Bayesian/MMSE Estimation for MIMO/OFDM Wireless Communications
Prof. Aditya K. Jagannatham
Department of Electrical Engineering
Indian Institute of Technology, Kanpur

Lecture – 25
Example of LMMSE/ MMSE Estimation for Multiple-input Multiple-Output (MIMO) Downlink Wireless Channel Estimation

Hello, welcome to another module in this massive open online course on Bayesian MMSE Estimation for Wireless Communication Systems or we are currently looking at the process of estimating or process of computing the LMMSE estimate or the MMSE estimate of a MIMO that is the multiple input multiple output wireless communication system that is a wireless system which has multiple transmit antennas, both multiple transmit antennas as well as multiple receive antennas, that is why it is known as a multiple input multiple output wireless communication system and that channel over which the communication takes place is known as the multiple input multiple output wireless channel.

So, we have derived the LMMSE estimate as well as the estimation with the co-variance for LMMSE estimation. Let us now do a simple example to understand this better.

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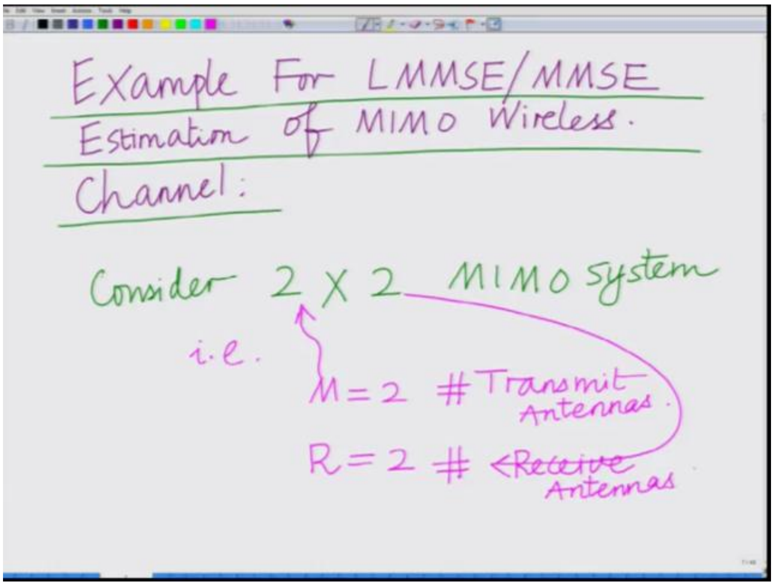

So, in today's module we are going to an example for the LMMSE or MMSE estimation, example for LMMSE or the MMSE estimation of a MIMO wireless channel. Let us consider a 2 cross 2 MIMO system, for this purpose consider a 2 cross 2 MIMO channel or basically a 2 cross 2 MIMO system, 2 cross 2 MIMO channel implies that is if you look at this, remember this is an M cross r where M is equal to 2, that is your number of transmit antennas, number of transmit antennas is equal to 2, number of receive antennas, that is here r equal to 2 which is the number of receive antennas.

So, we have a 2 cross 2 MIMO system, it is M cross r, M is a number of transmit antennas, r is the number of receive antennas, this is a 2 cross 2 MIMO wireless channel. Now, let us consider N equal to 3, all right in this a particular example.

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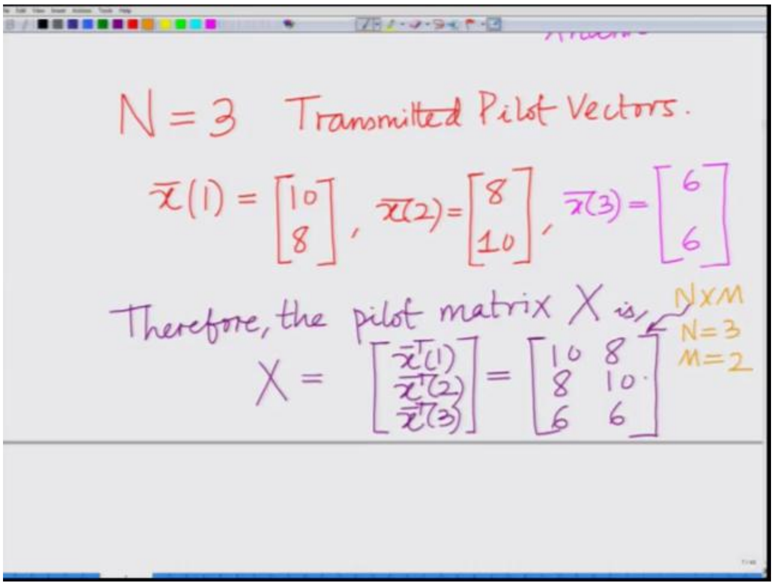

Let us say we have N equal to 3 transmitted pilot vectors. We have N equal to 3 transmitted pilot vectors, let them be X bar 1 that is the first pilot vector that is 10 8 X bar 2 that is the second pilot vector, second pilot vector that is eight, 10 X bar 3 that is the third pilot vector that is 6 . So, we have 3 pilot vectors that are transmitted. Now, from this 3 pilot vector remember N is a number of pilot vectors that we are considering from this 3 pilot vectors we will construct the pilot matrix. Therefore, the pilot matrix X is X equals the pilot matrix is X bar transpose 1 X bar transpose 2 X bar transpose 3 the pilot matrix is your 10, 8 X bar transpose 2 that is 8, 10 X bar transpose 3 that is 6, 6 and this pilot matrix is a N cross M pilot matrix, N is equal to 3, M is equal 2.

