

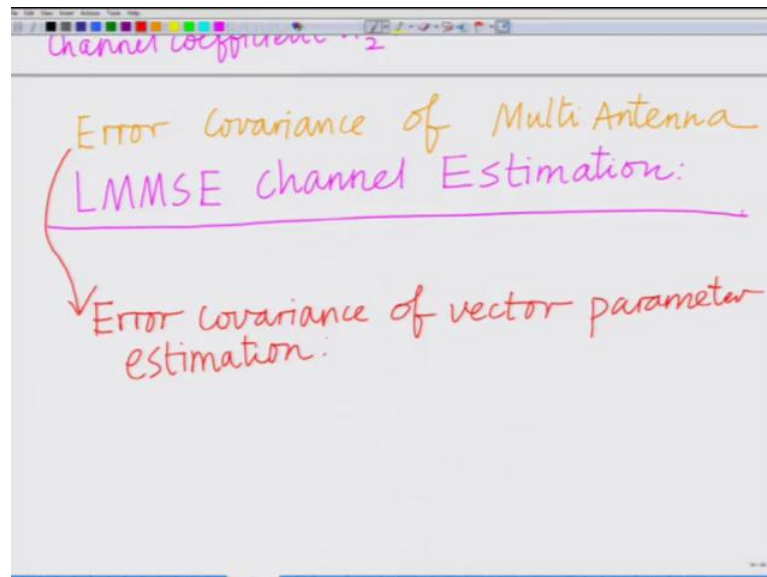
Bayesian/MMSE Estimation for MIMO/OFDM Wireless Communications
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Lecture - 21

Error Covariance Derivation and Example for Linear Minimum Mean Squared Error (LMMSE) Estimation of Multi-Antenna Downlink Wireless Channel

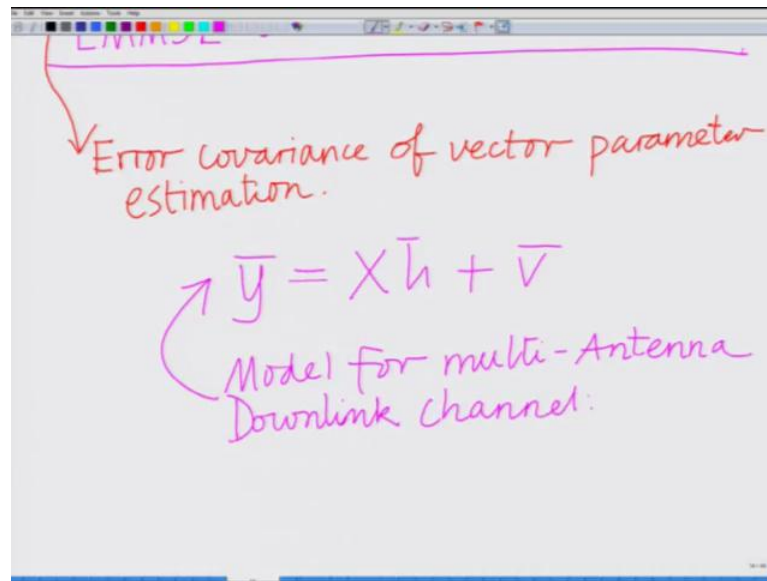
Hello. Welcome to another module, in this massive open online course, and Bayesian MMSE estimation for wireless communications. So, far, we have looked at estimation LMMSE or MMSE estimation, of a vector parameter, content, we illustrated this in the context of the, multichannel, eh, multi antenna, downlink wireless channel estimation. And we have also shown an example, to illustrate its application. So, now, what we are going to do is, we are going to derive the error covariance for the estimation of vector parameter, again illustrating within the context of, multilateral downlink channel estimation.

