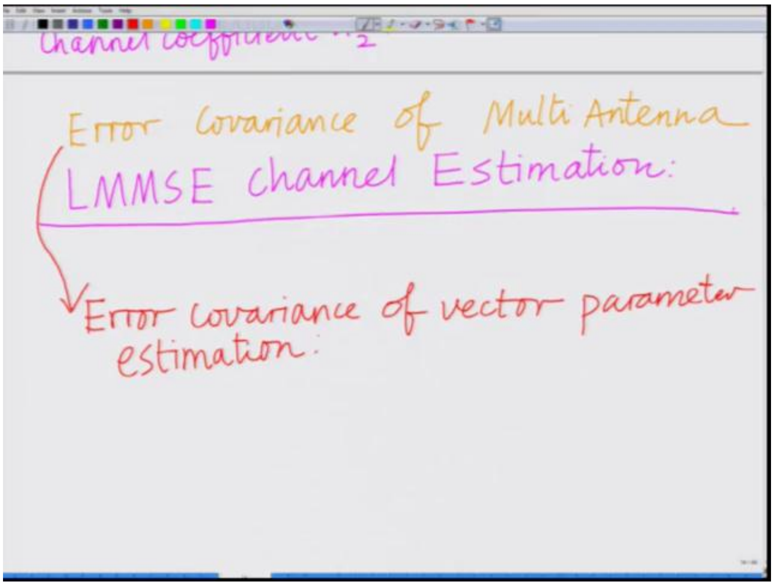
Bayesian/MMSE Estimation for MIMO/OFDM Wireless Communications
Prof. Aditya K. Jagannatham
Department of Electrical Engineering
Indian Institute of Technology, Kanpur

Lecture - 21
Error Covariance Derivation and Example for Linear Minimum Mean Squared Error (LMMSE) Estimation of Multi-Antenna Downlink Wireless Channel

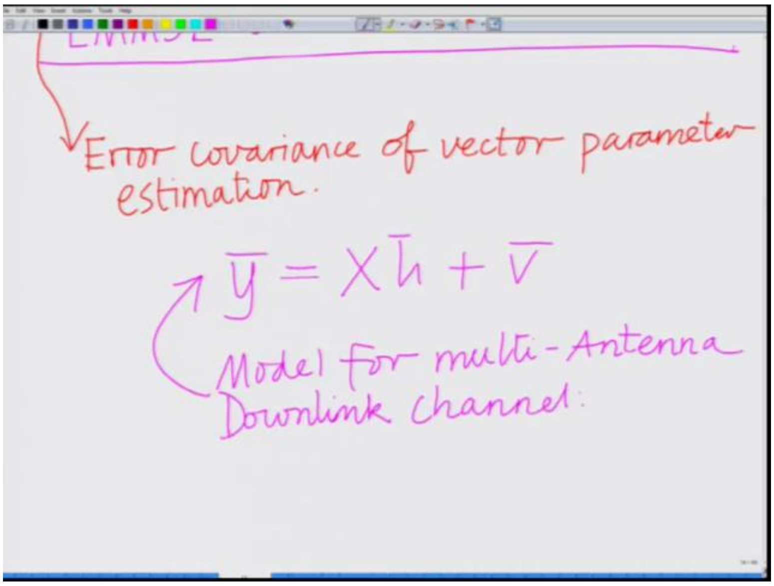
Hello. Welcome to another module, in this massive open online course, and Bayesian MMSE estimation for wireless communications. So, far, we have looked at estimation LMMSE or MMSE estimation, of a vector parameter, content, we illustrated this in the context of the, multichannel, eh, multi antenna, downlink wireless channel estimation. And we have also shown an example, to illustrate its application. So, now, what we are going to do is, we are going to derive the error covariance for the estimation of vector parameter, again illustrating within the context of, multilateral downlink channel estimation.

(Refer Slide Time: 00:51)



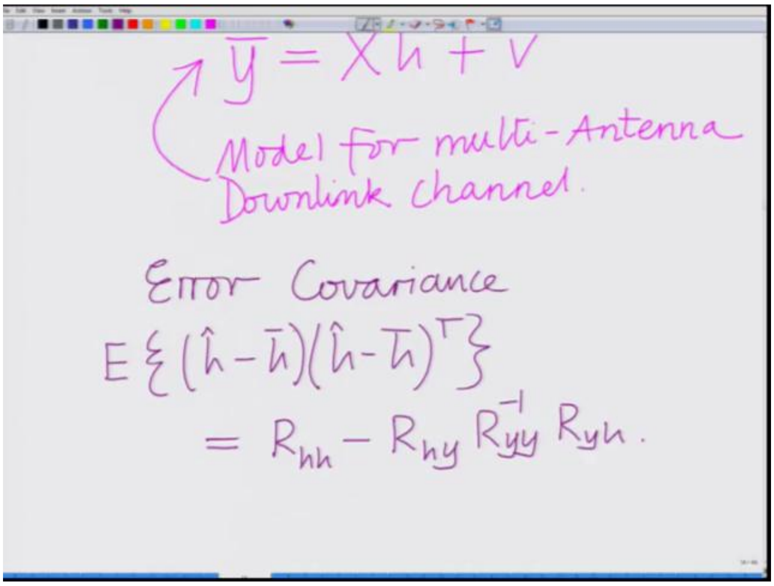
So, in this module, we are going to derive the error covariance, error covariance of multi antenna LMMSE channel estimation, or we can also think of this basically as, we can also think of this as, error covariance for vector parameter estimation - Error covariance of vector parameter estimation.