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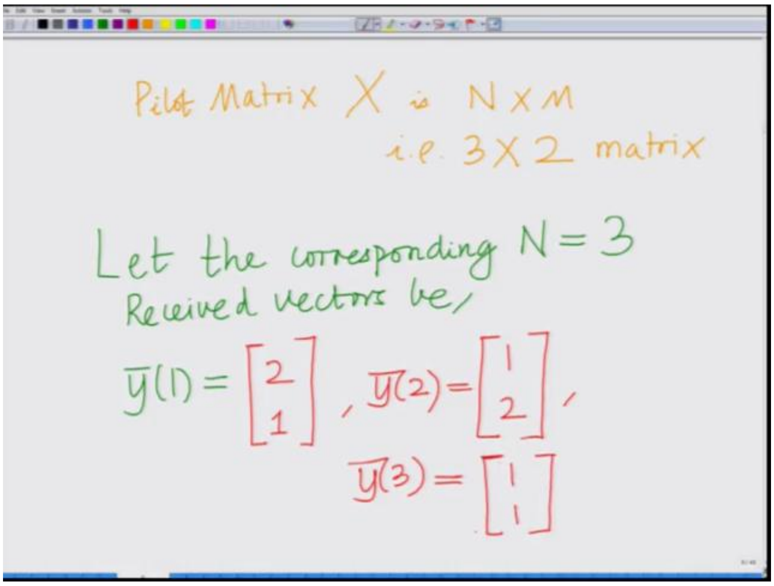

So, pilot matrix X N cross M that is your 3 cross 2, that is matrix of size 3 cross 2, that is the pilot matrix denoted by X is of size 3 cross 2, now since we have 3 transmitted N equal to 3 transmitted pilot vectors, we will have N equal to 3 corresponding receive pilot vectors. Now, observe that we have 2 receive antennas. So, each received vector will be of size 2 because of each receive antenna will have a symbol on each receive antenna alright so each received vector will be vector of size 2 that is a 2 cross 1 vector and we have N equal to 3, such vectors corresponding to 3 transmitted pilot vectors..

So, let the corresponding received vectors N equal to 3 receive vectors be the corresponding N equal to 3 received vectors, let the corresponding N equal to 3 received vectors be y bar of 1 this is equal to 2 1 y bar of 2 that is a second received vector is 1 2 and y bar of 3 that is equal to 1 that is N equal to 3 received vectors.

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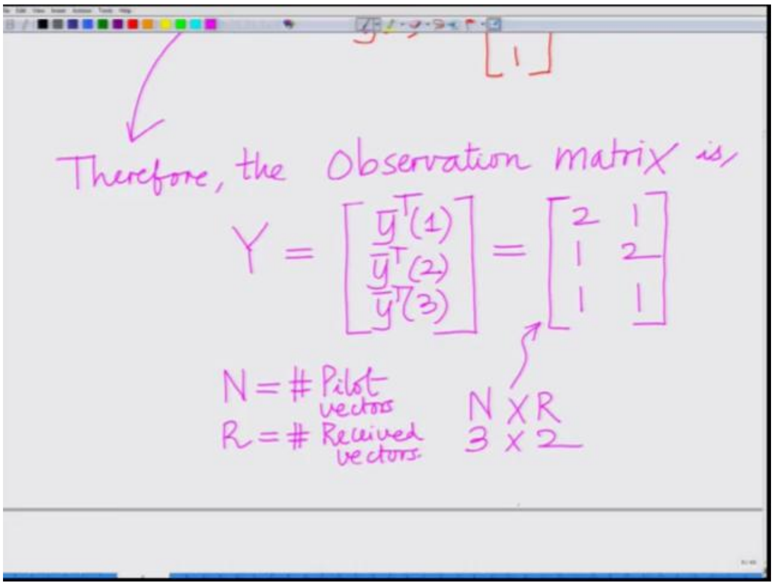

Therefore, from this received vectors, we can construct the observation matrix, the observation matrix is y equals y transpose y bar transpose 1 y bar transpose 2 y bar transpose 3 that is observation matrix which can be written as y bar transpose 1 that is 2, 1 y bar transpose 2 that is 1 comma 2 y bar transpose 3 that is 1 comma one, this is your observation matrix and remember this observation matrix is N cross r that is 3 cross 2 N equals number of pilot vectors r equals number of received or observed vectors.

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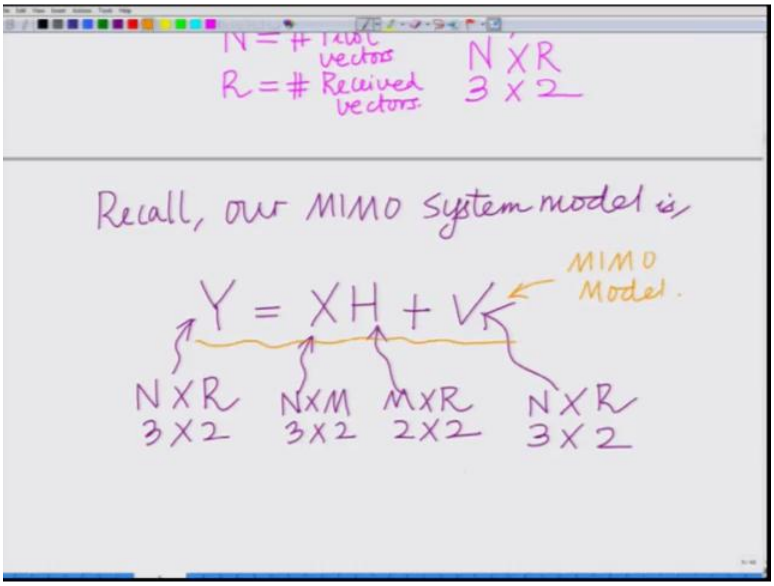

So, this is an N cross pilot matrix X this is an N cross r matrix alright there is the corresponding system model that we have derived earlier remember from the MIMO system model recall that the MIMO system model is our MIMO system model is y equals X h plus v this is our MIMO system model where y is N cross r that is in this case 3 cross 2 X is N cross m, in this case again 3 cross 2 h is M cross r, in this case 2 cross 2 and v is again N cross r, in this case it is a 3 cross 2 matrix, this is our MIMO system model that is a multiple input multiple output wireless channel model.

