**Bayesian/MMSE Estimation for MIMO/OFDM Wireless Communications**
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**Lecture – 18**
**Linear Minimum Mean Squared Error (LMMSE) Estimate for Multi-Antenna Downlink Wireless Channel Estimation – Part I**

Hello. Welcome to another module in this massive open online course on Bayesian MMSE Estimation for Wireless Communication Systems. Today let us start looking at LMMSE estimation for multi-antenna channel estimation, as we seen in the previous module the multi-antenna channel estimation problem correspondence to a scenario of vector parameter estimation, so now let us look at how we can apply or LMMSE principle that is a linear minimum mean squared error estimation principle to the estimation of a vector parameter in the contest of multi-antenna channel estimation.

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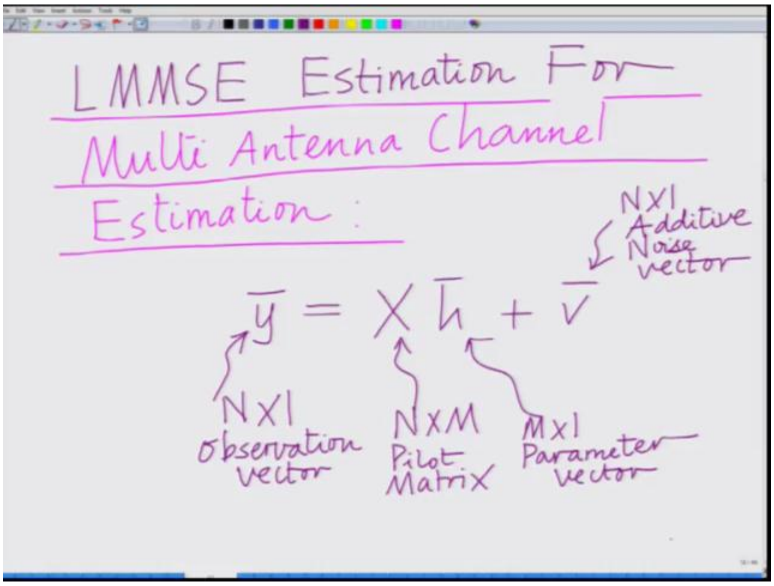


So, let us look at LMMSE, where LMMSE as you know stands for linear minimum squared error estimation for multi-antenna channel estimation. In fact, multi-antenna channels estimation you know wireless system. So, we are going to look at LMMSE estimation for a multi-antenna wireless channel.

And we are already seen that the problem so consider the multi-antenna estimation, so we have seen the problem can be expressed as y bar equals x h bar plus v bar, where this y bar this is your N cross 1 observation vector x is your N cross M pilot matrix and h bar is now your M cross 1 parameter vector previously you had a scalar parameter now you have an M cross 1 parameter vector and v bar is the additive noise; that is this is the N cross 1 additive. This is the same as before this is N cross 1 additive noise vector.

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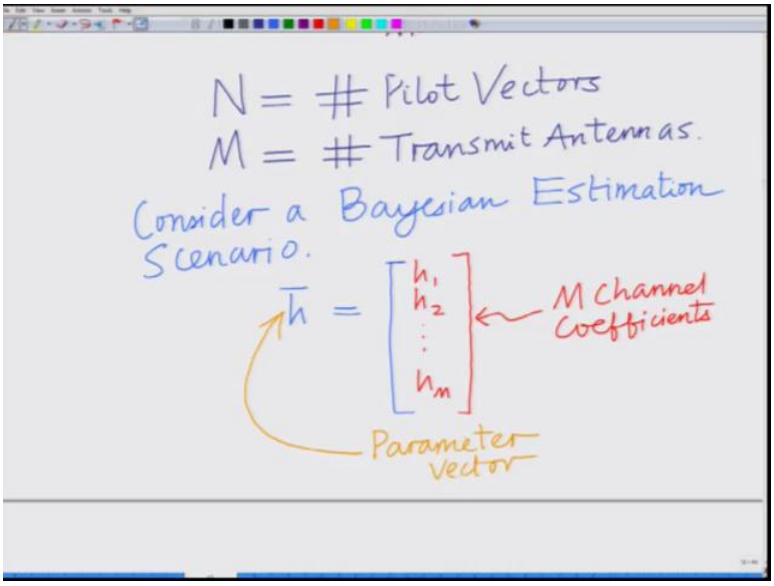


And now if we look at it what is N, Let us define this let us recall this quantity also N equals number of pilot vectors. N is basically the number of pilot vectors transmitted in the downlink by the (Refer Time: 03:15). Remember we are considering a downlink channel estimation scenario where N is number of pilot vectors, M is a number of transmit antenna there are multiple antennas M is number of transmit antennas. Recall that we are considering a system with M transmit antennas a single receive antenna correct and N transmitted pilot vectors. Therefore, the pilot matrix x is N cross M the observation vector y bar is N cross 1 the parameter vector h bar is M cross 1 right and v bar the noise vector is N cross one as you should as simplification we call it.