(Refer Slide Time: 02:08)



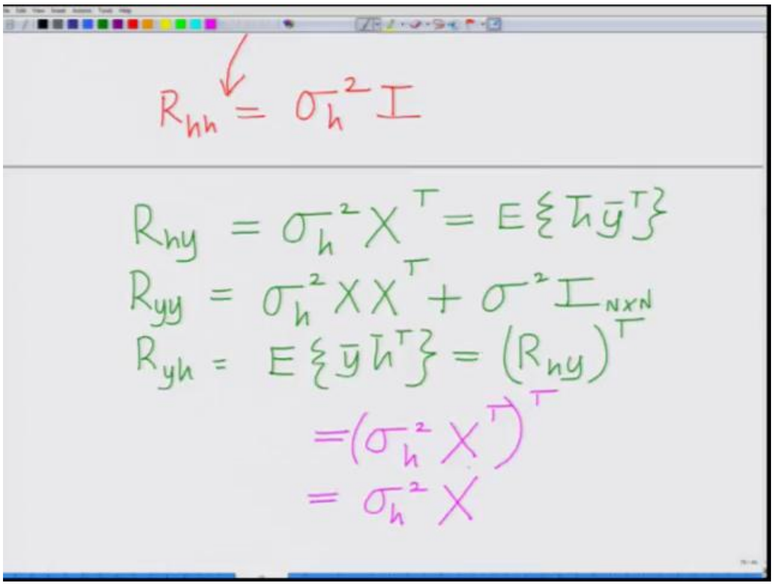
When we know that the system model, for multi antenna, for this multi antenna downlink channel, is given as y equals X h bar, plus v bar. This is the model for the multi antenna downlink channel. This is our model for the multi antenna downlink channel, and the error covariance explanation for the error covariance, this is given as, error covariance that is your expected value of h bar, minus h hat.

(Refer Slide Time: 02:46)



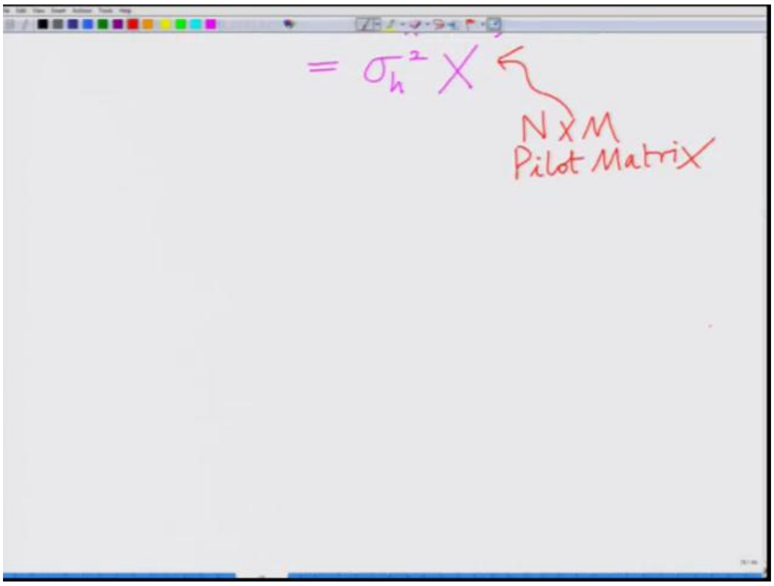
So, this is your expected value of h hat, that is, error vector h hat, minus the vector h bar, times h hat, minus h bar transpose, and this can be obtained as, this is given as again, Rhh minus Rhy into Ryy inverse in to Ry. This is the expression, for the error covariance, of course, for a scalar parameter, we is, previously we had look at only a scalar parameter, for a scalar parameter this will be the error variance, simply the error variance. Now we are looking at a vector parameter. So, we have to consider the covariance matrix, of the estimation error. And that is what we are going to now find, what is the covariance matrix of the estimation error, that is expected value of h hat, minus h bar, times h hat minus h bar, transpose.

(Refer Slide Time: 04:00)



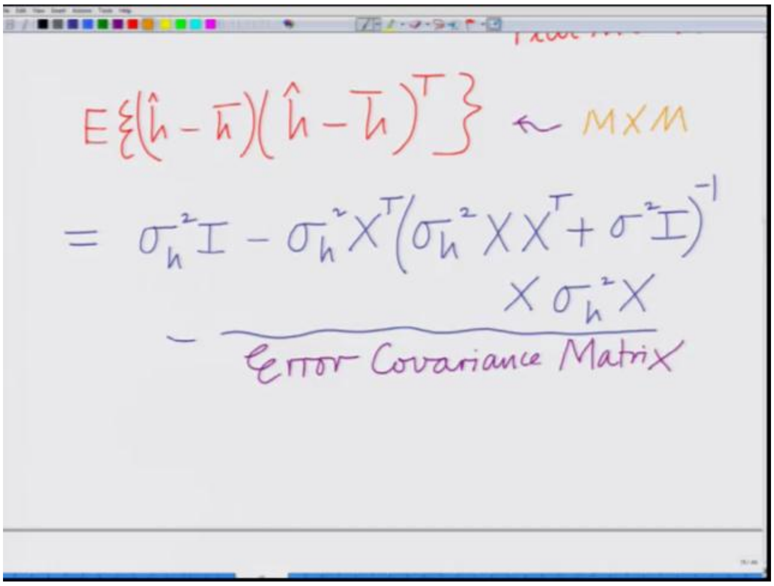
And now we know also we know each of these quantities. We know that Rhh, this Rhh, equal sigma h square, times identity, and we also know that, we also know that, Rhy equals, sigma h square, times, X transpose, Ryy equals, sigma h square, times XX transpose, plus sigma square times identity. And Ryy, where is identity, remember this is in n cross n, because X is n cross n, n cross n matrix. r , eh, we have look at now at Ryh, that is of course, expected value of, remember Rhy is expected value of, h bar into y bar transpose, Ryh is a different. This is the expected value of y bar, h bar transpose, which is basically transpose Ryh is transpose of Rhy.