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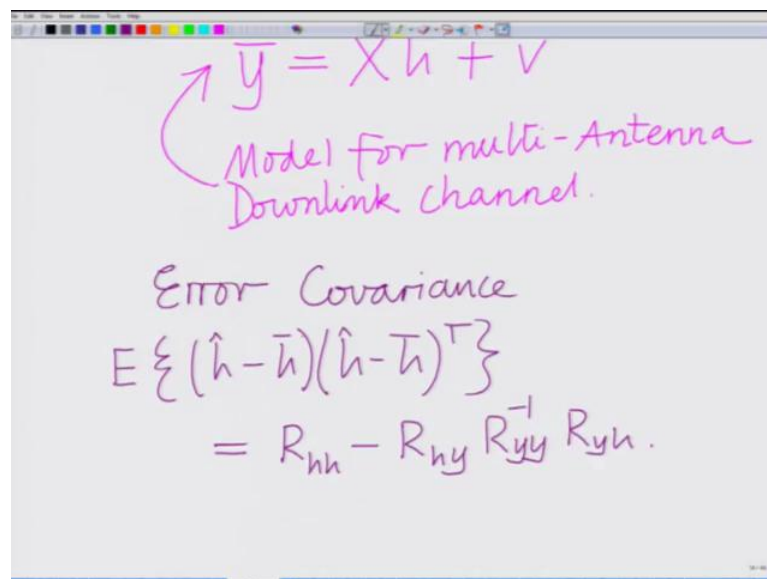
So, in this module, we are going to derive the error covariance, error covariance of multi antenna LMMSE channel estimation, or we can also think of this basically as, we can also think of this as, error covariance for vector parameter estimation - Error covariance of vector parameter estimation.

(Refer Slide Time: 02:08)



When we know that the system model, for multi antenna, for this multi antenna downlink channel, is given as y equals $X h$ bar, plus v bar. This is the model for the multi antenna downlink channel. This is our model for the multi antenna downlink channel, and the error covariance explanation for the error covariance, this is given as, error covariance that is your expected value of h bar, minus h hat.

(Refer Slide Time: 02:46)



So, this is your expected value of h hat, that is, error vector h hat, minus the vector h bar, times h hat, minus h bar transpose, and this can be obtained as, this is given as again,

R_{hh} minus R_{hy} into R_{yy} inverse in to R_y . This is the expression, for the error covariance, of course, for a scalar parameter, we is, previously we had look at only a scalar parameter, for a scalar parameter this will be the error variance, simply the error variance. Now we are looking at a vector parameter. So, we have to consider the covariance matrix, of the estimation error. And that is what we are going to now find, what is the covariance matrix of the estimation error, that is expected value of \hat{h} , minus \bar{h} , times \hat{h} minus \bar{h} , transpose.

(Refer Slide Time: 04:00)

$$R_{hh} = \sigma_h^2 I$$

$$R_{hy} = \sigma_h^2 X^T = E\{\bar{h} \bar{y}^T\}$$

$$R_{yy} = \sigma_h^2 X X^T + \sigma^2 I_{n \times n}$$

$$R_{yh} = E\{\bar{y} \bar{h}^T\} = (R_{hy})^T$$

$$= (\sigma_h^2 X^T)^T$$

$$= \sigma_h^2 X$$

And now we know also we know each of these quantities. We know that R_{hh} , this R_{hh} , equal σ_h^2 , times identity, and we also know that, we also know that, R_{hy} equals, σ_h^2 , times, X transpose, R_{yy} equals, σ_h^2 , times XX transpose, plus σ^2 times identity. And R_{yy} , where is identity, remember this is in n cross n , because X is n cross n , n cross n matrix. r , eh, we have look at now at R_{yh} , that is of course, expected value of, remember R_{hy} is expected value of, \bar{h} into \bar{y} transpose, R_{yh} is a different. This is the expected value of \bar{y} , \bar{h} transpose, which is basically transpose R_{yh} is transpose of R_{hy} .

(Refer Slide Time: 05:40)

$$= \sigma_h^2 X^T$$

$N \times M$
Pilot Matrix

So, this is basically your Rhy transpose, and indeed this is equal to sigma h square, X transpose, transpose of course, sigma X square is a scalar. So, this is simply sigma X square, X transpose - transpose is x, where X is, remember this is your pilot matrix. So, X is your n cross m, this is the n cross m, pilot matrix, now, therefore, substituting these quantities in this expression.

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$$E\{(\hat{h} - \bar{h})(\hat{h} - \bar{h})^T\} \leftarrow M \times M$$
$$= \sigma_h^2 I - \sigma_h^2 X^T (\sigma_h^2 X X^T + \sigma^2 I)^{-1} X \sigma_h^2 X$$

Error Covariance Matrix

Substituting in this expression over here, what we have is that basically, your error covariance. Expected value of h hat, minus h bar, into h hat, minus h bar transpose, this

is equal to, this is equal to, well, R_{hh} , which is σ_h^2 , σ_h^2 identity minus, R_{hy} , which is $\sigma_h^2 X^T$, R_{yy} inverse, that is σ_h^2 , XX^T plus, σ^2 identity, this is n crossed in identity matrix, times, R_{yh} which is σ_h^2 , times, simply the matrix x .