And now we are going to demonstrate or illustrate how to estimate this channel vector h bar and this is a vector parameter. So, now consider a Bayesian scenario; remember this is a base we are talking about base and estimation considers a Bayesian scenario. Bayesian scenario implies there is prior information about the parameter Bayesian estimation scenario. Bayesian estimation scenario implies there is prior information about the parameter. And this case since we have a parameter vector h bar equals what is this we have a parameter vector h 1 h 2 up to h M, where these are the M channel coefficients correct. These are the M channel coefficients varying channel coefficients rather these are the M channel coefficients and therefore h bar is a collection of M parameters, so there h bar this is your parameter vector. These are some things to remember this is a parameter vector; we are considering the estimation of a vector parameter.

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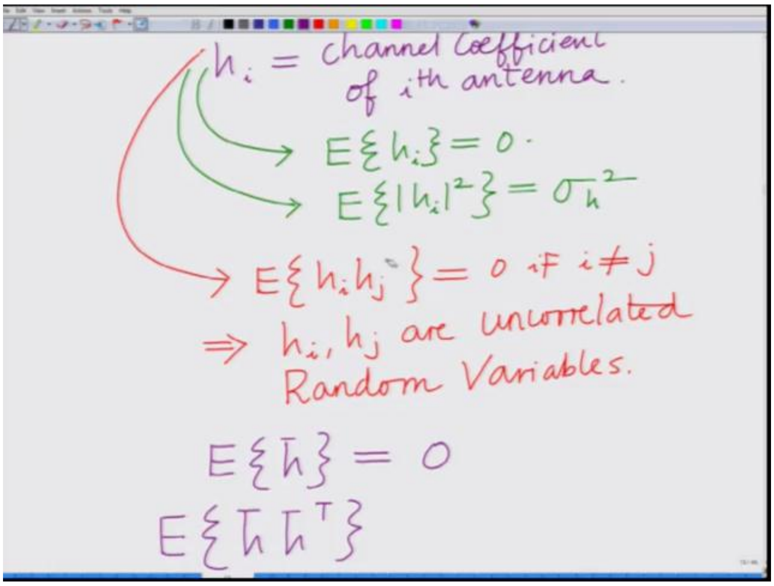


And h i is the channel coefficient corresponding to the ith antenna, h i is a channel coefficient of the ith antenna. Now, previously remember whenever we consider previously so far when we looked at Bayesian estimation we have looked at MMSE estimation and we consider the parameter vector parameter h to be Gaussian in nature. However, in previous module we have introduce the LMMSE estimation, and for the LMMSE the LMMSE is valued for an arbitrarily distributed observation and parameter. Therefore, we no longer need to consider the parameter or the parameter vector to be

Gaussian in nature as long as we are willing to simply accept that it is only the LMMSE estimator and it is a best amongst the class of linear estimators.

Since we are going to be using the LMMSE estimate we now are not going to assume that we are going to assume in the h the parameter vector h bar can be (Refer Time: 06:57) any arbitrary distribution not necessarily Gaussian distributed. And therefore we can simply use the LMMSE estimate without any problem. So, we are not going to assume any prior distribution for h we are simply going to say that h is any arbitrary distribution with each coefficient h i having 0 mean power average gain or expected value of magnitude h i square equals sigma h square. Further, we are also going to say that this expected value of h i into h j conjugate that is channel coefficient of two different antennas this is equal to 0 if i not equal to j. Implies h i and h j are uncorrelated random variables; these are since the covariance is 0.

And once this is the expected value of h i h j conjugate is 0 we can only say that these are uncorrelated, because we have not assuming h i h j to be Gaussian we cannot say that these are independent. All we can say at this point since we are not made any assumption about the prior distribution of h; we can only assume they are uncorrelated. And that is sufficient for the purpose of LMMSE estimation. We do not need to nearly assume that they are independent. As long as they are uncorrelated we are only concerned with the uncorrelated nature, we are not nearly bothered about whether they are independent or not because the LMMSE estimates simply considers the covariance information it does not need information about the probability density functions of h the parameter vector h in fact also the observation vector y bar.

So, that is something keeps in mind. And therefore, we have since each element h i is 0 mean if we look at expected h bar naturally since each component is 0 mean expected value of this parameter vector h bar is going to be 0.