(Refer Slide Time: 05:40)



So, this is basically your Rhy transpose, and indeed this is equal to sigma h square, X transpose, transpose of course, sigma X square is a scalar. So, this is simply sigma X square, X transpose - transpose is x, where X is, remember this is your pilot matrix. So, X is your n cross m, this is the n cross m, pilot matrix, now, therefore, substituting these quantities in this expression.

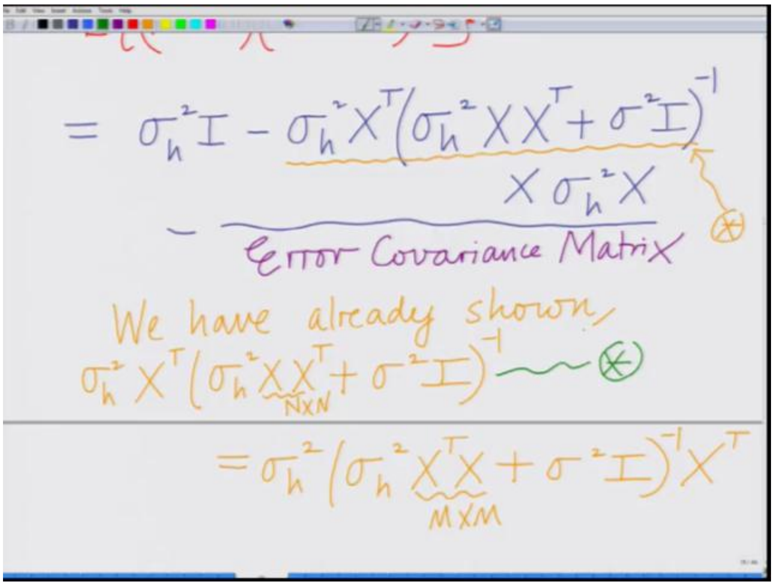
(Refer Slide Time: 06:11)



Substituting in this expression over here, what we have is that basically, your error covariance. Expected value of h hat, minus h bar, into h hat, minus h bar transpose, this is equal to, this is equal to, well, Rhh, which is sigma square, sigma h square identity minus, Rhy, which is sigma h square X transpose, Ryy inverse, that is sigma h square, XX transpose plus, sigma square identity, this is n crossed in identity matrix, times, Ryh which is sigma h square, times, simply the matrix x.

So, that is expression that is expression that we have and now, we are going to simplify. So, this is the expression for the error covariance, correct? This is the expression, initial expression, for the error covariance, rather error covariance matrix, and remember this error covariance matrix, this is of size, because h is of size, is an m dimensional parameter vector, naturally, this will be of size m cross m, this is an m cross m error covariance matrix. And what we are going to do now, is we are going to simplify this expression for the error covariance matrix, get into a nice compact form, a, basically which gives us, what is the expression for the error covariance.

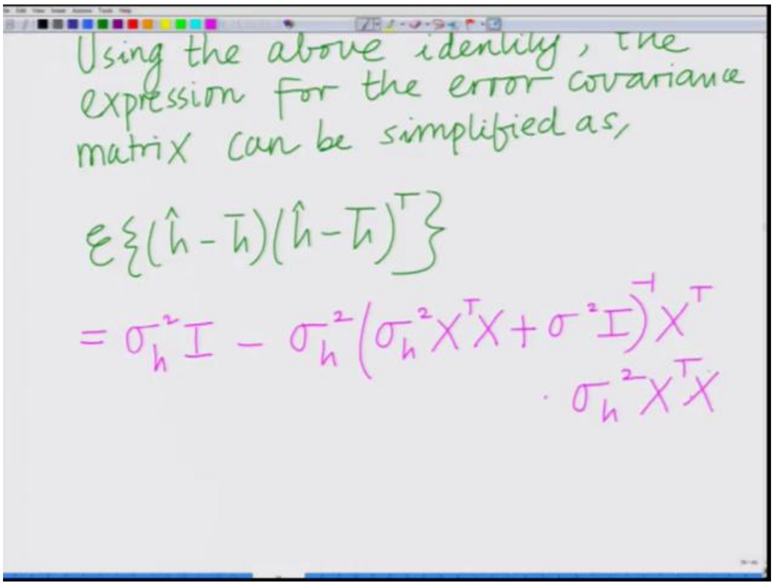
(Refer Slide Time: 08:03)



So, have to again simplify this, we will use the principle, or there is a result that we have did earlier. So, observe, we have already shown, or we have already shown, in the previous module, or in one of the previous module, during the simplification of the LMMSE estimate. We had shown this property, that is sigma h square, X transpose, times, sigma h square, XX transpose, plus sigma square identity, inverse, is equal to, sigma h square, sigma h square X transpose x, plus sigma square, identity inverse, times, X transpose, and remember XX transpose is n cross n matrix, X transpose n, X transpose

X is in m cross m, this is the result that we have already shown. And now if you look at this, this part, sigma h square X transpose sigma h square XX transpose sigma square identity, if you call this as star. If you call this as star, this is exactly your star.

