Bayesian/MMSE Estimation for MIMO/OFDM Wireless Communications
Prof. Aditya K. Jagannatham
Department of Electrical Engineering
Indian Institute of Technology, Kanpur

Lecture-24
LMMSE/ MMSE Estimation for Multiple-input Multiple-Output (MIMO) Downlink Wireless Channel Estimation

Hello. Welcome to another module in this massive open online course on Bayesian MMSE Estimation for Wireless Communications. We are looking at MIMO channel estimation and we have look at the statistics of the MIMO channel matrix H at the noise matrix V. Now, let us derive the MMSE or the LMMSE estimate. MMSE estimate for a Gaussian prior and LMMSE estimate for any other general non Gaussian prior of the MMSE or LMMSE estimate of the MIMO channel matrix H.

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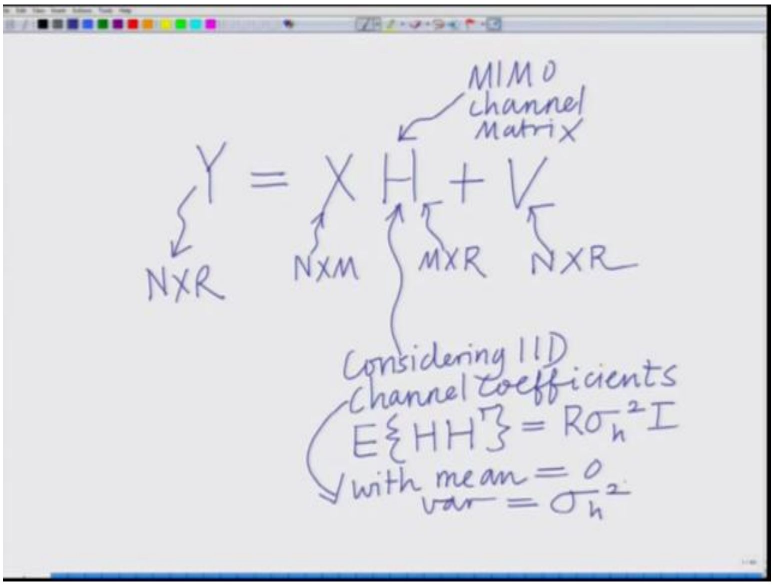

So, we have the MIMO channel estimation scenario where we have Y equals XH plus V this is your MIMO channel matrix this is N cross R, N cross M, M cross R and N cross R. Considering IID channel coefficients we have shown that expected value of HH transpose equals R sigma h square IID. IID channel coefficients with mean 0 variance sigma h square each considering this we have shown. That it is basically expected value of the covariance expected value of HH transpose is R sigma h square times identity.

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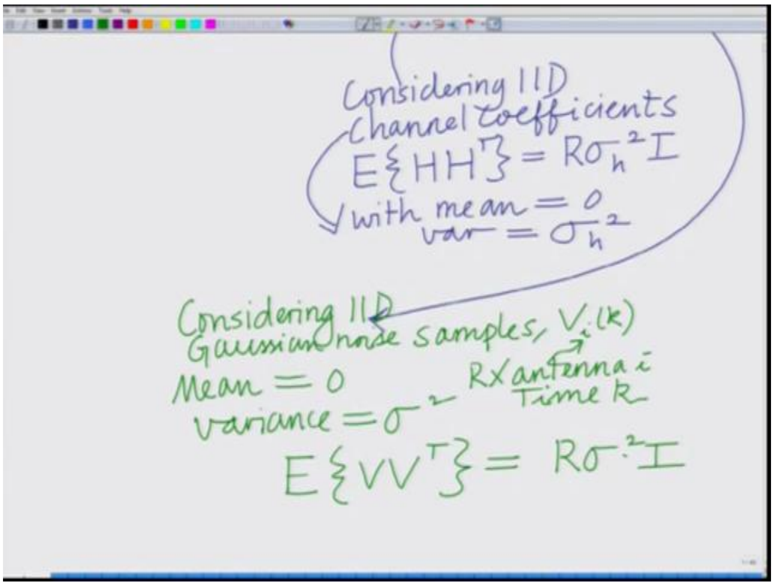

Similarly, considering IID Gaussian noise samples V i of k V i is a noise on antenna receive antenna i that is received antenna i at time k considering IID Gaussian noise samples mean equal to 0 and variance equal to sigma square, we have shown that the noise covariance expected VV transpose equal to sorry not 0 expected VV transpose equals R sigma square times identity matrix.