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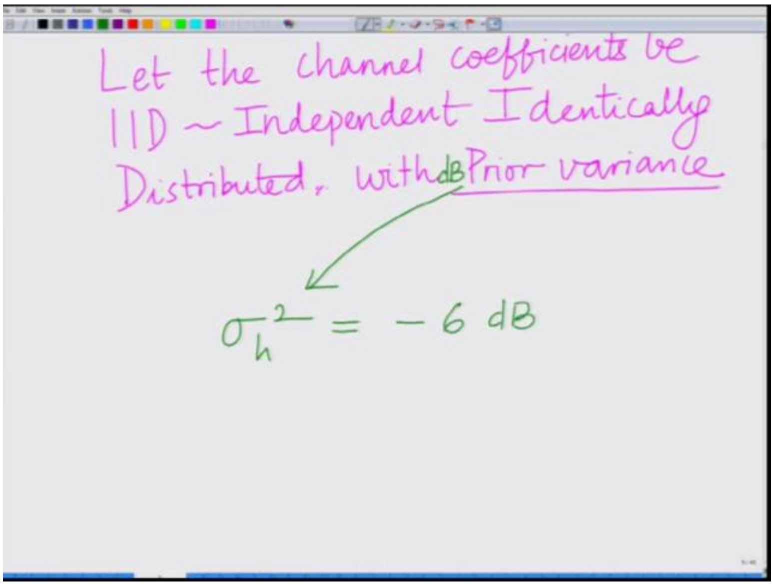

Now, let the channel coefficients be IID, let us assume IID channel coefficients with prior variance sigma X square with identical prior variance sigma h square, let the channel coefficients be independent be IID that is independent identically distributed that is independent identically distributed with each having prior variance. Remember, we are denoting the prior variance, this prior variance we are denoting this by sigma h square or let us say dB prior variance sigma h square equals minus 6 dB.

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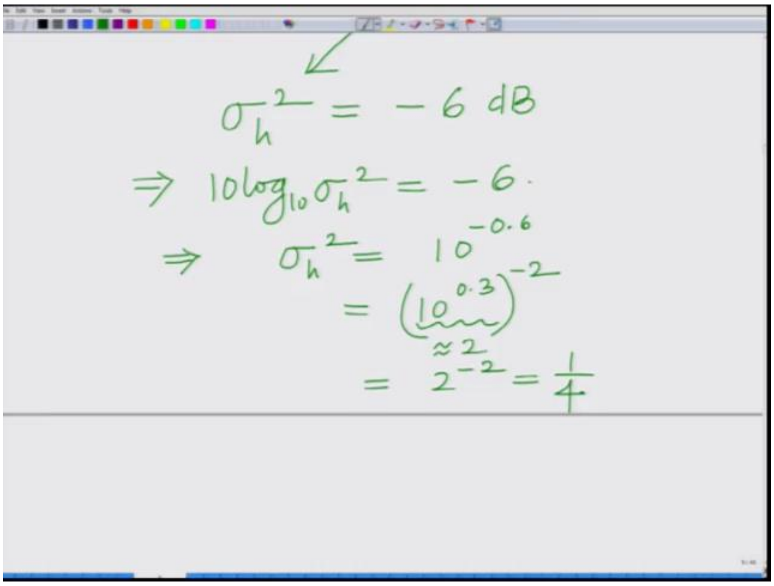

The prior variance sigma h square is minus 6 dB implies if you look at 10 log 10 sigma h square that is equal to minus 6 implies sigma h square is 10 raise to power minus point 6 equals 10 raise to the power of point 3 to the power of minus 2 and we know 10 raise to the power of point 3 there is approximate equal to root 2 because 3 dB is approximately 2. So, this is 2 raise to the power of minus 2 equals 1 by 4. So, we are prior variance in dB sigma h square which is minus 6 dB is approximately this is 1 by 4 that is sigma h square where each IID channel coefficient of the channel matrix h is 1 by 4.