(Refer Slide Time: 09:38)

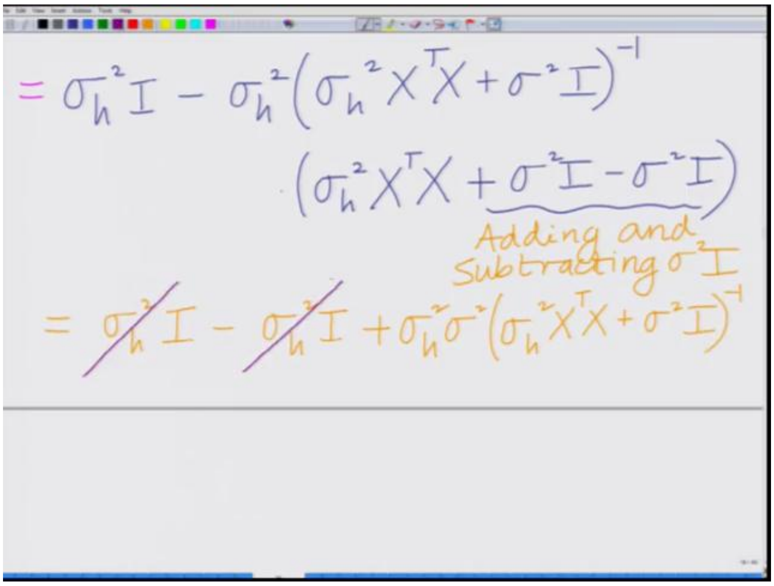


So, I can replace this quantity, this star, by sigma h square, by sigma h square, X transpose x, plus, sigma square identity, inverse, times, X transpose, therefore, now the error covariance using the above principle, or using the above identity, using the above identity, the expression for the error covariance matrix can be simplified as, the expression for the error covariance matrix, this can be simplified as, well, I have expected value of h hat, minus h bar, into h hat, minus h bar transpose, this is equal to sigma h square, identity, minus.

Now, instead of sigma h square, X transpose, sigma h square, XX transpose sigma X square identity inverse, I am going to employ, sigma h square, X transpose X plus, of course, this is sigma h square over here, all right, we just write it clearly, this is going to be, sigma h square, X transpose X plus, sigma square identity inverse, into X transpose times, sigma, a times we are going to have, well times we are going to have, sigma h square, X transpose of x, what we are doing is, basically this sigma h square. So, basically what we are doing, we are replacing this sigma h square h transpose, sigma X square, XX transpose, the sigma square identity, inverse, by this quantity, and there is another sigma h square, outside, which will basically continue to be over here.

Now, to this X transpose x, what I am going to do, is I am going to add and subtract sigma square identity, of this X transpose x, and that is have I am going to simplify.

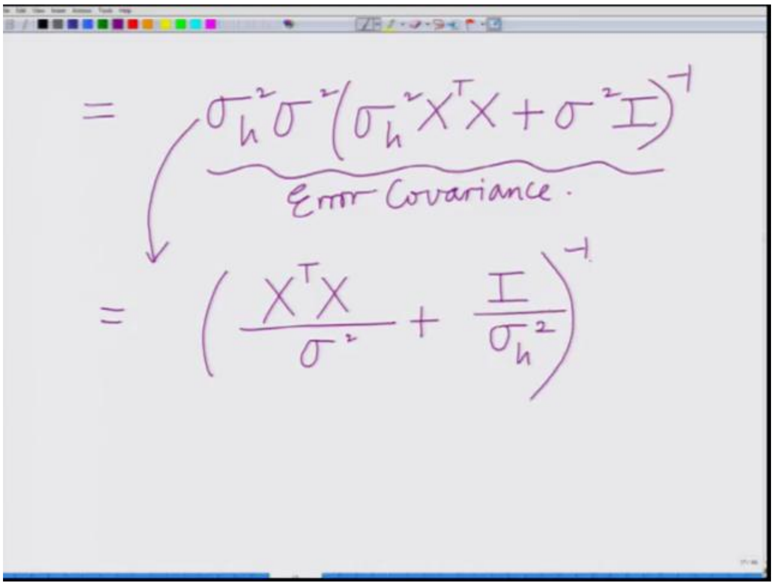
(Refer Slide Time: 12:17)



So, adding and subtracting sigma square identity, now this is a slightly tricky step. So, please pay attention. So, this will be sigma h square identity, minus sigma h square, into sigma h square X transpose x, plus, sigma square identity, inverse, sorry, this is only going to be, and now into these X transpose x, times, I am going to have, sigma h square, X transpose x, plus sigma square identity, minus sigma square identity, adding and subtracting sigma square. So, this is basically your adding, adding and subtracting sigma square identity.