So, that is expression that is expression that we have and now, we are going to simplify. So, this is the expression for the error covariance, correct? This is the expression, initial expression, for the error covariance, rather error covariance matrix, and remember this error covariance matrix, this is of size, because h is of size, is an m dimensional parameter vector, naturally, this will be of size m cross m , this is an m cross m error covariance matrix. And what we are going to do now, is we are going to simplify this expression for the error covariance matrix, get into a nice compact form, a, basically which gives us, what is the expression for the error covariance.

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The image shows a whiteboard with handwritten mathematical derivations. At the top, the expression for the error covariance matrix is given as:

$$= \sigma_h^2 \mathbf{I} - \sigma_h^2 X^T (\sigma_h^2 X X^T + \sigma^2 \mathbf{I})^{-1} X \sigma_h^2 X$$

The term $\sigma_h^2 X^T (\sigma_h^2 X X^T + \sigma^2 \mathbf{I})^{-1} X \sigma_h^2 X$ is underlined in orange and labeled "Error Covariance Matrix" in purple. A circled 'X' is next to it. Below this, the text "We have already shown," is written in orange. This is followed by the expression:

$$\sigma_h^2 X^T (\sigma_h^2 X X^T + \sigma^2 \mathbf{I})^{-1} X \sigma_h^2 X$$

where $X X^T$ is labeled "N x N" and $X X^T + \sigma^2 \mathbf{I}$ is labeled "M x M". A circled 'X' is next to this expression. Below a horizontal line, the simplified expression is shown:

$$= \sigma_h^2 (\sigma_h^2 X^T X + \sigma^2 \mathbf{I})^{-1} X^T$$

where $X^T X$ is labeled "M x M".

So, have to again simplify this, we will use the principle, or there is a result that we have did earlier. So, observe, we have already shown, or we have already shown, in the previous module, or in one of the previous module, during the simplification of the LMMSE estimate. We had shown this property, that is σ_h^2 , X^T , times, σ_h^2 , XX^T plus σ^2 identity, inverse, is equal to, σ_h^2 , $\sigma_h^2 X^T X$ plus σ^2 identity inverse, times, X^T , and remember XX^T is n cross n matrix, X^T is n , X is n cross m

X is in m cross m, this is the result that we have already shown. And now if you look at this, this part, sigma h square X transpose sigma h square XX transpose sigma square identity, if you call this as star. If you call this as star, this is exactly your star.

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Using the above identity, the expression for the error covariance matrix can be simplified as,

$$E\{(\hat{h} - \bar{h})(\hat{h} - \bar{h})^T\}$$

$$= \sigma_h^2 I - \sigma_h^2 (\sigma_h^2 X^T X + \sigma^2 I)^{-1} X^T \cdot \sigma_h^2 X^T X$$

So, I can replace this quantity, this star, by sigma h square, by sigma h square, X transpose x, plus, sigma square identity, inverse, times, X transpose, therefore, now the error covariance using the above principle, or using the above identity, using the above identity, the expression for the error covariance matrix can be simplified as, the expression for the error covariance matrix, this can be simplified as, well, I have expected value of h hat, minus h bar, into h hat, minus h bar transpose, this is equal to sigma h square, identity, minus.

Now, instead of sigma h square, X transpose, sigma h square, XX transpose sigma X square identity inverse, I am going to employ, sigma h square, X transpose X plus, of course, this is sigma h square over here, all right, we just write it clearly, this is going to be, sigma h square, X transpose X plus, sigma square identity inverse, into X transpose times, sigma, a times we are going to have, well times we are going to have, sigma h square, X transpose of x, what we are doing is, basically this sigma h square. So, basically what we are doing, we are replacing this sigma h square h transpose, sigma X square, XX transpose, the sigma square identity, inverse, by this quantity, and there is another sigma h square, outside, which will basically continue to be over here.

Now, to this $X^T x$, what I am going to do, is I am going to add and subtract $\sigma^2 I$, of this $X^T x$, and that is what I am going to simplify.