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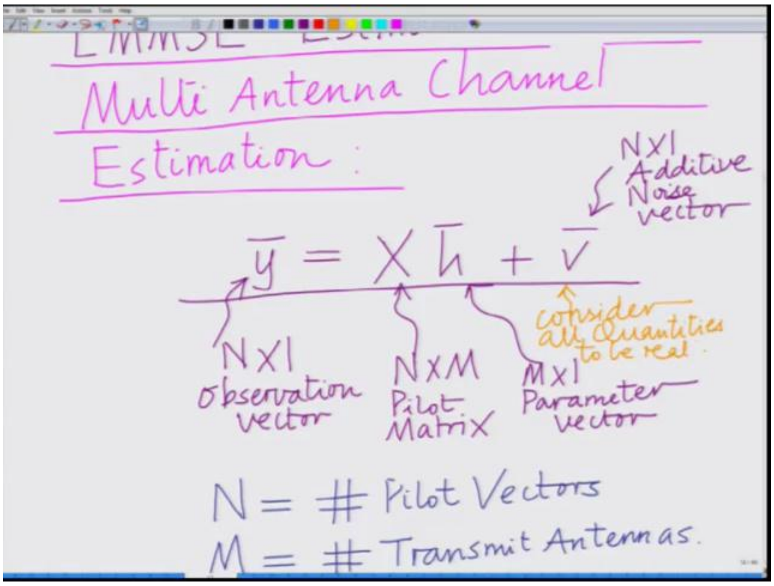


Further, if we look at this covariance matrix, to consider a simple scenario I am simply considering a real scenario. Let us simply consider a real scenario I am going to remove this complex conjugate just to make this estimate paradime simple consider all quantities to be real to begin with. Consider all this quantities to be real. We are going to illustrate later how you can extend this to complex scenario.

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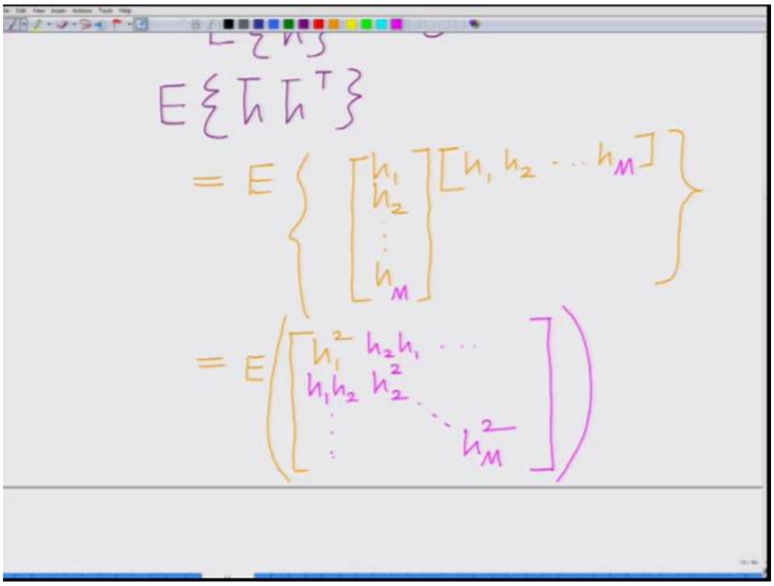

Then what is going to happen this expected value of h bar h bar transpose, if you expand this what is a expected value of h bar this is expected value of well take the column vector h 1 h 2 h L times h bar transpose which is a row vector h 1 h 2 up to h L. Now you can see when you multiply this column vector by the row vector what you are going to have is all the diagonal elements are going to be elements of the form h 1 square h 2 square so on. The off diagonal elements are going to the form h 1 h 2 h 2 h 1, so those expectations are going to be 0, because we are considering the different the channel coefficient corresponding to the different antennas to be uncorrelated.

So, just write this more explicitly I think probably if writing one more step will help expected value of h 1 square the second diagonal element will be h 2 square, so on the last diagonal element will be of course this is M, this is h M square and these entries are going to be of the form h 1 h 2 h 2 h 1 and so on.

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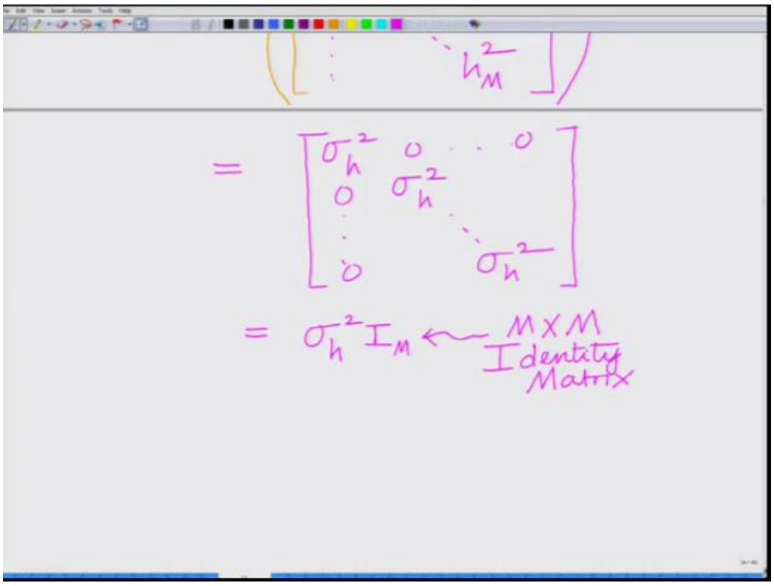


And if you look at the expected value of these the expected value of the off diagonal element are going to 0 the expected value of the diagonal element expected value of h 1 square equals expected value of h 2 square so on. So, this diagonal elements are going to be equal to sigma x square or the off diagonal elements are going to be 0, so this is going to be sigma x square times the identity matrix and this is the just to the be clear this is an

M dimensional identity matrix. When I write I M this means an M cross M identity matrix, because identity matrix is the square matrix that is well known so I am going to simply write the number of rows and number of columns they are both equal. So this is I M basically denotes the M cross M identity matrix.

