let us explore this quantity, let us explore this quantity, expected value of x k into x bar transpose k, this is equal to this is equal to expected value of well x k, the symbol x k times of row vector x bar k x k times x k plus 2, x k plus 1, x k into x k minus 1 which is basically expected value of course, now if you expand this you can see this is x k. If you expand this you can simply see that all the terms accept the expected value of x square those will vary expected value x k into x k plus 2 is 0 x k expected x k plus 1 is 0 expected value of x k of k is P d expected value of x k to x k minus 1 is 0. So, this is equal to P d into 0 0 1 0 and we will denote this matrix as vector 1 bar 2 transpose. So, this we can call it as P d times 1 bar 2 transpose because if you look at this vector one bar two the definition 1 bar 2 is 0 0 1 0 and if you start counting from the first position as zero this is the 0th position, this is the first position, this is the second position and this is the third position.

So, this vector one bar two has all zeros expect one in the second position. So, if you look at this vector one bar two that we have defined this is all zero vector expect it has one in the second position. So, this is the vector 1 bar 2 and naturally therefore, now this is P d times one bar two transpose, this will be your P d 1 bar 2 transpose times H transpose.

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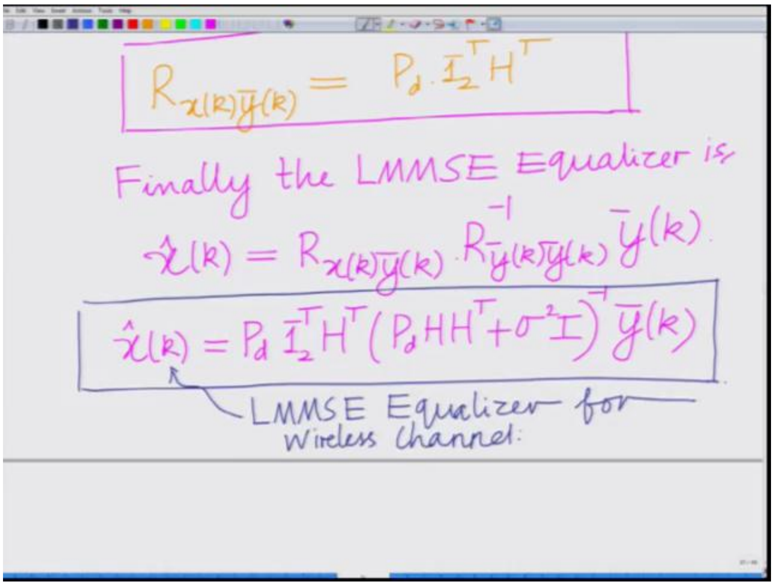

Therefore, finally, what we have is R the cross covariance R x k y bar k is P d 1 bar 2 transpose into H transpose this is the cross covariance. And now we can find the estimate finally, the MMSE equalizer, LMMSE equalizer - x hat of k equals R, x k y bar of k into R, y bar of k if y bar of k inverse at y bar of k and therefore, x hat of k equals P d 1 bar 2 transpose H transpose into P d H H transpose plus sigma square identity inverse into y bar of k and therefore, this is your expression for - this is a neat elegant expression. So, this is the LMMSE equalizer for your wireless. So, this is the LMMSE equalizer for wireless channel.

And couple of points before we conclude, basically though we have illustrated this for a simple case with 2 channel taps and a 3 tap equalizer you, as we can see if a naturally extended to scenario with any number of channel taps and equalizer of any order or any number of taps in the equalizer. All the changes is basically the effective channel matrix has to be appropriately constructed for the (Refer Time: 30:46) of the channel filter with the given number of channel taps and the equalizer of the desired order alright. So, this can be naturally extended and two, also again we have consider a simple scenario with IID symbols, but if there is in principle, if there is that is if there is some correlation between the different symbols that also can be handled because what changes is basically only the covariance matrix that is expected value of x bar x bar k transpose. If you construct the covariance matrix appropriately and plug that in basically all the expressions that we can handle even the case where the symbols are correlated and so on and so forth.

So, this basically illustrates the simple example or a simple scenario of LMMSE equalizer design for 2 channel taps and 3 tap equalizer alright and this can be readily extended to other scenarios very easily. So, this illustrates basically the concept of LMMSE equalizer design and LMMSE equalization for a wireless communication channel with inter symbol interference towards the basically the equalization is towards removing the effect of inter symbol interference. So, we conclude the module here.

Thank you.