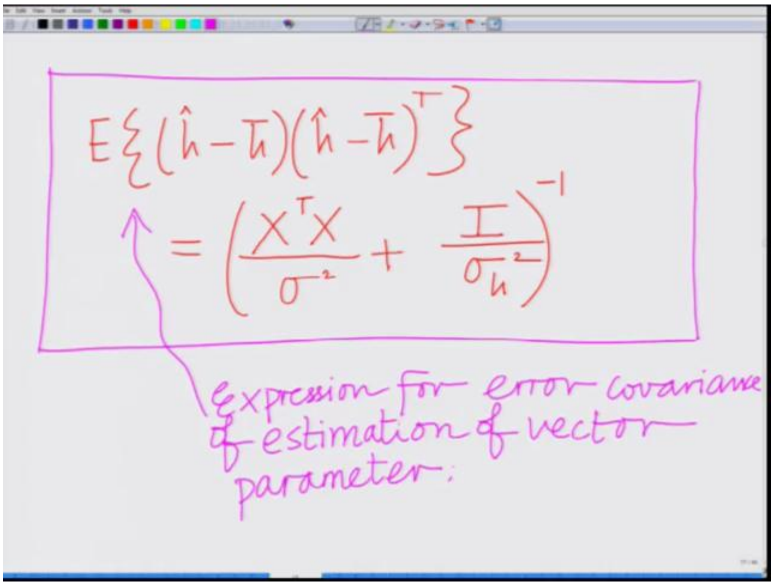
Now, we can see, the sigma square h square, X transpose, sigma square identity, into this first part, sigma h square, X transpose, sigma square identity, these 2 are inverse of each other. So, multiplication of these 2 will give identity. So, so we will have sigma h square, I. So, what we will have is, net, sigma h square I, minus of course, sigma h square, times, these 2 will give identity. So, you are left with sigma h square, times, identity, plus, what we have is sigma h square, sigma h square, X transpose x, plus, sigma square identity, into sigma square identity. So, what we will have net is, sigma h square, sigma square, into sigma h square, X transpose x, plus, sigma square identity, inverse, and that is what we are left. And now you can see, these 2 quantities, sigma h square identity, minus, sigma h square, identity, these 2 cancel, and what we have left with is basically the error covariance, is sigma h square, sigma square, sigma h square, X transpose x, plus sigma square, identity, inverse, this is the expression for the error covariance.

(Refer Slide Time: 14:34)



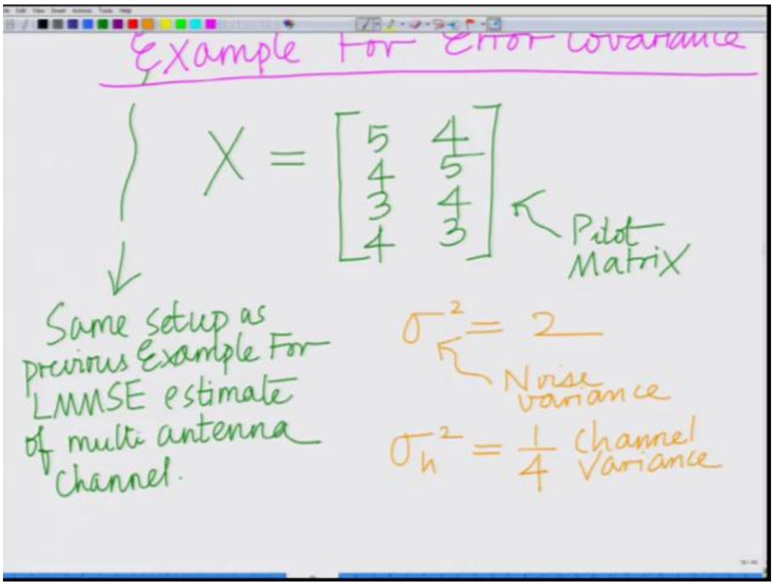
So, what we have is X sigma square, into sigma square, times, sigma h square X transpose x, plus, sigma square identity, inverse. Now what we are going to do, we are going to take the sigma, the sigma is the h square, into sigma square, this is a scalar factor. So, I can take it inside the matrix inverse. if I take it inside the matrix inverse, taking this sigma h square, sigma square, inside the matrix inverse, I get the final expression, that is X transpose x, divided by sigma square, plus, identity, divided by sigma h square, inverse.

(Refer Slide Time: 15:44)



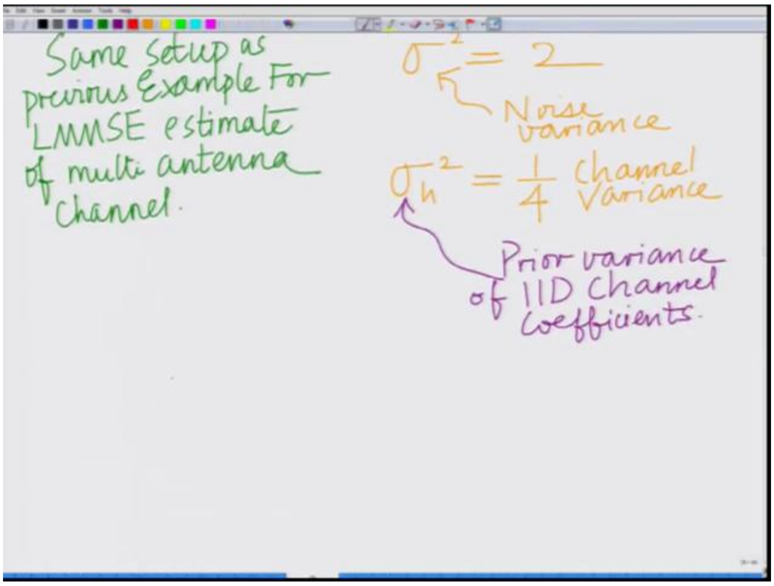
So, this is your final expression, for the error covariance. So, error covariance, expected value of h hat, minus h bar, into h hat, minus h bar transpose, equals, X transpose x, by sigma square, plus, I, divided by sigma h square, inverse. This is the expression, and this is the error covariance, for your vector parameter models. So, this is the covariance. Of course, we are done this is in the context of, your, downlink multi antenna channel estimation, but realize that, this is the general expression, for an error covariance, for the estimation of a vector parameter. So, this is the expression for error covariance of estimation, of a vector parameter, that is X transpose x, divided by sigma square, plus, I, divided by sigma h square, whole inverse. This is the error covariance for the estimation of a vector parameter.