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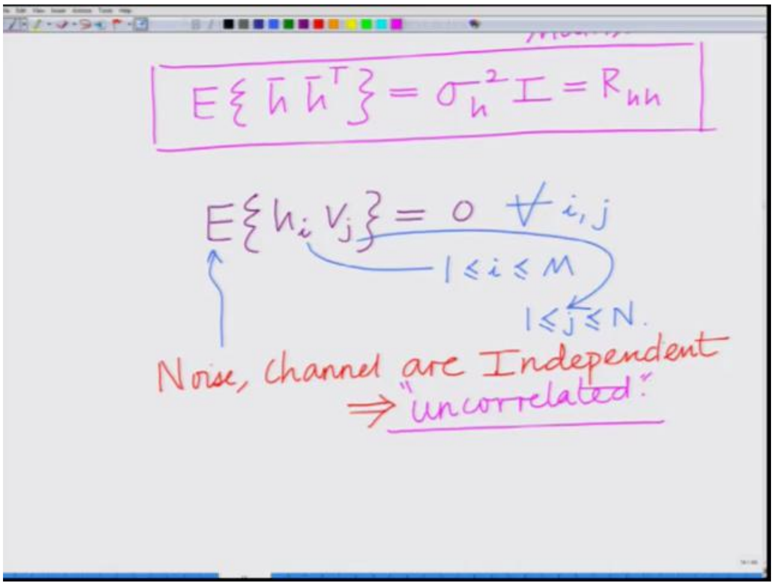


So, we can say that expected value of h bar h bar transpose equals sigma h square times the identity matrix. You can call this as in this case R hh, since we have parameter vector you have the covariance matrix R h h expected value of h bar h bar transpose considering uncorrelated channel coefficient each with gains sigma h square each with average power sigma h square each antenna. That means, basically the covariance matrix expected value of h bar h bar transpose that is R hh is sigma square times sigma h square times identity.

Now, another assumption that we are going to make which is also valid assumptions is that the channel coefficient and the noise samples are uncorrelated and this is a valid assumptions because the channel coefficient are related to the channel coefficients of the radio propagation of the wireless environment. And the noise samples are the noise samples which arise because of the thermal noise at the receiver. These two phenomena are basically independent one the channel coefficients are arising because of the scattering nature of the wireless environment, because of the reflections and the propagation in the wireless environment. And the Gaussian noise at the receiver is arising basically because of the thermal property the thermal noise in the cerckets at the receiver. And these two phenomena are independent so we are going to assume that the channel coefficients and the noise samples are independent.

In fact, as we already said they are independent, but for purpose we do not need independence as long as they are uncorrelated that is sufficient. So, all though they are independent we are going to simply use the fact that they are uncorrelated, which means that if you take any channel coefficient expected value of h i times any noise samples v j; expected value of h i times of v j this is equal to 0 for all i comma j. Remember we have M channel coefficients of 1 less than equal to i less than the all, so i can be any value between 1 and M and we have N noise samples so j can be any value between 1 and N. And therefore this basically arises because of the noise comma channel are independent, of course independence means uncorrelated. For any random variables if they are independent then naturally they are uncorrelated. So, this is something to keep in mind.

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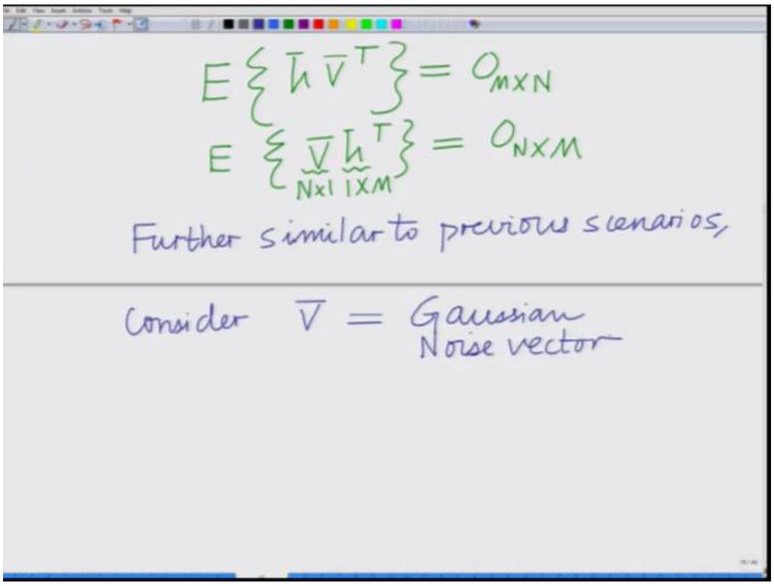


Which means now since any channel coefficient is uncorrelated with any noise samples if I look at expected value of h bar times v bar transpose this is equal to 0, in fact this will be M cross N expected value of v bar times h bar transpose. Remember v bar is N cross 1 h bar is 1 h bar transpose is 1 cross m so this will be 0 N cross M, because the channel vector and the noise samples are uncorrelated for expected value of h bar v bar transpose expected value of v bar h bar transpose both are 0.