These are the statistical properties of the channel matrix and the noise channel matrix H and the noise matrix V that we have derived in the previous module. Now, let us compute the MIMO channel matrix estimate LMMSE estimate of the MIMO channel matrix.

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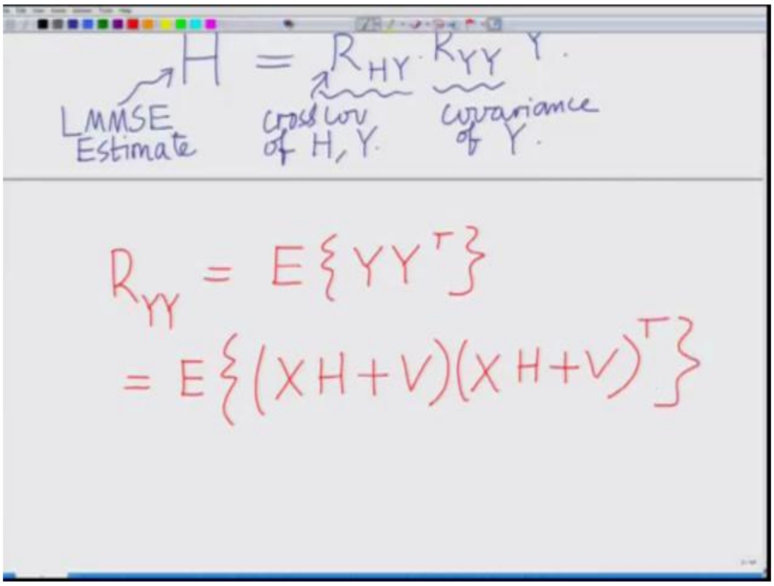

We know from our previous discussion that the LMMSE estimate for any general parameter the LMMSE estimate this is equal to R HY R YY inverse into the observation matrix Y. So, we need to compute these quantities R YY which is the covariance of Y and R HY this is the cross covariance of channel matrix H comma observation matrix Y. So, h hat we know as R HY R YY inverse time square.

So, let us start by computing the covariance matrix R YY. We know R YY is expected YY transpose of course it is complex Y then it is YY hermitian. Let us consider to make scenario consider simple scenario. Let us start with real observation matrices Y which is expected value of now I am going to substitute and we have done this several times before again it is going to be very similar XH plus V times XH plus V transpose.

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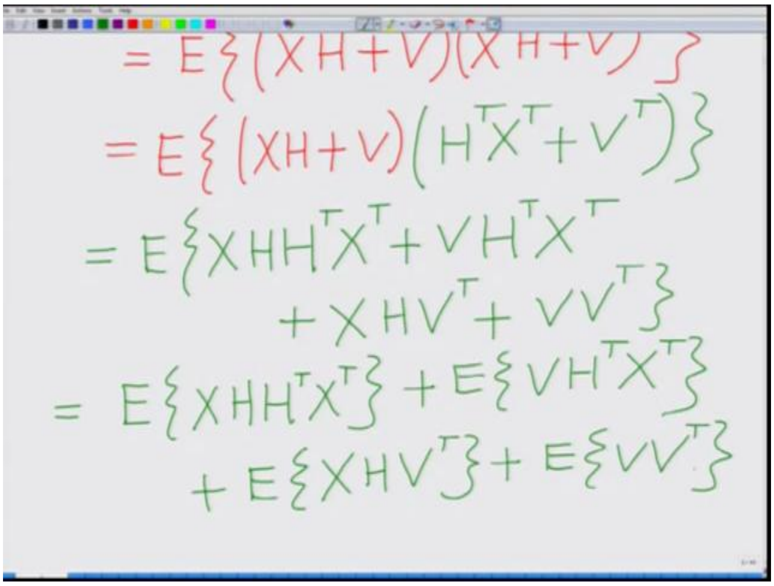

Which is equal to expected value of XH plus V times x plus h XH plus V transpose is H transpose X transpose plus V transpose this is equal to again now taking this term expanding this term by term H H transpose X transpose VH transpose X transpose plus XH V transpose plus VV transpose which we can simplify as now taking this term by term expected value of X HH transpose X transpose plus expected value of V H transpose X transpose plus expected value of XH V transpose plus expected value of VV transpose.