(Refer Slide Time: 12:17)

$$= \sigma_h^2 I - \sigma_h^2 (\sigma_h^2 X^T X + \sigma^2 I)^{-1}$$

$$(\sigma_h^2 X^T X + \underbrace{\sigma^2 I - \sigma^2 I})$$

Adding and Subtracting $\sigma^2 I$

$$= \cancel{\sigma_h^2 I} - \cancel{\sigma_h^2 I} + \sigma_h^2 \sigma^2 (\sigma_h^2 X^T X + \sigma^2 I)^{-1}$$

So, adding and subtracting $\sigma^2 I$, now this is a slightly tricky step. So, please pay attention. So, this will be $\sigma_h^2 I$, minus σ_h^2 , into $\sigma_h^2 X^T X$, plus $\sigma^2 I$, inverse, sorry, this is only going to be, and now into these $X^T x$, times, I am going to have, σ_h^2 , $X^T x$, plus $\sigma^2 I$, minus $\sigma^2 I$, adding and subtracting $\sigma^2 I$. So, this is basically your adding, adding and subtracting $\sigma^2 I$.

Now, we can see, the $\sigma_h^2 X^T X$, $\sigma^2 I$, into this first part, $\sigma_h^2 I$, $X^T X$, $\sigma^2 I$, these 2 are inverse of each other. So, multiplication of these 2 will give identity. So, so we will have $\sigma_h^2 I$. So, what we will have is, net, $\sigma_h^2 I$, minus of course, σ_h^2 , times, these 2 will give identity. So, you are left with σ_h^2 , times, identity, plus, what we have is σ_h^2 , $\sigma_h^2 X^T X$, plus, $\sigma^2 I$, into $\sigma^2 I$. So, what we will have net is, σ_h^2 , σ^2 , into $\sigma_h^2 X^T X$, plus, $\sigma^2 I$, inverse, and that is what we are left. And now you can see, these 2 quantities, $\sigma_h^2 I$, minus, $\sigma_h^2 I$, these 2 cancel, and what we have left with is basically the error

covariance, is σ_h^2 , σ^2 , σ_h^2 , $X^T X$, plus σ^2 , identity, inverse, this is the expression for the error covariance.

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The image shows a whiteboard with handwritten mathematical equations. The first equation is:

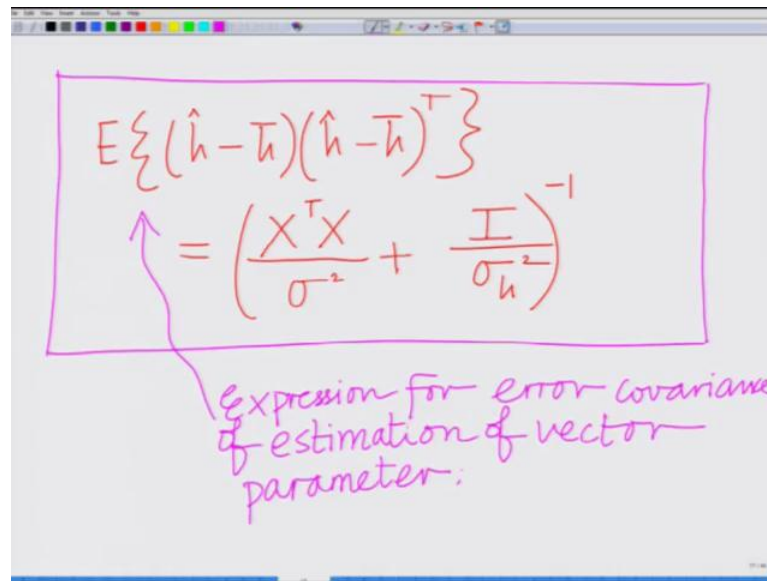
$$= \sigma_h^2 \sigma^2 (\sigma_h^2 X^T X + \sigma^2 \mathbf{I})^{-1}$$

Below the term $(\sigma_h^2 X^T X + \sigma^2 \mathbf{I})^{-1}$ is a wavy line and the text "Error Covariance". An arrow points from this term to the second equation:

$$= \left(\frac{X^T X}{\sigma^2} + \frac{\mathbf{I}}{\sigma_h^2} \right)^{-1}$$

So, what we have is $X^T X$ σ^2 , into σ^2 , times, σ_h^2 $X^T X$ plus, σ^2 identity, inverse. Now what we are going to do, we are going to take the σ_h^2 , the σ_h^2 is the σ_h^2 , into σ^2 , this is a scalar factor. So, I can take it inside the matrix inverse. if I take it inside the matrix inverse, taking this σ_h^2 , σ^2 , inside the matrix inverse, I get the final expression, that is $X^T X$ divided by σ^2 , plus, identity, divided by σ_h^2 , inverse.