(Refer Slide Time: 17:37)



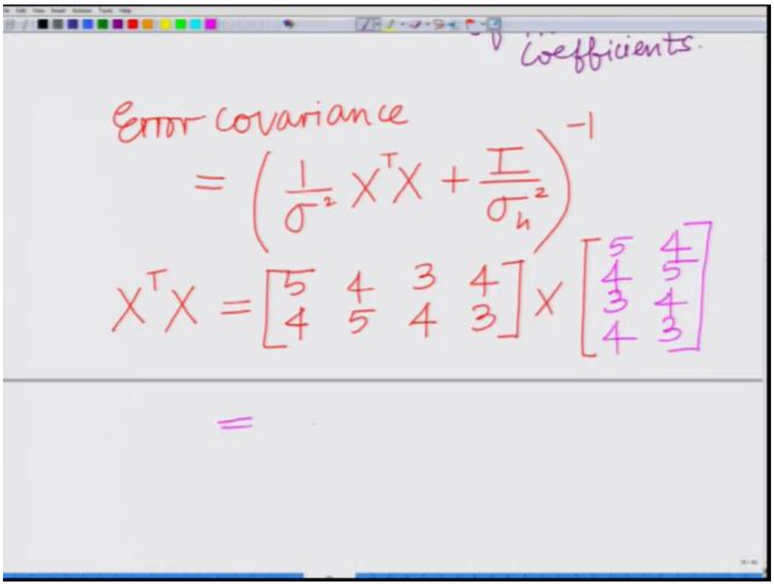
So, now we have this error covariance. So, now, let us do an example to understand this better, let us do an example to understand this better. So, let us do an example. So, let us do an example, to compute this, to compute, this error covariance for the vector parameter, and what we consider is, let us consider the same parameters, or the same values, or the same set up, as the previous example, that we have done for the LMMSE, LMMSE estimate, of the vector parameter. This is a multi antenna downlink channel. So, we are considering the pilot matrix 5 4, 4 5, 3 4, 4 3, basically this is the same, same set up as previous example for LMMSE estimate, of your LMMSE estimate, of the multi antenna channels.

(Refer Slide Time: 19:51)



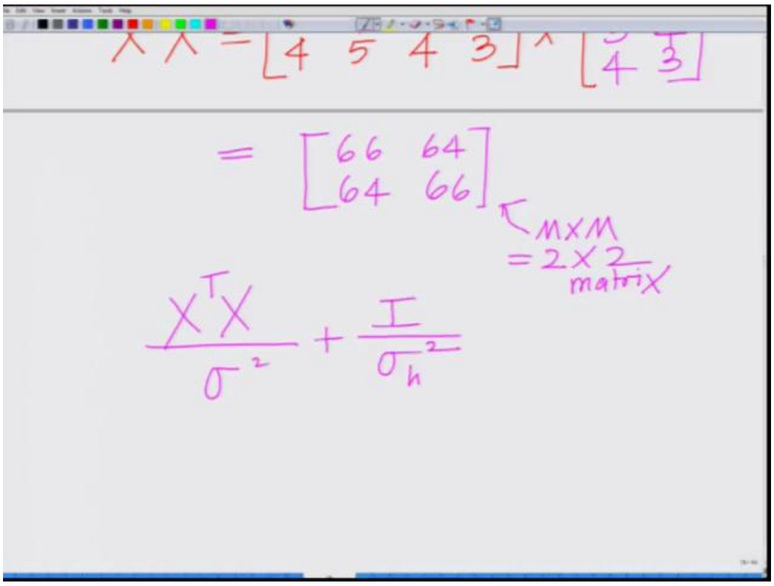
So, the pilot matrix, this is your pilot matrix, this is the pilot matrix. We consider sigma square equals, 2, this is your noise variance, this is your noise variance, sigma h square, equals 1 by 4, this is the, this is the channel variance, correct? Sigma square equals 1 by 4, this is the channel variance, this is the channel variance, or the prior variance of the channel, rather the prior variance, the prior variance of your IID, prior variance of your IID channel coefficients therefore, what we get is that, your error covariance, this is equal to, as we had just derived, that is, 1 over sigma square, X transpose x, plus I over, sigma h square, inverse. That is the expression, that we have we have computed, X transpose x, in the previous example, where X is the pilot matrix.

(Refer Slide Time: 20:18)



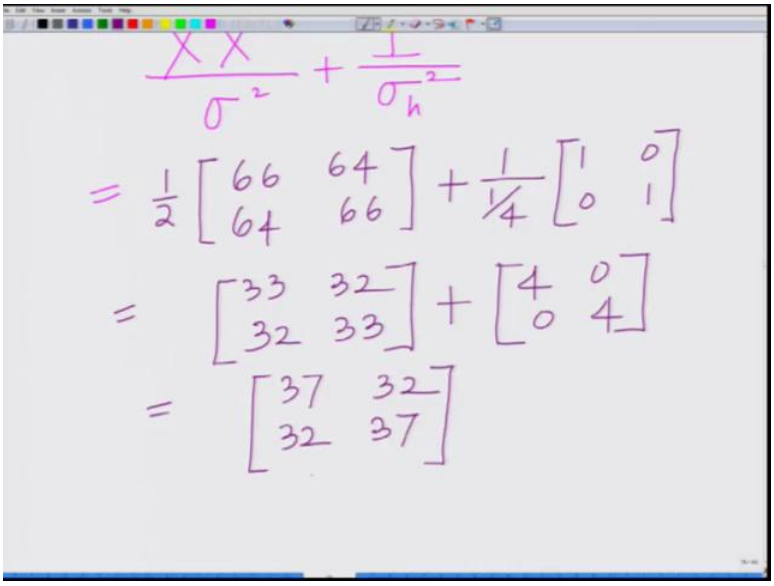
So, I am going to reuse those values. So, X transpose x, which is the transpose of the matrix x, times, itself. Remember this is m cross m matrix, it is a 2 cross 2 matrix, which is basically your, 5 4, 4 5, 3 4, 4 3, times, X, which is the pilot matrix.