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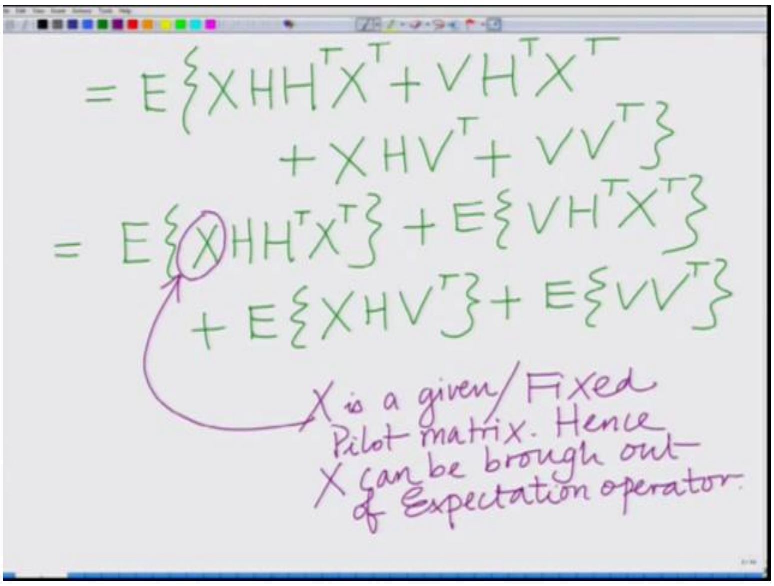

Of course now what we know x is a deterministic matrix given pilot matrix so this can be brought out of the expectation that X is a fixed pilot matrix this is nothing random about X. So, X can be brought out of the expectation operator, so x is a deterministic pilot matrix x is a given pilot matrix x is a given or basically a fixed pilot matrix. Hence, X can be brought out of expectation operator.

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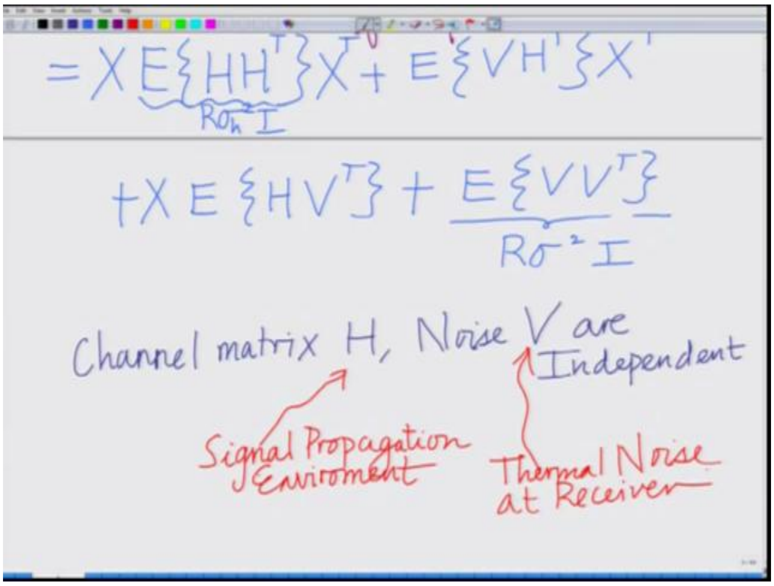

So, I can bring out X out of the expectation operator. So, bringing X out we can write this as now bring X out X expected value of HH transpose X transpose we know expected value of HH transpose is R sigma h square identity plus expected value of VH transpose X transpose plus expected value of or X times expected value of HV transpose plus expected value of VV transpose, and we know expected value VV transpose this is R sigma square times identity.

And now we also know that H is the channel matrix that arises of the environment that is arises out of the propagation environment and V is noise sample Gaussian noise sample arising due to the thermal noise at the thermal noise at the receiver. And these two random phenomenon that is the scattering environment, the propagation environment, the signal propagation environment and the phenomenon that lead to a noise at the receiver that is the thermal noise at the receiver these two are independent.

Hence, naturally the channel matrix H and noise V are independent. So we have channel matrix H comma noise V are these are independent, because the channel matrix this arises out of your this arise out of your signal propagation environment. For instance what is this depend on, this basically depends on a the signal propagation depend environment depends on the number of obstruction which is reline of side path or it is a no line of side path, the buildings, the trees which are also called as the scattering, so it depends upon the signal propagation environment; the distance and the a the varies the locations of these scatters and also the delays that introduced by the distances of these scatters which translated into phases etcetera.

And the noise V this arises due to the thermal noise which arises due to thermal effects at the receiver thermal noise at the receiver and these two are independent. Therefore, we have the channel matrix H and noise V are independent and therefore we can that since they are independent we have expected value of HV transpose if they are independent that is equal to expected value of h into expected value of V transpose and both of these are 0 means; so naturally this is equal to 0.