Now let us look at the covariance of the noise, and this is something that we already looked at that is the covariance that is we are going to consider the noise samples for the similar previous scenario. Similar to previous scenarios consider v bar equals Gaussian noise vector with covariance expected value of v bar v bar transpose equals sigma square times identity. This implies basically that expected value of each we are going to assume 0 mean noise.

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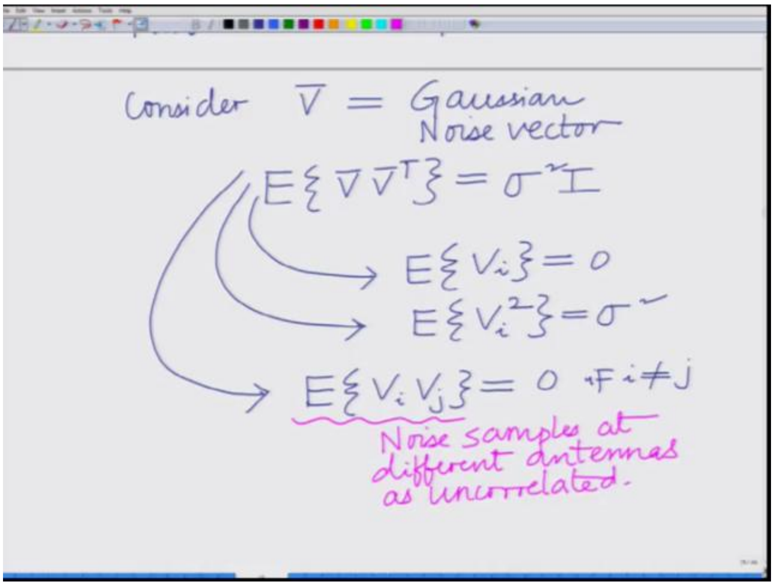


So, expected value of each v i is 0, the each noise is of variance expected value of v i square equals sigma square. Further, we are going to assume that the noise samples at two different antennas are uncorrelated. Since, the noise is Gaussian this implies also independence, if i is not equal to j noise samples at two different antennas are uncorrelated.

Noise samples at the different antennas these are uncorrelated. So, this is (Refer Time: 019:06) now describe the statistics of the problem we are describe the covariance matrix of the channel vector covariance matrix of the noise vector cross covariance of the channel and the noise. So, now, let us find the LMMSE estimate. Now we have all the ingredients to find the LMMSE estimate of the channel vector h bar.

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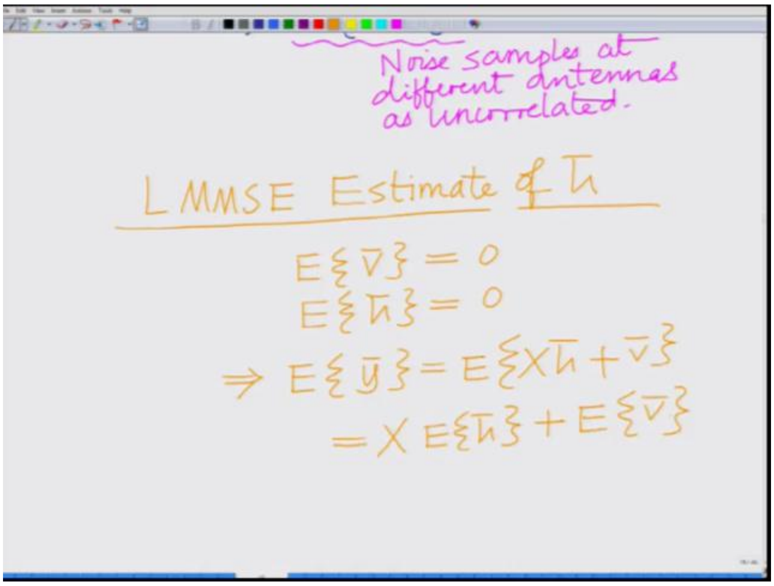


So, let us now proceed to find LMMSE estimate and we know we have derived this expression first observe now that we have noise is 0 mean expected of v bar is equal to 0; we have channel is 0 mean expected h bar equal to 0. This implies expected y bar equals expected x h bar plus v bar equals x times expected h bar plus expected v bar.