(Refer Slide Time: 21:26)



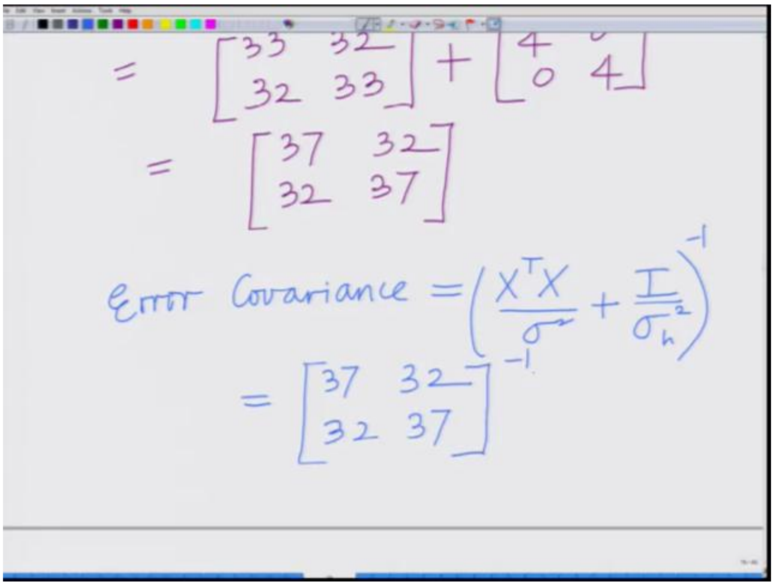
Once again, 5 4, 4 5, 3 4, 4 3, which is equal to, which is equal to, 66, 64, this is the matrix, that we have already computed 66, 64, 64, 66, this is m cross m, equals, basically your 2 cross 2 matrix, and X transpose x, divided by sigma square, plus I, divided by sigma h square, this is half, where sigma square is 2.

(Refer Slide Time: 22:11)

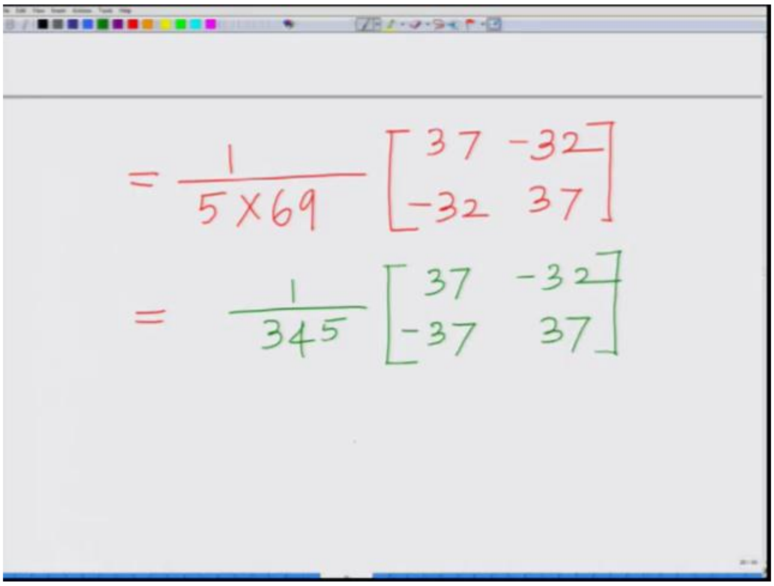


So, this is 1 of sigma square is half, 66, 64, 64, 66, plus 1, over sigma h square, is 1 by 4. So, 1 over 1 by 4, 2 cross 2 identity matrix, 1 0 0 1. So, this will be basically your 33, 32, 32, 33, plus 4 times the identity matrix, that is 4 0 0 4 this will be, 37, 32, 32, 37, and for therefore, the error covariance equals now, equals well, X transpose x, by sigma square plus, I, divided by sigma h square, inverse, which is basically inverse of this matrix, your 37,32, 32, 37, inverse, this is equal to, well there are 2 cross 2 matrix, so, inverse is easy to compute 1 over the determine that 37 square minus 32 square, yesterday we saw, that is 37 minus 32, times 37 plus 32, that is, a 69 times, swapped off diagonal elements, both them are same.

(Refer Slide Time: 23:08)

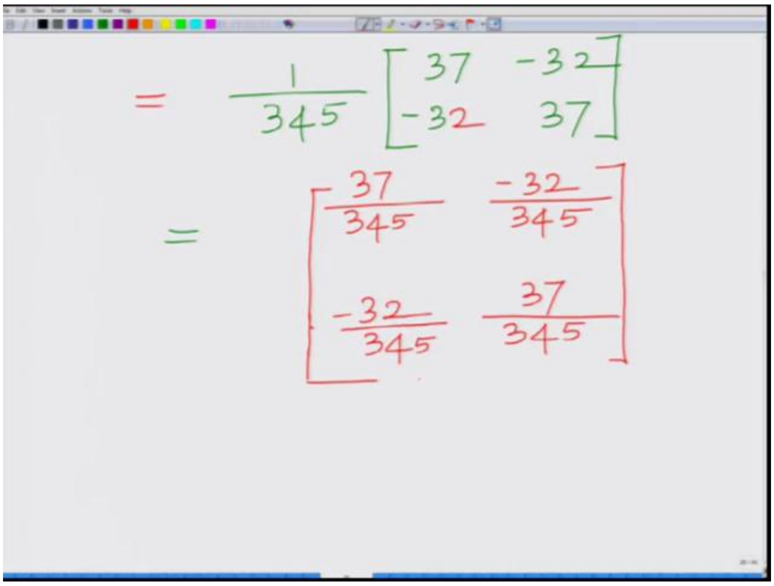


(Refer Slide Time: 23:42)