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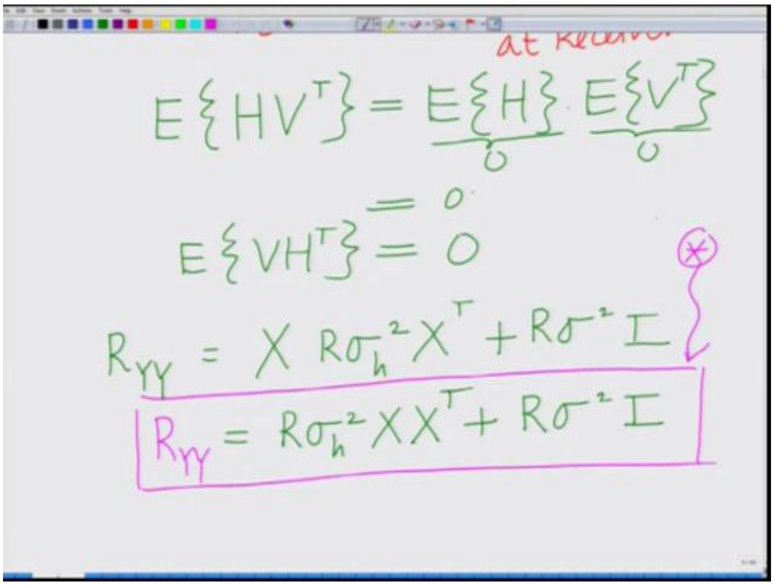

Similarly, expected value of V times H transpose this is also equal to 0, because the channel matrix and the noise are independent and therefore we have these two terms VH expected value of VH transpose expected value of HV transpose are 0. And therefore, what we have is that your covariance matrix R YY equals X R sigma h square X transpose plus R sigma square identity which is basically R sigma h square XX transpose plus R sigma square identity. This is the expression for your output covariance matrix R YY. You can call this as star because we will need this expression later R YY, the expression of R YY we are going to substitute it in the expression for the LMMSE estimate of H hat derive the LMMSE estimate. But before that we also need to compute the cross covariance R HY.

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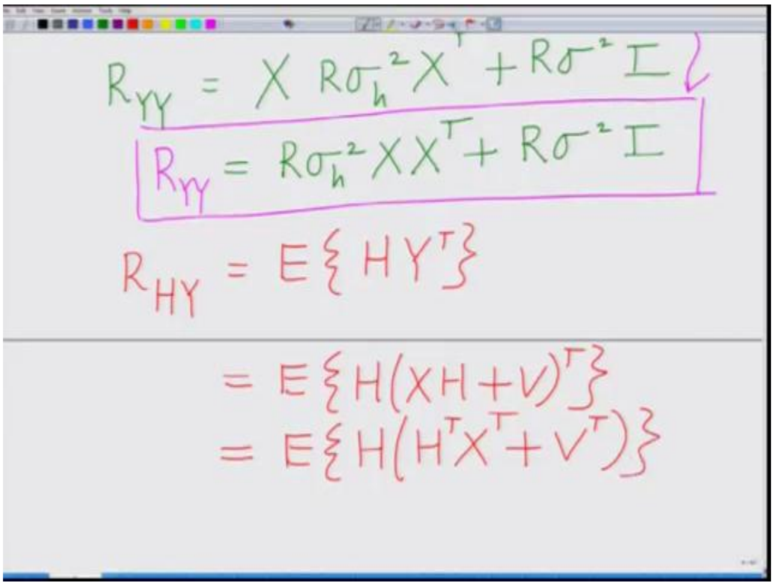

So R HY that is given as R HY, again this is similar to what we do for the multi antenna it is a downlink channel estimation scenario that is expected value of H into Y transpose which is expected value of H into XH plus V transpose expected value of H into H transpose X transpose plus V transposes.