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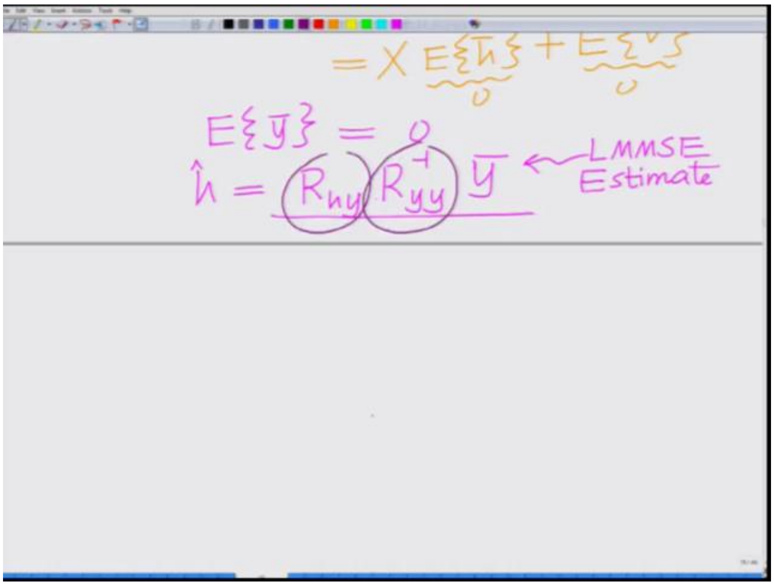


Now, expected h bar is 0 expected v bar is 0 so this is basically 0, so we have expected value of y bar is equal to 0. Basically, we have expected value of y bar equal to 0 expected value of h bar equal to 0. So, the parameter expected is 0 mean observation vector is 0 mean, so we can use the expression for the LMMSE estimate that we have derived previously. We have h hat will be R hy into R yy inverse into y bar, this is the this is the expression for the LMMSE estimate.

So, now we have find these two quantities; we have to find R hy and R yy and R yy inverse we have to find these two quantities.

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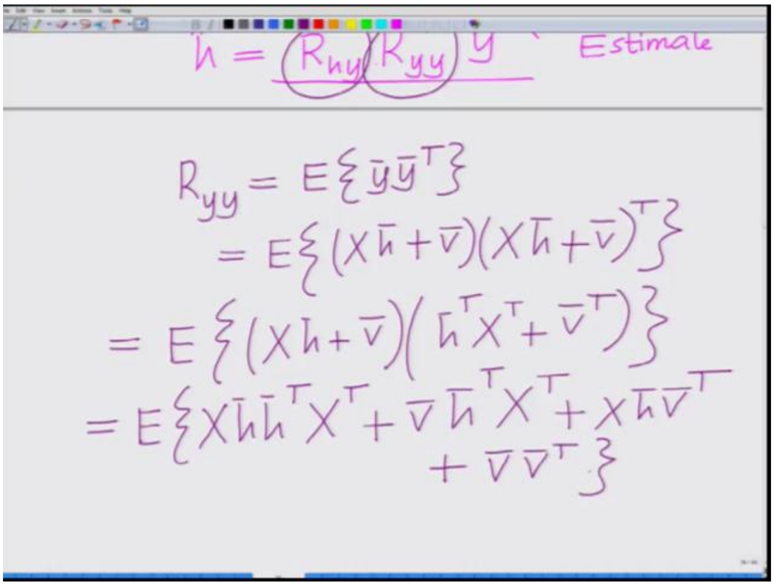


$$\hat{h} = R_{hy} R_{yy}^{-1} \bar{y}$$

$$R_{yy} = E\{\bar{y}\bar{y}^T\}$$
$$= E\{(X\bar{h}+\bar{v})(X\bar{h}+\bar{v})^T\}$$
$$= E\{(X\bar{h}+\bar{v})(\bar{h}^T X^T + \bar{v}^T)\}$$
$$= E\{X\bar{h}\bar{h}^T X^T + \bar{v}\bar{h}^T X^T + X\bar{h}\bar{v}^T + \bar{v}\bar{v}^T\}$$

So, now let us start with R yy; R yy equals expected value of y bar y bar transpose which is equal to expected value of x h bar plus v bar into x h bar plus v bar transpose using the properties of the transpose this is basically x h bar plus v bar times h bar transpose times x transpose plus v bar transpose. This is equal to expected value of x. Now, let us take the product term by term x h bar h bar transpose x transpose plus v bar h bar transpose x transpose plus x h bar, I am taking the product term by term taking the product v bar v bar transpose.

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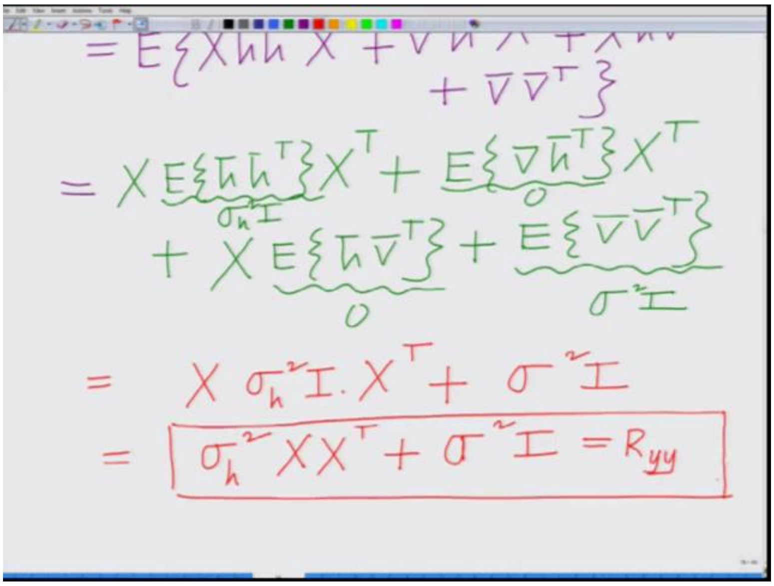