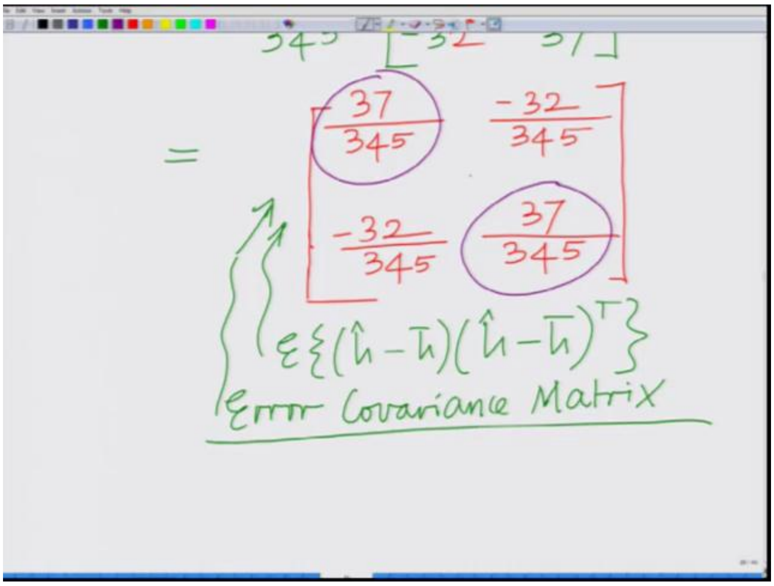
So, that will be no change, negative, swap of the diagonal elements, negative of the off diagonal elements, minus 32 minus 32.

(Refer Slide Time: 24:31)



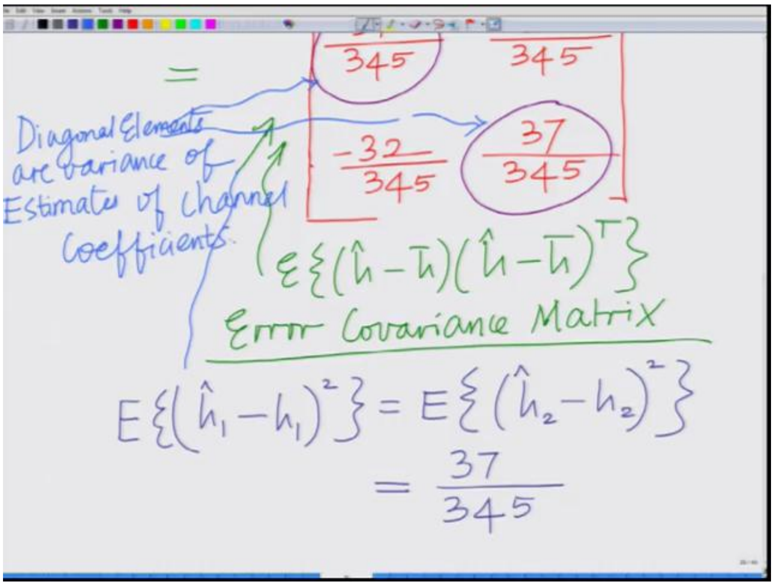
The diagonal elements are 37, 37. So, this will be now, simplifying this, this will be 1 over 345 times, well 37, minus 32, minus 32, 37 and therefore, finally, bringing this thing inside, that we can write this, as your 37 by 345, minus 32 by 345, minus of course, this as to be minus 32, minus 32, by 345, 37 by 345, and this is basically the expression for, this is basically the expression for the error covariance matrix, expected value of h hat, minus h bar, into h hat, minus h bar transpose.

(Refer Slide Time: 25:05)



So, this is the error covariance, this is your, this is the expression for the error covariance matrix. And of course, now if we look at these diagonal elements, diagonal elements, these diagonal elements are the MMSE, of the individual channel coefficients, right? This is the error covariance, we considered, considered basically, has both estimate of the variances, and also the, the co-area, the cross correlation, the cross co-relation between the errors of the estimates of channel coefficients.

(Refer Slide Time: 26:15)



So, now what you see is, it the diagonal elements, are basically the variances, in the estimates of the channel coefficients, h one h 2, which means, basically, your expected value of h 1 hat, minus h 1 square, this is equal to expected value of estimate of h 2 hat, minus h 2 square, and this is equal to, this is basically equal to 37, divided by 345, all right? The diagonal elements are the variances in the estimates of the channel coefficients.

Let me write that down, just write that down, for the sense of completion. These diagonal elements are basically variances, of your estimates, are the variances of the estimates, variances of the estimates, of your channel, and variances of the estimates of the channel coefficients. So, the diagonal estimates are the covariance. So, basically what we have done in this module is, we have derived the expression, for the error covariance matrix, of the estimate of a vector parameter, the vector parameter is h bar, which is basically the channel vector, corresponding to the multi antenna downlink channel.

Previously we have looked at the LMMSE estimate, now we have also derived the error covariance expression, for the error covariance, and we are simplified the expression for the error covariance, to do an elegant closed form expression, and we have also, with the aid of an example, demonstrated how to compute this error covariance, and from the error covariance, how to extract the information about the, variances of estimates of the individual coefficients, of the individual components, the estimate of the individual components, of the vector parameter.

So, we will stop this module here, and continue with other aspects in the subsequent modules.

Thank you very much.