(Refer Slide Time: 15:44)



The image shows a handwritten mathematical derivation on a whiteboard. The equation is enclosed in a purple rectangular box. The equation is:

$$E\left\{(\hat{h} - \bar{h})(\hat{h} - \bar{h})^T\right\} = \left(\frac{X^T X}{\sigma^2} + \frac{I}{\sigma_h^2}\right)^{-1}$$

Below the box, there is a purple arrow pointing from the text to the left side of the equation. The text reads:

expression for error covariance of estimation of vector parameter.

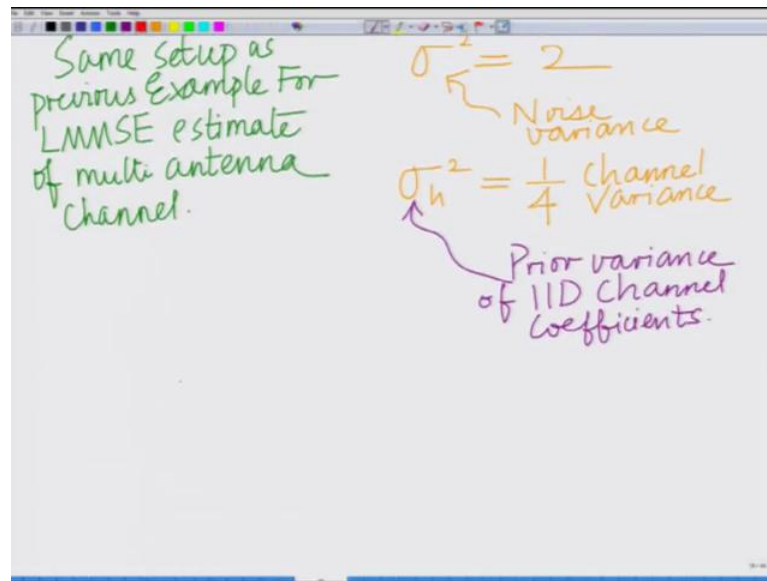
So, this is your final expression, for the error covariance. So, error covariance, expected value of \hat{h} minus \bar{h} , into \hat{h} minus \bar{h} transpose, equals, X transpose x , by sigma square, plus, I , divided by sigma h square, inverse. This is the expression, and this is the error covariance, for your vector parameter models. So, this is the covariance. Of course, we are done this is in the context of, your, downlink multi antenna channel estimation, but realize that, this is the general expression, for an error covariance, for the estimation of a vector parameter. So, this is the expression for error covariance of estimation, of a vector parameter, that is X transpose x , divided by sigma square, plus, I , divided by sigma h square, whole inverse. This is the error covariance for the estimation of a vector parameter.

(Refer Slide Time: 17:37)

The slide is titled "Example For Error Covariance" in pink. It features a handwritten matrix $X = \begin{bmatrix} 5 & 4 \\ 4 & 5 \\ 3 & 4 \\ 4 & 3 \end{bmatrix}$ labeled as the "Pilot Matrix". Below the matrix, it states "Same setup as previous Example For LMMSE estimate of multi antenna channel." To the right, it specifies the noise variance as $\sigma^2 = 2$ and the channel variance as $\sigma_h^2 = \frac{1}{4}$.

So, now we have this error covariance. So, now, let us do an example to understand this better, let us do an example to understand this better. So, let us do an example. So, let us do an example, to compute this, to compute, this error covariance for the vector parameter, and what we consider is, let us consider the same parameters, or the same values, or the same set up, as the previous example, that we have done for the LMMSE, LMMSE estimate, of the vector parameter. This is a multi antenna downlink channel. So, we are considering the pilot matrix 5 4, 4 5, 3 4, 4 3, basically this is the same, same set up as previous example for LMMSE estimate, of your LMMSE estimate, of the multi antenna channels.