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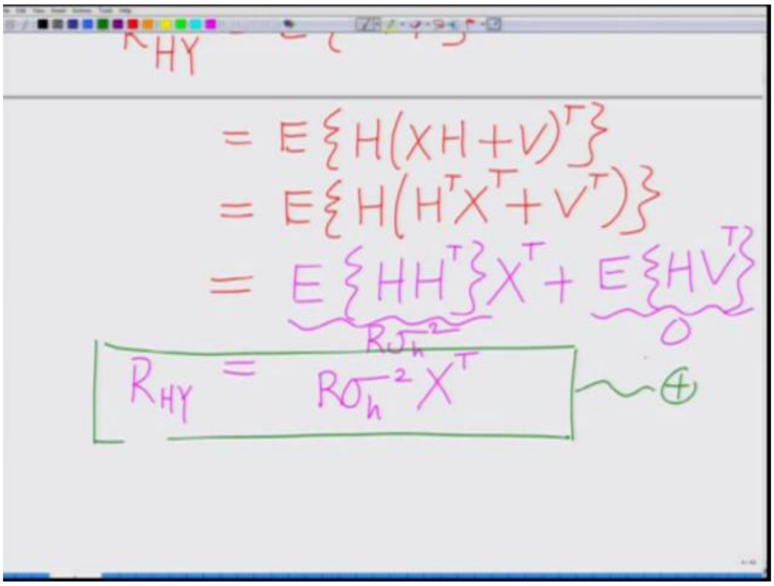

Which is expected value of well once again you can take X outside the expectation of the written HH transpose into X transpose plus expected value of HV transpose, of course expected value of HH transpose that we know is R sigma h square. So, this is R sigma h square X transpose expected value of HV transpose is 0, so this is your R HY. R HY is R sigma h square. This again you call basically your plus. Now, you can take R HY and this star R YY and substituted all the way back in this expression that is your H hat equal to R HY in to R YY inverse into Y.

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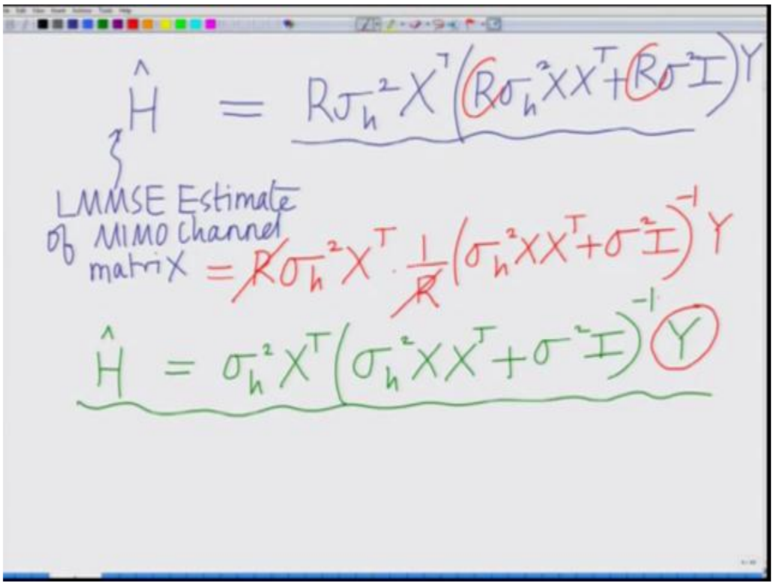

So, what we get now once you substitute these expressions is basically you have H hat the estimate of the LMMSE estimate of the MIMO channel matrix is R HY that is R sigma h square X transpose into R YY inverse that is R sigma h square XX transpose plus R sigma square identity inverse times your matrix times observation matrix Y, this is what we have. And now observe that that this R is a scalar quantity, so I can bring it outside of this inverse and when you bring it outside by this inverse it becomes 1 over R. So, this R and R outside will cancel and what you will have is sigma h square.

Let me just write one more step. So, this is R sigma h square X transpose type once you bring R outside this become 1 over R sigma h square XX transpose plus sigma square identity inverse into Y. And now we have the R and R cancel and what you have is sigma h square X transpose sigma h square XX transpose plus sigma square identity inverse into Y, and this is the expression for your LMMSE estimate of H hat.

And now observe that this LMMSE estimate of H hat is identical to that of the LMMSE estimate of the downlink multi antenna channel estimation scenario except that vector Y bar has now been replaced by the observation matrix Y. So, the pilot matrix X is the same and expression in terms of pilot matrix is the same, simply the observation vector Y bar has been replaced by the matrix Y.