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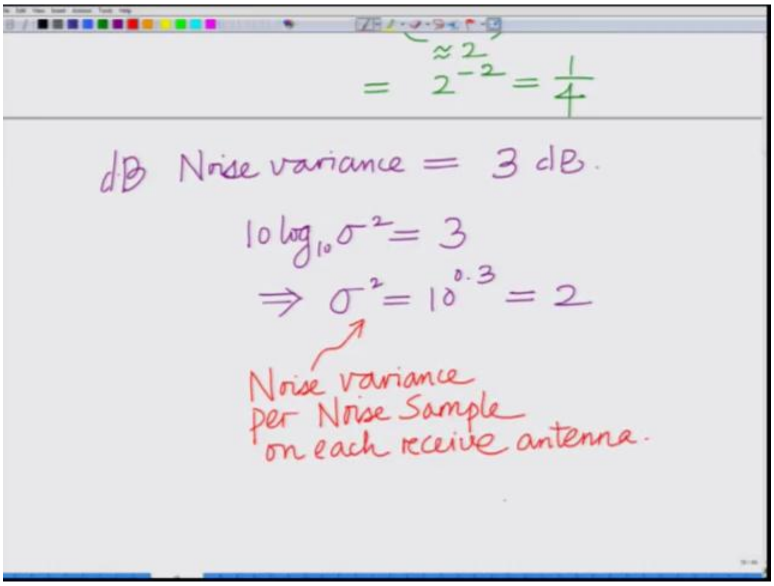

Now, let us look at the noise variance, let us assume that the prior noise variance or not prior noise variance, but rather the dB noise variance equals 3 dB, this is straight forward this is 10 because we know 10 log 10 sigma square equals 3 dB or rather 3, this implies sigma square equals 10 to the power of point 3 equals 2 because we know 3 dB corresponds to 2. So, sigma square that is the noise variance per noise element that is the noise variance sigma square per noise sample on each receives at remembers we are assuming and we have gone through this when we discuss the noise, it is a noise variance per noise sample on each receive antenna that is equal to 2.

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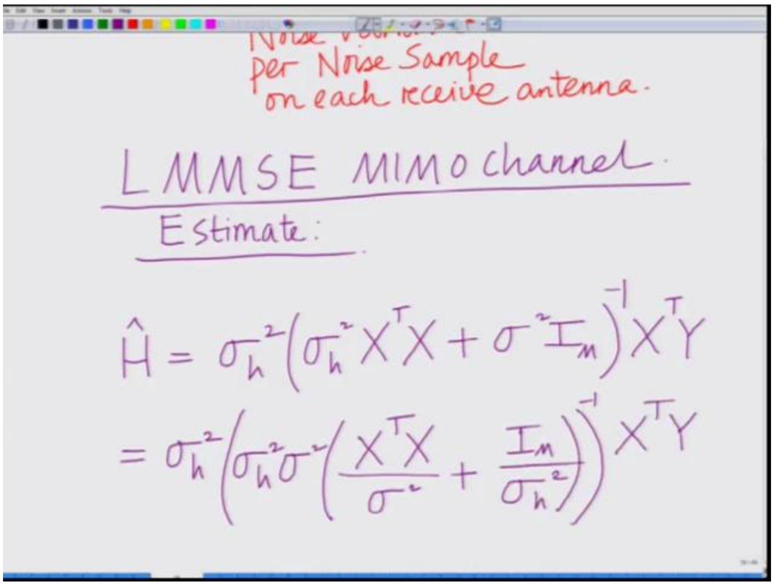

Now, let us compute X transpose, first let us simplify the LMMSE channel estimate. We have to compute the LMMSE MIMO channel estimate, let us compute remember the LMMSE MIMO channel estimate LMMSE of course, if it is Non-Gaussian, MMSE if it is Gaussian that becomes your h hat equals sigma h square sigma h square X transpose X plus sigma X h square identity of size M cross M inverse into X transpose y. We will just simplify this a little bit, I can write this also as sigma h square, in this term I can take sigma h square, in term inside brackets I can take sigma h square sigma square common. So, that becomes X transpose X divided by sigma square plus I m divided by sigma h square inverse into X transpose y.

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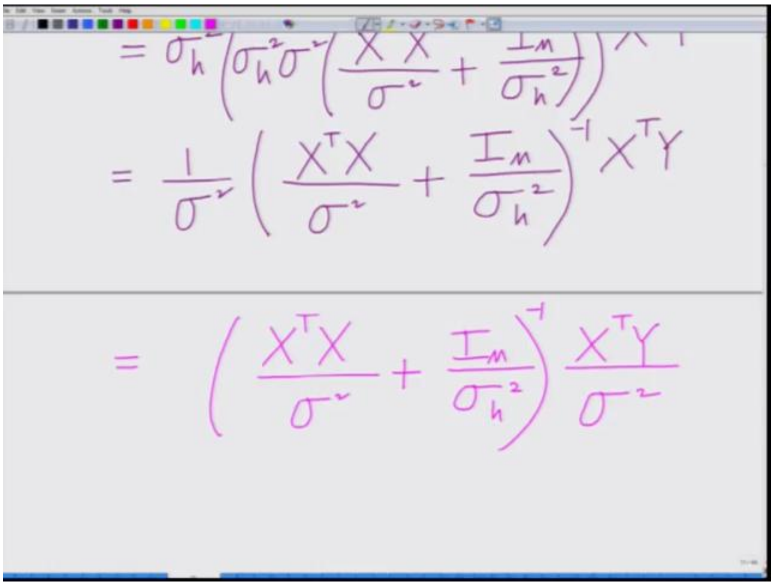

Now, when the sigma X square sigma square comes out of the bracket because there is inverse it goes into the denominator. So, I have sigma h square divided by sigma h square into sigma square that becomes 1 by sigma square X transpose X divided by sigma square plus I M divided by sigma h square into X transpose y and now I can write this as bringing the sigma square just for convenience bringing the sigma square, we need X transpose y I can write this as X transpose X divided by sigma square plus I M divided by sigma h square inverse into X transpose y divided by sigma square there is just an equivalent way of writing this because this is going to be convenient for us in the future.