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So, the pilot matrix, this is your pilot matrix, this is the pilot matrix. We consider sigma square equals, 2, this is your noise variance, this is your noise variance, sigma h square, equals 1 by 4, this is the, this is the channel variance, correct? Sigma square equals 1 by 4, this is the channel variance, this is the channel variance, or the prior variance of the channel, rather the prior variance, the prior variance of your IID, prior variance of your IID channel coefficients therefore, what we get is that, your error covariance, this is equal to, as we had just derived, that is, $\frac{1}{\sigma^2} X^T X + I$ over, sigma h square, inverse. That is the expression, that we have we have computed, $X^T X$, in the previous example, where X is the pilot matrix.

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The image shows a whiteboard with handwritten mathematical expressions. At the top right, the word "Coefficients" is written in purple. The main expression is for the Error Covariance matrix, written in red:

$$\text{Error Covariance} = \left(\frac{1}{\sigma^2} X^T X + \frac{I}{\sigma_h^2} \right)^{-1}$$

Below this, the matrix $X^T X$ is calculated as the product of a 2x4 matrix and a 4x2 matrix:

$$X^T X = \begin{bmatrix} 5 & 4 & 3 & 4 \\ 4 & 5 & 4 & 3 \end{bmatrix} X \begin{bmatrix} 5 & 4 \\ 4 & 5 \\ 3 & 4 \\ 4 & 3 \end{bmatrix}$$

Below this, there is a purple equals sign followed by a blank space.

So, I am going to reuse those values. So, $X^T X$, which is the transpose of the matrix X , times itself. Remember this is m cross m matrix, it is a 2 cross 2 matrix, which is basically your 5 4, 4 5, 3 4, 4 3, times X , which is the pilot matrix.

(Refer Slide Time: 21:26)

The image shows a whiteboard with handwritten mathematical expressions. At the top, the calculation of $X^T X$ is shown:

$$X^T X = \begin{bmatrix} 5 & 4 \\ 4 & 5 \\ 3 & 4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 5 & 4 \\ 4 & 5 \\ 3 & 4 \\ 4 & 3 \end{bmatrix}$$

Below this, the result is shown as a 2x2 matrix:

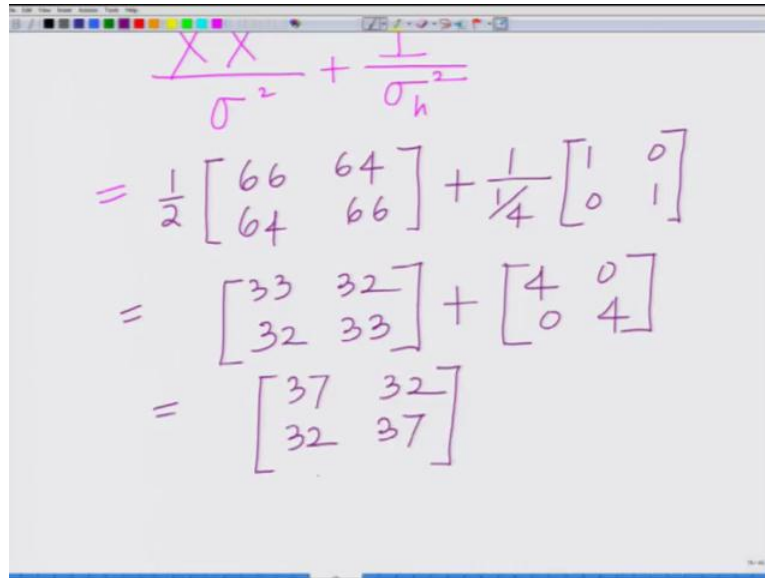
$$= \begin{bmatrix} 66 & 64 \\ 64 & 66 \end{bmatrix}$$

To the right of this matrix, a purple arrow points to it with the text "MxM = 2x2 matrix". Below this, the expression for the Error Covariance matrix is written in purple:

$$\frac{X^T X}{\sigma^2} + \frac{I}{\sigma_h^2}$$

Once again, 5 4, 4 5, 3 4, 4 3, which is equal to, which is equal to, 66, 64, this is the matrix, that we have already computed 66, 64, 64, 66, this is m cross m , equals, basically your 2 cross 2 matrix, and $X^T X$, divided by sigma square, plus I , divided by sigma h square, this is half, where sigma square is 2.