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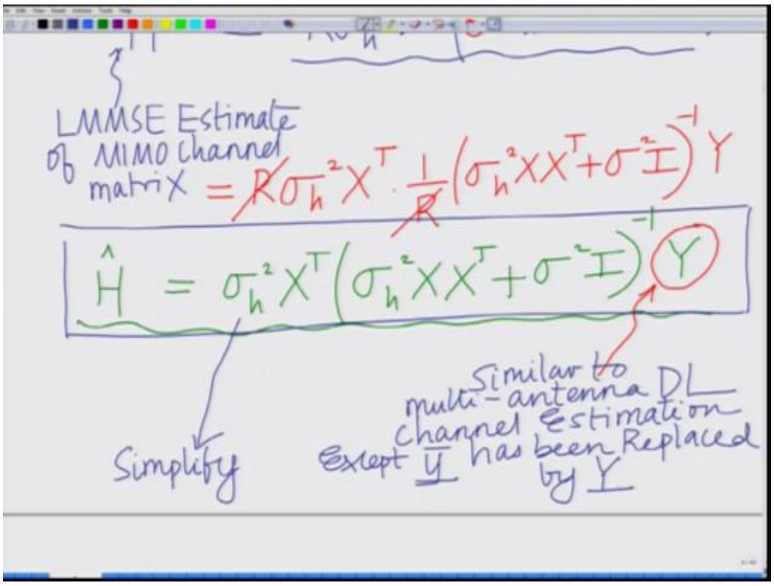

So, the net what you observe is that this is similar to your multi antenna downlink channel estimation except Y bar has been replaced by your Y observation vector Y bar has been replaced by Y. This is basically your expression for the LMMSE estimate. Of course, we can again simplify this expression.

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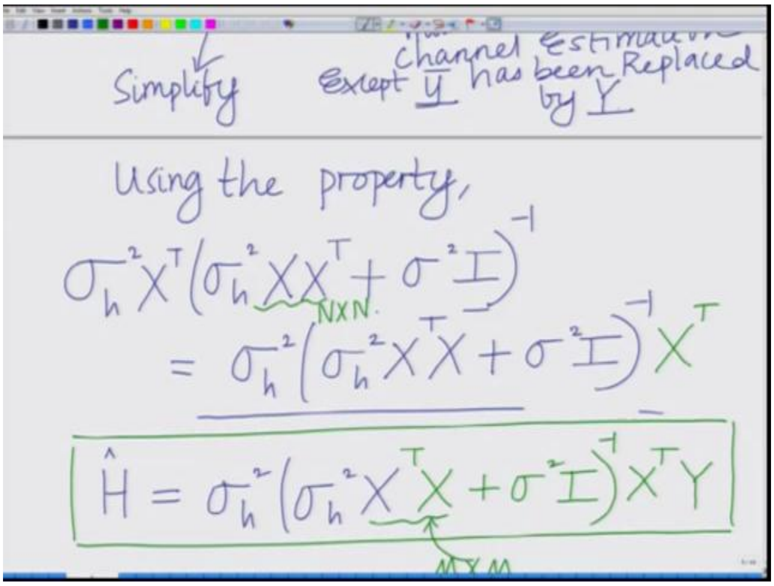

We can simplify this expression by again using the property that from your multi antenna downlink channel estimation we know sigma h square X transpose into sigma h square X transpose times sigma h square XX transpose plus sigma square identity inverse this is equal to well sigma h square X transpose X plus or sigma h square X transpose X rather sigma h square X transpose X plus sigma square identity inverse. And therefore employing this principle we have H hat equals I am just going to substitute instead of this sigma h square X transpose the first term I am going to substitute the second term and what I have is sigma h square into sigma h square X transpose x plus sigma square, I am sorry there as to be N X transpose here sigma x sigma square I inverse X transpose Y. Again basically if you see XX transpose X this is much simpler a this is much simpler to compute because X transpose X is a M cross M matrix, but x which is much smaller but XX transpose this is N cross N matrix.

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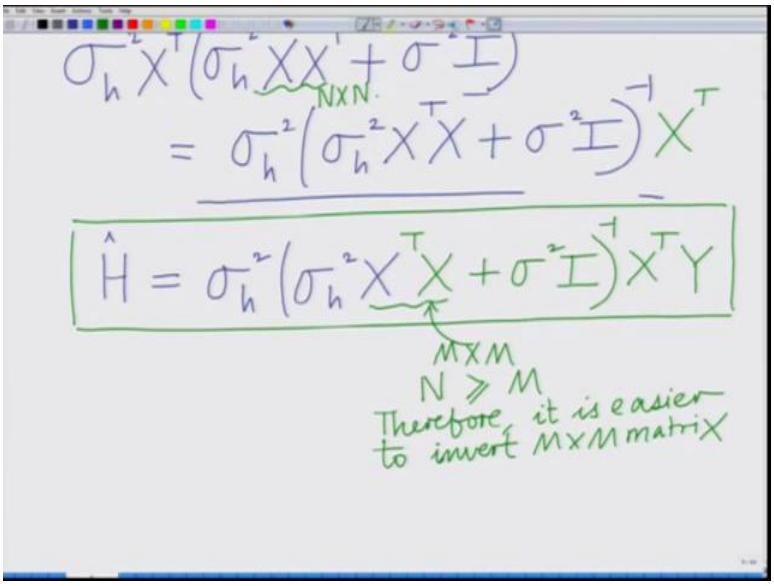