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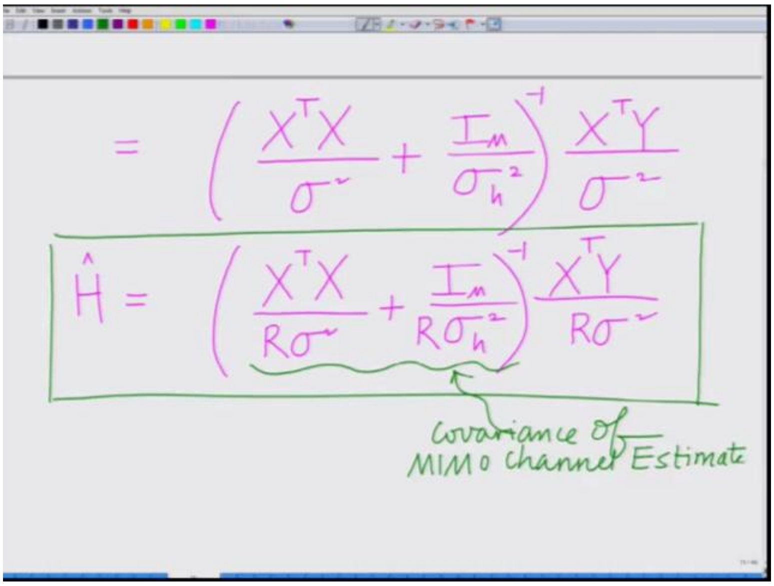

Now, what I am going to do is I am going to divide and multiply by r the number of receive antennas and what that is going to give me is this is X transpose X divided by sigma square into I M divided by sigma h sigma h square inverse into X transpose y divided by r sigma square and I take that the r in the numerator because I divide and multiply by r the multiplied r I take inside the inverse. So, it again comes in the denominator that becomes r in the denominator.

So, that because X transpose x, the final expression for the MIMO channel estimate becomes just an equivalent way of writing it at X transpose X divided by r sigma square plus I M divided by r sigma square inverse into X transpose y divided by r sigma square and I am saying this is particularly convenient because if you look at this part, this part is nothing but the co-variance of the MIMO, this is the co-variance this X transpose X by r sigma square plus I M that is identity of size M cross M divided by r sigma h square inverse. This whole part is nothing but the co-variance of the estimation of the MIMO channel matrix.

So, 1 need not re-compute the co-variance all over again, 1 can use this part that is computed for the LMMSE estimate reuses it for the co-variance alright. So, that it just convenient because it saves some computational burden and remember again r this is nothing new, r is the number of receive antennas that you should be well aware by now, r equals the number of receive antennas in the MIMO system. Now, let us start computing each of these quantities, of course we need X transpose X.

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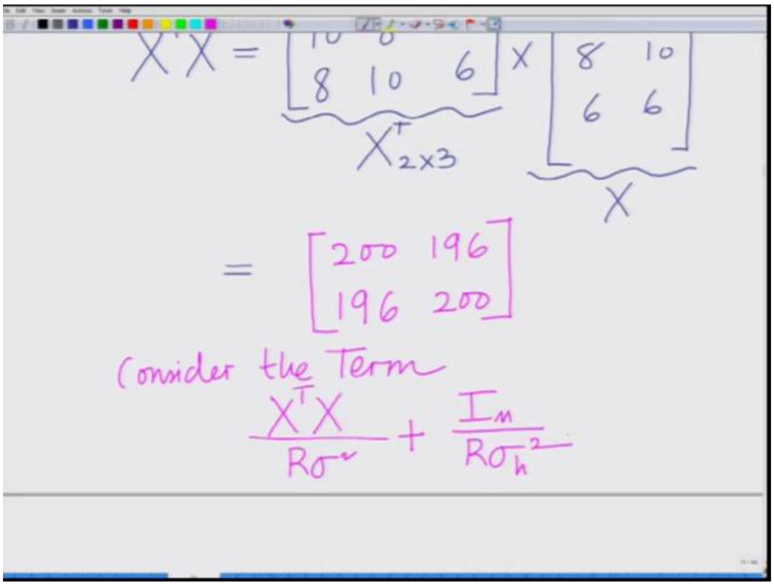

So, let us start by computing X transpose X, X transpose is the transpose of X of course, you take X make the rows into columns, columns into rows that becomes X transpose X is the 3 cross 2 matrix. So, X transpose is naturally going to be a 2 cross 3 matrix times X which is basically your pilot matrix 10 8 8 10 6 comma 6, this is your pilot matrix X this becomes your 200 196, 196 200 and now observe that the LMMSE estimate.

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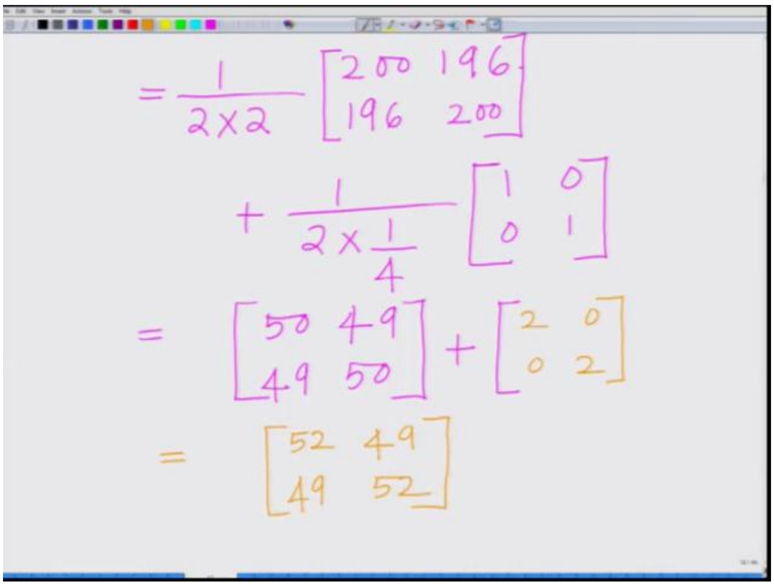