Now if you take the expectation operator inside you will get x expectation of h bar h bar transpose x transpose plus expected value of v bar h bar transpose x transpose plus x expected value of h bar v bar transpose plus expected v bar v bar transpose. Now we have already said that the noise and the channels are uncorrelated so therefore expected value of v bar h bar transpose expected value of h bar v bar transpose is are both 0. Expected value of h bar h bar transpose this is sigma h square times identity, we have already derived that expected value of v bar v bar transpose this is square time identity this also we have already derived, so this becomes sigma h squares.

So, let me write this x times sigma h square times identity matrix into x transpose plus sigma square times identity which is basically sigma h square is a scalar so bring it out this becomes x x transpose plus sigma square times the identity matrix this is your covariance matrix R yy; this is the covariance matrix R yy. R yy is the covariance matrix which expected value of y bar y bar transpose is sigma x square x x transpose plus sigma square times identity, where x of course remember I can recall this is the pilot matrix.

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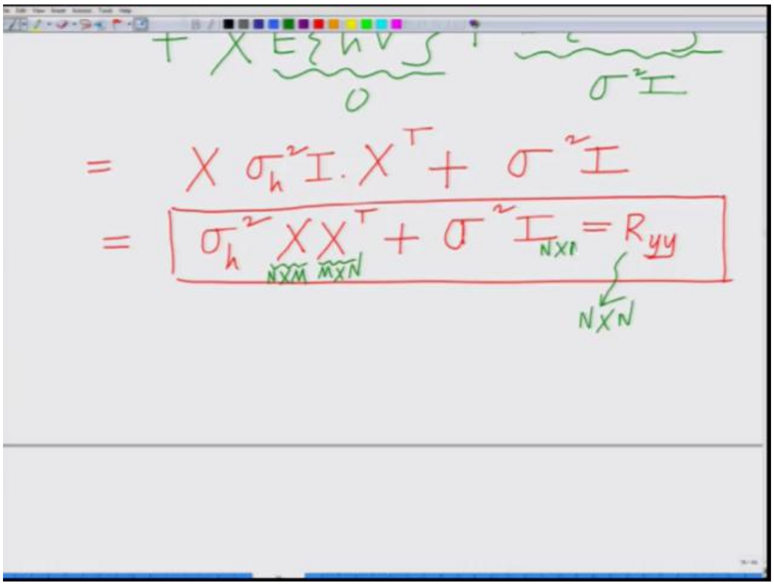


And observe that of this matrix observe that this is N cross M this is N cross M, so R yy this is N cross M M cross N, so R yy is basically N cross N this is an identity matrix is naturally also N cross N.

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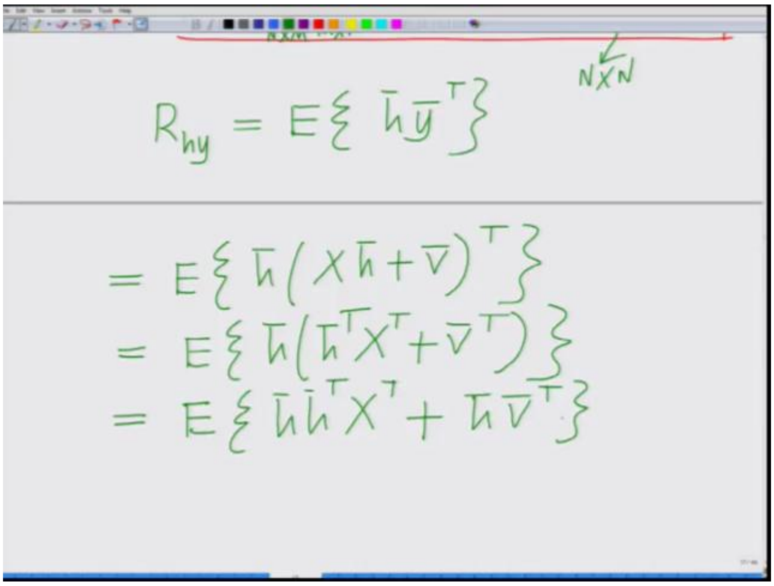

Now, let us look at R hy. R the cross covariance matrix of h comma y R hy is expected value of h y bar h bar y bar transpose this is equal to expected value of h bar times x h bar plus v bar transpose which is equal to expected value of h bar h bar transpose x transpose plus v bar transpose equals expected value of h bar h bar transpose x transpose plus h bar v bar transpose expanded it term by term.