And therefore since typically N is much greater than M or let just say typically N is greater than or equal to M. Therefore, this it is easier to invert M cross M. The second estimate that is sigma h square times sigma h square X transpose x plus sigma square identity inverse into X transpose Y this is much more easier to compute than the first one.

So, this is basically the simplified expression for the LMMSE estimate of the MIMO channel matrix H, so we have extended this to basically MIMO channel estimation. Let us also now complete this by computing the error covariance.

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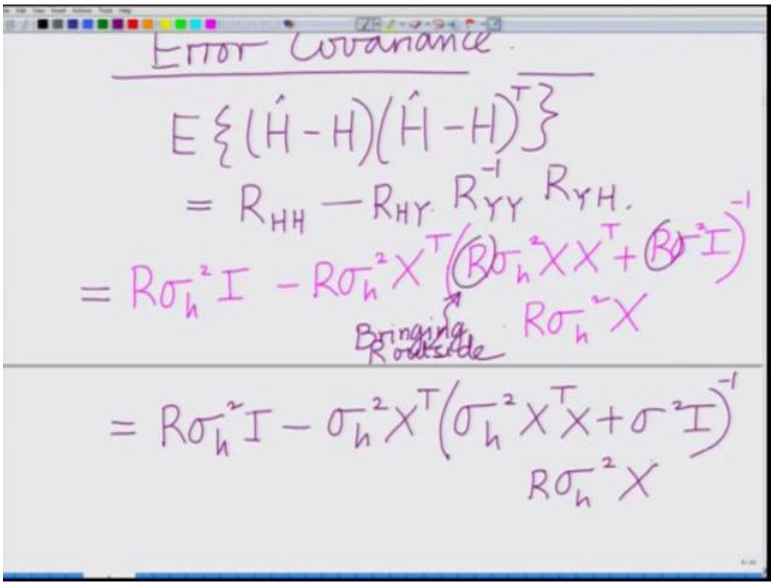

So, one can also compute the error covariance matrix. The error covariance that is the expected value of H hat minus H this is equal to H hat minus H into expected value of H hat minus H into H hat minus H transpose is equal to R HH minus R HY R YY inverse into R YH. I am going to substitute the different components that is we have R HH which is R sigma h square I minus R HY R is sigma h square X transpose R YY inverse R sigma h square XX transpose plus R sigma square identity inverse times R YH which is R sigma h square times X.

Now taking R outside again bringing R outside R becomes R inverse so this becomes R sigma h square I minus this R and R inverse this cancel so we have sigma h square X transpose sigma h square X transpose X plus sigma square identity inverse times R sigma h square X.

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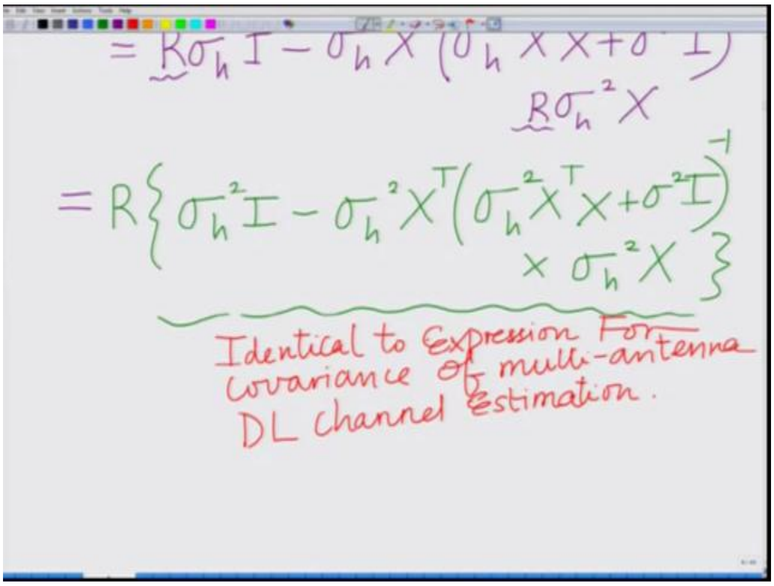

Now, let us take this R and R outside then I have R sigma h square I minus sigma h square X transpose sigma h square X transpose X plus sigma square identity inverse into sigma h square X. And now you can see this part that is sigma h square I minus sigma h square X transpose into sigma h square X transpose X plus sigma square I inverse into sigma h square X, now this is identical to multi antenna. This is identical you can verify this to expression for covariance of multi antenna downlink channel estimation.