Now consider the term X transpose X by r sigma square plus M cross M identity that is 2 cross 2 identity divided by r sigma h square which in this case 1 over r sigma square 1 over r into sigma square is 2 r equal to 2 sigma square is equal 2 into X transpose X that is 200 196, 196 200 plus 1 over r sigma h square 1 over 2 into sigma h square is 1 by 4 times identity 2 cross 2 1 0 0 1 and if you compute this, this will be basically 50 49, 49 50 plus 2 0, 0 2 that is your 52 49, 49 52.

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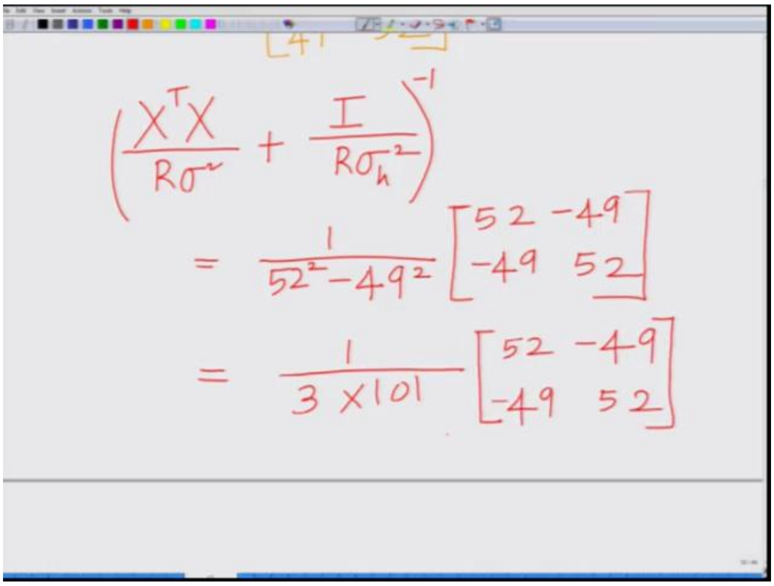

And therefore, now we compute the inverse of this quantity X transpose X divided by r sigma square plus I divided by r sigma h square inverse which becomes, now it is a 2 cross 2 matrix. So, inverse is 1 over the determinant which 52 square minus 49 square times interchange the diagonal elements, diagonal elements of same so no effect 52 of diagonal elements take the negative minus 49 minus 49. So, this becomes basically 1 by 3 times 101 into 52 minus 49 minus 49 52. So, this becomes 1 over 3 times 101 to 52 minus 49 minus 49 52, this is the expression for that the matrix X transpose X divided by r sigma square plus identity matrix size 2 cross 2 divided by r sigma h square whole inverse.

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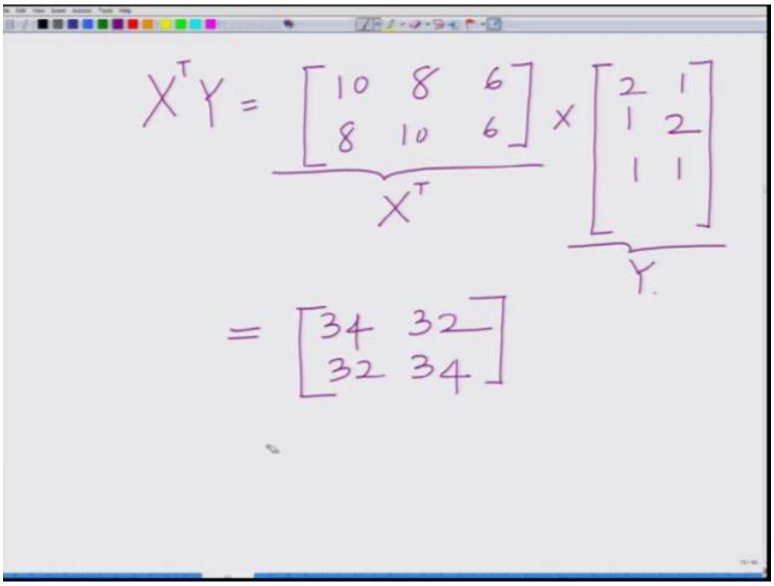

Now, we will compute the other component which is X transpose y alright and then you have to multiply by X transpose y divided by r sigma square. Now, let us compute the other component X transpose y, this is again X transpose which is a 2 cross 3 matrix 10 8 8 10 6 6 X transpose into your observation matrix y 2 1 1 2 into your observation matrix y which is 34 32 32 34.