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$$\begin{aligned} & \frac{XX}{\sigma^2} + \frac{I}{\sigma_h^2} \\ &= \frac{1}{2} \begin{bmatrix} 66 & 64 \\ 64 & 66 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 33 & 32 \\ 32 & 33 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 37 & 32 \\ 32 & 37 \end{bmatrix} \end{aligned}$$

So, this is 1 of sigma square is half, 66, 64, 64, 66, plus 1, over sigma h square, is 1 by 4. So, 1 over 1 by 4, 2 cross 2 identity matrix, 1 0 0 1. So, this will be basically your 33, 32, 32, 33, plus 4 times the identity matrix, that is 4 0 0 4 this will be, 37, 32, 32, 37, and for therefore, the error covariance equals now, equals well, X transpose x, by sigma square plus, I, divided by sigma h square, inverse, which is basically inverse of this matrix, your 37,32, 32, 37, inverse, this is equal to, well there are 2 cross 2 matrix, so, inverse is easy to compute 1 over the determine that 37 square minus 32 square, yesterday we saw, that is 37 minus 32, times 37 plus 32, that is, a 69 times, swapped off diagonal elements, both them are same.

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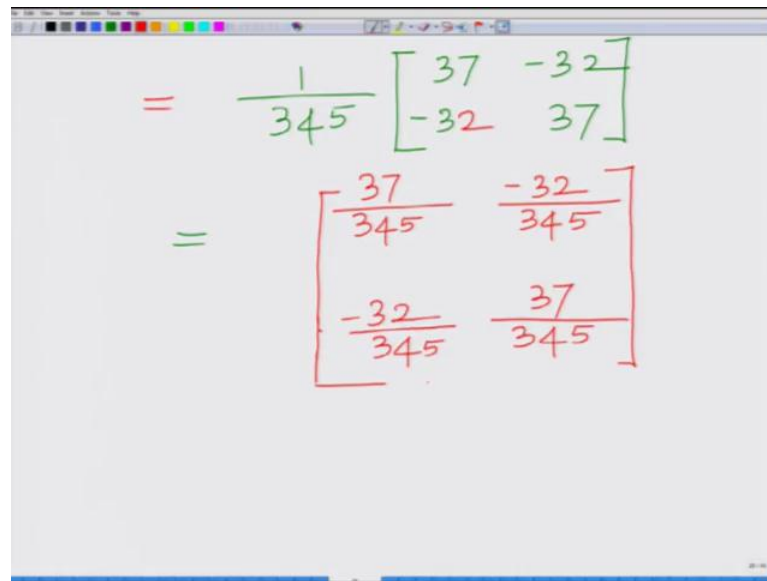
$$\begin{aligned} &= \begin{bmatrix} 33 & 32 \\ 32 & 33 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 37 & 32 \\ 32 & 37 \end{bmatrix} \\ \text{Error Covariance} &= \left(\frac{X^T X}{\sigma^2} + \frac{I}{\sigma_h^2} \right)^{-1} \\ &= \begin{bmatrix} 37 & 32 \\ 32 & 37 \end{bmatrix}^{-1} \end{aligned}$$

(Refer Slide Time: 23:42)

$$\begin{aligned} &= \frac{1}{5 \times 69} \begin{bmatrix} 37 & -32 \\ -32 & 37 \end{bmatrix} \\ &= \frac{1}{345} \begin{bmatrix} 37 & -32 \\ -32 & 37 \end{bmatrix} \end{aligned}$$

So, that will be no change, negative, swap of the diagonal elements, negative of the off diagonal elements, minus 32 minus 32.

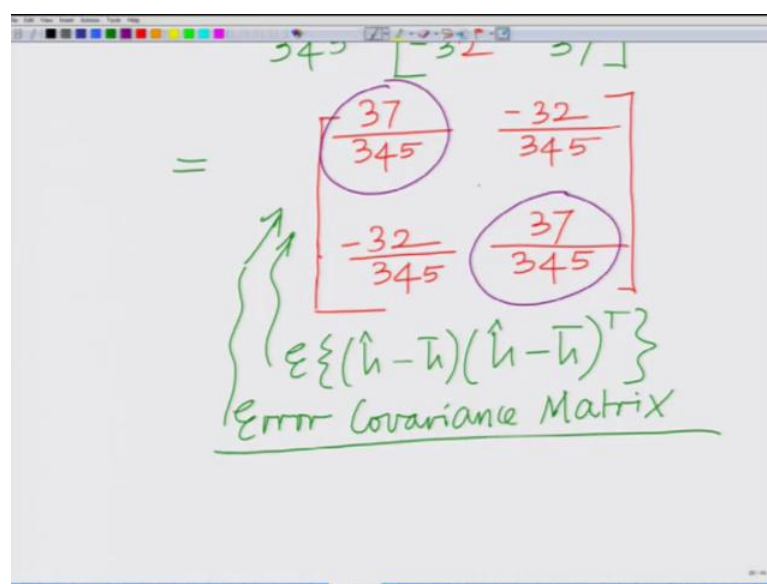
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The image shows a whiteboard with handwritten mathematical expressions. The first expression is
$$= \frac{1}{345} \begin{bmatrix} 37 & -32 \\ -32 & 37 \end{bmatrix}$$
 The second expression is
$$= \begin{bmatrix} \frac{37}{345} & \frac{-32}{345} \\ \frac{-32}{345} & \frac{37}{345} \end{bmatrix}$$