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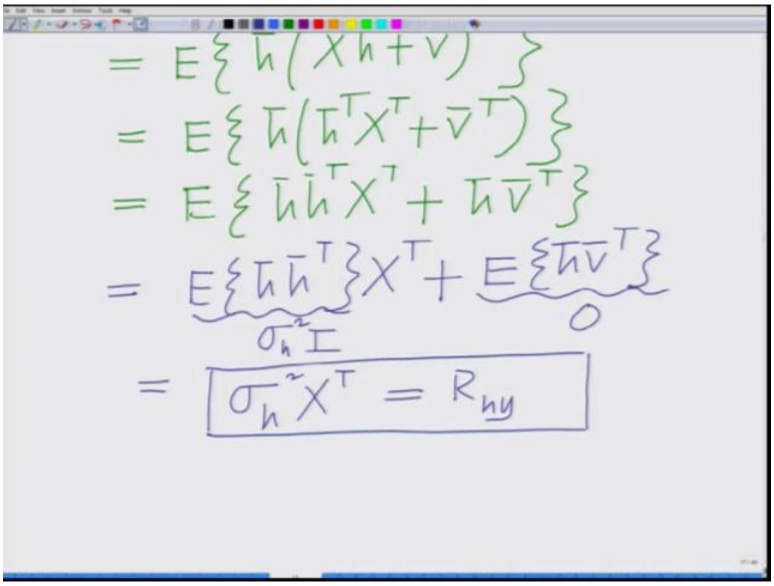


Again take the expectation operator inside this is going to be expected value of h bar h bar transpose x transpose plus expected value of h bar v bar transpose, noise and channels are uncorrelated expected value of h bar v bar is 0 this is equal to expected value of h bar h bar transpose. You already see this is sigma h square times identity, so therefore this becomes sigma h square times x transpose.

This is your R hy. So, we have now both the quantities we have R hy we have R yy therefore the channel estimate is h hat is equal to R hy into R yy inverse into y bar that is the LMMSE estimate. Now we are going to substitute these quantities to derive the LMMSE estimates.

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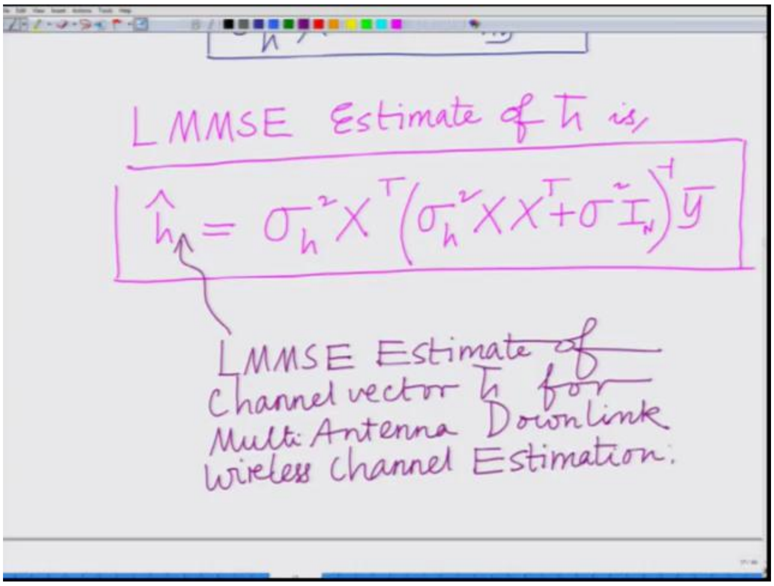


So therefore, now LMMSE estimate is given as; LMMSE estimate of h bar is h hat this is h hat equal to R hy which is sigma h square times x transpose R yy inverse sigma h square x x transpose plus sigma square times identity this is N cross N identity matrix inverse into y bar this is you expression for the LMMSE estimate this is the expression for LMMSE estimate of the channel vector h bar. Let us write this down again LMMSE estimate of channel vector h bar for multi-antenna downlink wireless. So, multi-antenna downlink wireless channel estimation that is h hat equals sigma h square x transpose times sigma h square h x x transpose plus sigma square identity inverse times the vector y bar y bar is the observation vector.

So, we have derived the expression for the LMMSE estimate of the channel vector. And therefore, you will illustrate basically the general principle of LMMSE that is linear minimum mean squared error estimation of a channel vector. Previously we have only considered scalar parameter now we have extended to a scenario with a vector parameter. Further, there is a way to simplify this expression for the estimate further which we are going to do in the subsequent module. So, this module basically involved applying the principle of LMMSE estimation to derive the channel derived the estimate of the channel vector of a multi-antenna wireless communication downlink scenario.

So, we will stop here we will simplify this expression for the estimate further in the next module and we will also derive the mean squared error, that is the mean squared error or the mean squared error of this LMMSE estimate. So, we will stop here.

Thank you very much.