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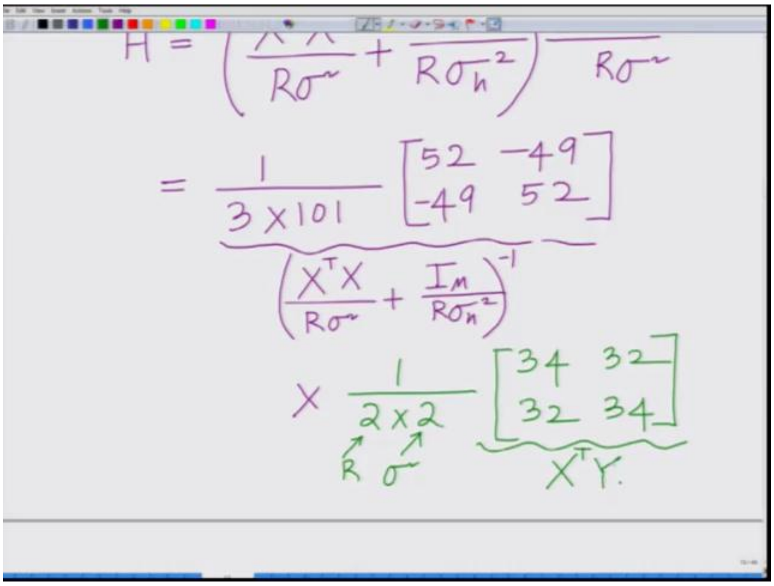

Therefore, now your MIMO channel estimate is given as h hat equals X transpose X given by r sigma square plus I M divided by r sigma h square into X transpose y divided by r sigma square which is equal to 1 over 3 into 101 52 minus 49 minus 49 52, this is your matrix X transpose X divided by r sigma square plus I M divided by r sigma h square inverse times of course, we need to have the other part which is 1 over r sigma square 1 over r into sigma square this is of course r, this is your sigma square which corresponds to 2, which corresponds to 3 dB X transpose y which is basically 34 32 32 34. This is of course, your matrix X transpose y.

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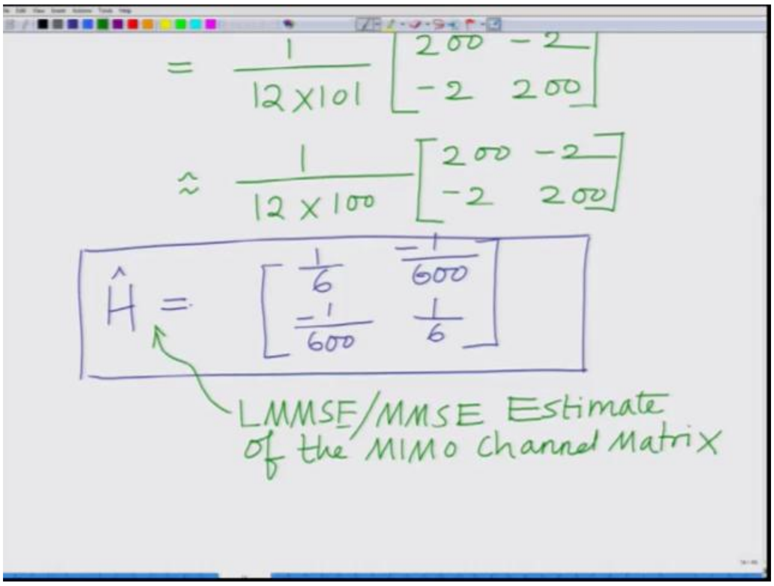

And now you put this all together and what you get is 1 over 12 times 101 into 200 minus 2 minus 2 200.

Now, approximating this 101 as 100, what I am going to get is this approximately equal to 1 over 12 into 100 200 minus 2 minus 2 comma 200 and if you look at this, this will be equal to basically, you can see this will be equal to 1 over 6 1 over 6 minus 1 over 600 minus 1 over 600 1 over 6 and that is the estimate of your MIMO channel matrix.

So, this is basically estimate of the MIMO channel matrix or. In fact, we can write this as the LMMSE slash MMSE estimate of the MIMO channel matrix for the given problem. This is the LMMSE slash MMSE estimate of the MIMO channel matrix, this is 1 over 6 minus 1 over 600 minus 1 over 600 1 over 6. So, for this example this is a MMSE or LMMSE estimate of course that is LMMSE if the prior probability density function of the channel coefficient of h is Non-Gaussian. The MMSE estimate if the prior probability density function is Gaussian that has to be clear to you at this point.

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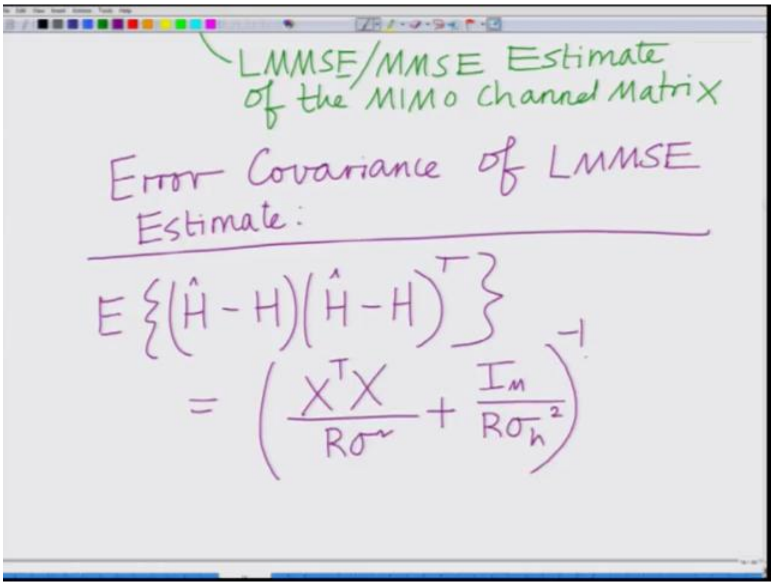

Now, let us compute error co-variance as I said, I don't need to separately compute the error covariance again because I have already computed, it is a part of the calculation of the LMMSE estimate, error covariance of your LMMSE estimate, that is your expected value of h hat minus h into h hat minus h transpose that is equal to and we have already computed this that is X transpose X divided by r sigma square plus I M divided by r sigma h square inverse.