The diagonal elements are 37, 37. So, this will be now, simplifying this, this will be 1 over 345 times, well 37, minus 32, minus 32, 37 and therefore, finally, bringing this thing inside, that we can write this, as your 37 by 345, minus 32 by 345, minus of course, this as to be minus 32, minus 32, by 345, 37 by 345, and this is basically the expression for, this is basically the expression for the error covariance matrix, expected value of \hat{h} minus \bar{h} , into \hat{h} minus \bar{h} transpose.

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The image shows a whiteboard with handwritten mathematical expressions. The first expression is
$$= \begin{bmatrix} \frac{37}{345} & \frac{-32}{345} \\ \frac{-32}{345} & \frac{37}{345} \end{bmatrix}$$
 The diagonal elements $\frac{37}{345}$ and $\frac{37}{345}$ are circled in red. Below the matrix, there is a green arrow pointing to the matrix and the text
$$\{E\{(\hat{h} - \bar{h})(\hat{h} - \bar{h})^T\}$$
 Error Covariance Matrix

So, this is the error covariance, this is your, this is the expression for the error covariance matrix. And of course, now if we look at these diagonal elements, diagonal elements, these diagonal elements are the MMSE, of the individual channel coefficients, right? This is the error covariance, we considered, considered basically, has both estimate of the variances, and also the, the co-area, the cross correlation, the cross co-relation between the errors of the estimates of channel coefficients.

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The slide shows a handwritten derivation of the error covariance matrix. At the top, a 2x2 matrix is written in red ink:

$$\begin{bmatrix} 345 & 345 \\ -\frac{32}{345} & \frac{37}{345} \end{bmatrix}$$

Blue arrows point from the text "Diagonal Elements are variance of Estimates of channel coefficients." to the diagonal elements of the matrix. Below the matrix, the expression for the error covariance matrix is written in green ink:

$$E\{(\hat{h} - \bar{h})(\hat{h} - \bar{h})^T\}$$

This is labeled as the "Error Covariance Matrix". Below this, the variance of the first channel coefficient estimate is calculated:

$$E\{(\hat{h}_1 - h_1)^2\} = E\{(\hat{h}_2 - h_2)^2\} = \frac{37}{345}$$

So, now what you see is, it the diagonal elements, are basically the variances, in the estimates of the channel coefficients, h_1 h_2 , which means, basically, your expected value of h_1 hat, minus h_1 square, this is equal to expected value of estimate of h_2 hat, minus h_2 square, and this is equal to, this is basically equal to 37, divided by 345, all right? The diagonal elements are the variances in the estimates of the channel coefficients.

Let me write that down, just write that down, for the sense of completion. These diagonal elements are basically variances, of your estimates, are the variances of the estimates, variances of the estimates, of your channel, and variances of the estimates of the channel coefficients. So, the diagonal estimates are the covariance. So, basically what we have done in this module is, we have derived the expression, for the error covariance matrix, of the estimate of a vector parameter, the vector parameter is \bar{h} , which is basically the channel vector, corresponding to the multi antenna downlink channel.

Previously we have looked at the LMMSE estimate, now we have also derived the error covariance expression, for the error covariance, and we are simplified the expression for the error covariance, to do an elegant closed form expression, and we have also, with the aid of an example, demonstrated how to compute this error covariance, and from the error covariance, how to extract the information about the, variances of estimates of the individual coefficients, of the individual components, the estimate of the individual components, of the vector parameter.

So, we will stop this module here, and continue with other aspects in the subsequent modules.

Thank you very much.