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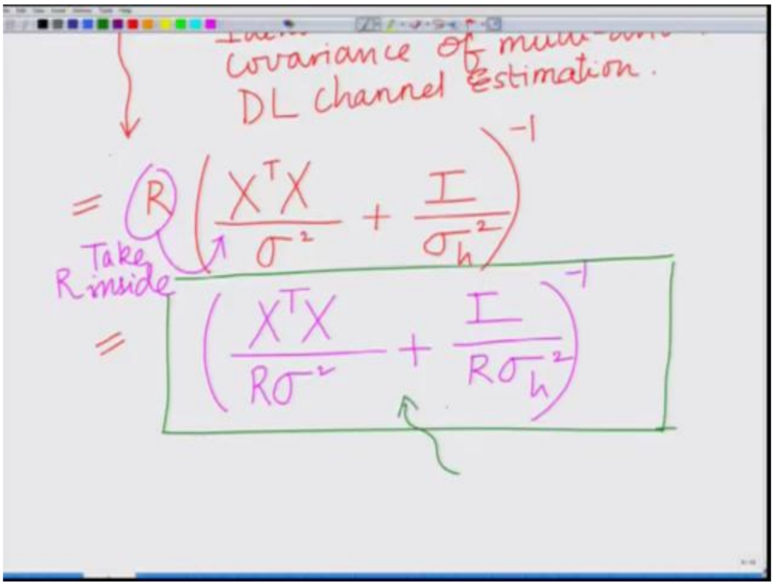

Therefore, this is equal to R times X transpose X divided by sigma square plus I divided by sigma h square inverse. And now take R inside, taking R inside R becomes 1 over R and what happens is this becomes X transpose X divided by sigma square. In fact, R sigma square I divided by R sigma h square inverse.

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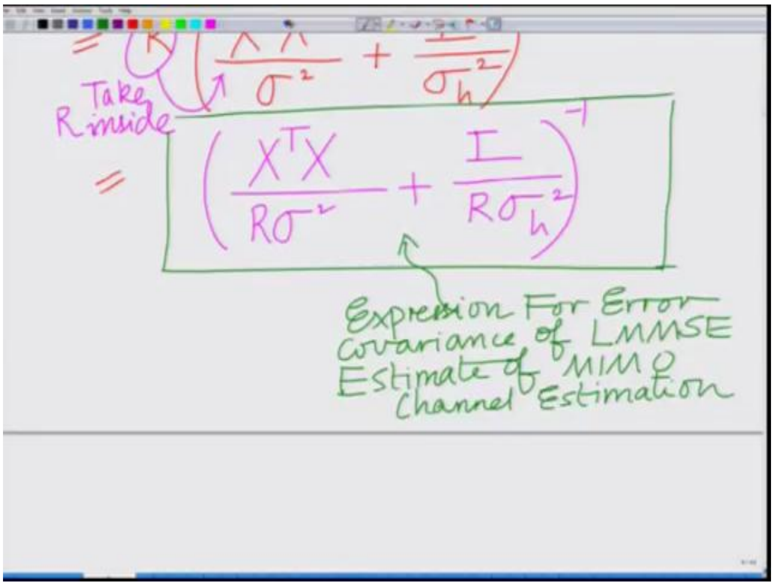

This is the expression for the error covariance of the LMMSE estimate of the MIMO channel estimation. That is what we have basically done in today's module is basically we completed this for the derivation of the LMMSE estimate of the MIMO channel estimation matrix H also derived the expression for the error covariance of the MIMO channel matrix X.

In the subsequent in the next module we are going to do simple example to illustrate how to apply this. So, that will basically complete a this portion of LMMSE estimate which is very important because MIMO channel estimation and MIMO channel or MIMO wireless communication is getting prominence is become is a big part of 3G and 4G wireless communication also moving forward in 5G wireless communication. So, MIMO channel estimation is of course one of the key components to enable MIMO wireless communication. Therefore this process of computing LMMSE estimate has the significant role to play in the context of wireless communication system implementation. So, we will stop here and continue in subsequent modules.

Thank you very much.