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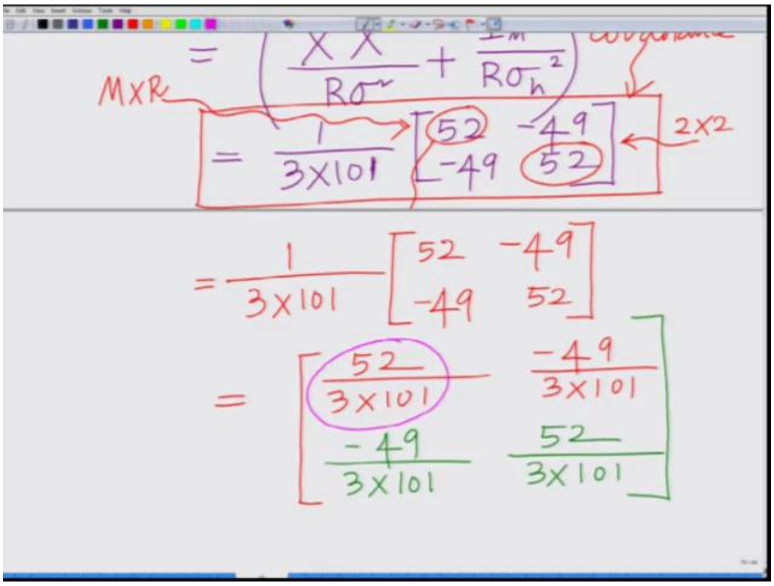

This we have already computed this is equal to 1 over 3 times 101 into 52 minus 49 minus 49 52 alright, this is the error covariance and also realize that this is the error covariance which is the 2 cross 2 we are looking at h hat minus h into h hat minus h transpose.

So, the diagonal elements correspond to the error co variances of each receive antenna. In general look at this there is h minus h into h minus h hat transpose, this is going to be a T cross, this is going to be an M cross r, this is going to be M cross r matrix. So, this is the M cross r matrix. So, each element is basically the sum of the co variances corresponding to sum of the error co variances corresponding to each transmit antenna if you look at each row alright. So, each row of basically corresponds to the receive antennas coefficients on the all the receive antennas for each transmit antenna for instead of you look at the first row, first row corresponds to first transmit antenna all the receive antennas and when you do h hat minus h into h hat minus h transpose, your summing across all the receive antennas therefore, is the sum of the each diagonal element is the sum of the variances corresponding to all the receive antennas for every transmit antenna.

This 52 or if you look at, let me write this 1 over 3 times 1 cross 1 into 52 minus 49 minus 49 52, this is equal to well let me bring that factor inside 52 divided by 3 into 1 cross 1 minus 49 divided by 3 into 1 cross 1 minus 49 divided by 3 into 1 cross 1 52 divided by 3 into 1 cross 1 and this is basically sum of variances corresponding to first transmit antenna sum of variances of all receive antennas for first transmit antenna and this is sum of variances of all r X antennas on second transmit antenna.

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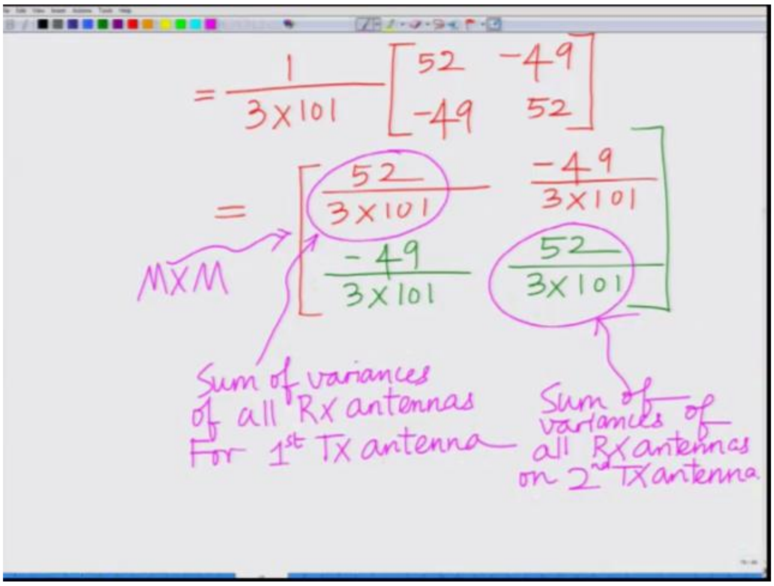

So, that is what we have alright so this is M cross M matrix, the error covariance that we have computed is an M cross M matrix, of course, one can generalize this notion and compute the error covariance for a general that is remember if you look at the MIMO channel matrix we said it is M cross r matrix. So, it has M r channel coefficient. So, 1 can compute the general error covariance matrix which is M r times M r that is compute the error variance of each element and error the cross correlation of the cross co variances of each of the covariance corresponding to each elements separately, but here we are doing with simply computing the sum of the error variances of all the channel coefficients corresponding to each transmit and all the receive antennas. So, that is the only difference of course, you can think a little bit of how to extend it to compute the error covariance of the M r, the net M r cross M r covariance corresponding to the error covariance corresponding to all the M r channel coefficients.

So, this example clearly illustrates how to apply the principle of LMMSE estimation, to compute the MIMO channel estimate and also the error covariance of the MIMO channel estimate for the give example set up. So, we will stop here and continue with other aspects in this module.

Thank you.