Bayesian/MMSE Estimation for MIMO/OFDM Wireless Communications

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Lecture – 16
Linear Minimum Mean Squared Error (LMMSE) Estimate Derivation - Part II

Hello, welcome to another module in this massive open online course on MMSE Estimation for Wireless Communication Systems. So, we are looking at the linear MMSE estimator that is the linear minimum mean square error estimator or also known as the LMMSE estimator and we are looking at finding the optimal waiting vector c bar that is the combining vector c bar

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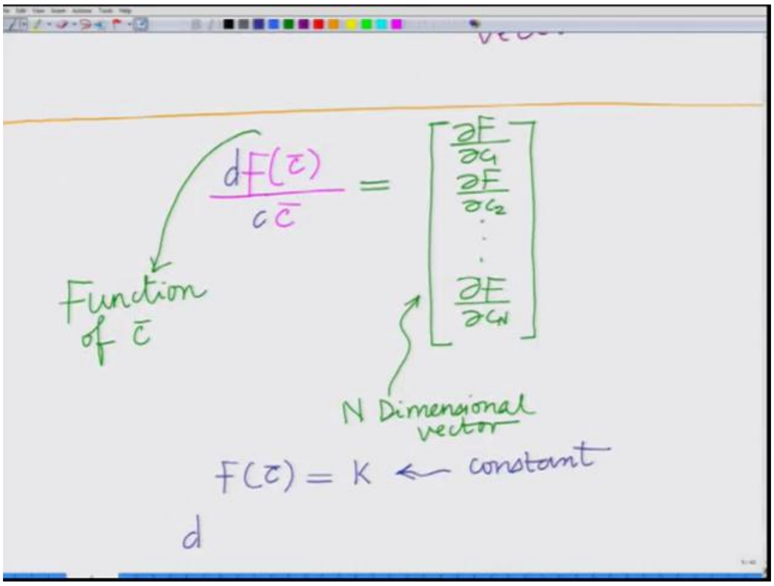

and we said that if you have a function f of c bar function f of the vector c bar, then the derivative with respect to c bar that is this is a function of c bar and it is derivative with respect to c bar will be dou f by dou c 1 dou f by dou c two so on dou f by dou c N that is derivative with respect to each components. So, naturally this is an n dimensional vector where n itself is the dimension of this vector c bar for instance, let us take an example let us consider a simple scenario where this f of c bar equals this function is k, this function of c bar is k which is a constant.

Then dou f, then d f in fact, this d f we can write here instead of dou, since we are considering the derivative this d f by d c bar.

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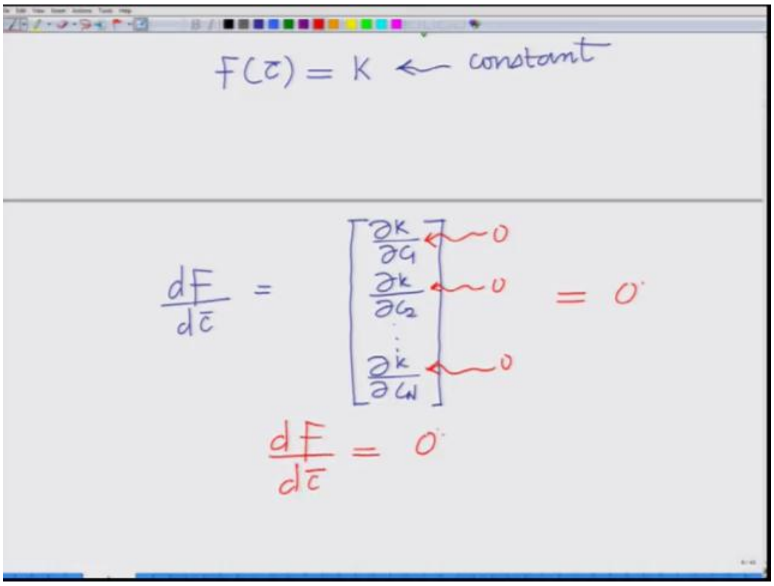

So, I will write d f by d c bar that will be d of that is that will be equal to let us write it over here your d f by d c bar will be partial derivative of k with respect to c 1 derivative of k with respect to c 2 derivative of k with respect to c N and each of these is equal to 0 since, this is a derivative of a constant with respect to c 1 c 2 c n therefore, each of these are equal to 0, this equal to 0 which means d f by d c bar is equal to 0, the derivative of this function f if f is a constant. We already know this means very straight for the derivative of constant with respect to a scalar quantity is 0 and now we are naturally extending to a vector, we are saying that this derivative of this constant with respect to this vector is zero.

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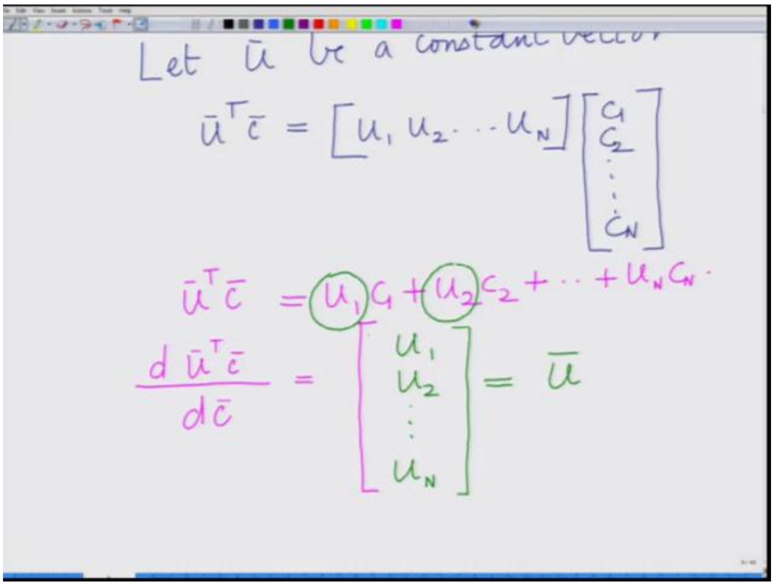

Let us now consider a slightly more advance scenario, slightly more complex scenario. Let us consider a vector in a product with a vector. So, let us say u is a constant vector, let u bar be a constant vector, then the inner product is u bar transpose c bar which is the row vector u 1 u 2 so on up to u N times the column vector c 1 c 2 so on up to c N and this is equal to u 1 c 1 plus u 2 c 2 plus u N c N, we already said this u bar is a constant vector. So, this is my u bar transpose c bar, hence the derivative of this quantity. Now, this u bar transpose c bar with respect to this quantity with respect to the vector c bar.

Now, you can see this is nothing but the derivative of u bar transpose c bar with respect to c 1, the derivative you can see is u 1 with respect to c 2 and the partial derivative will be u 2 so on with respect to c N, the partial derivative is u N.

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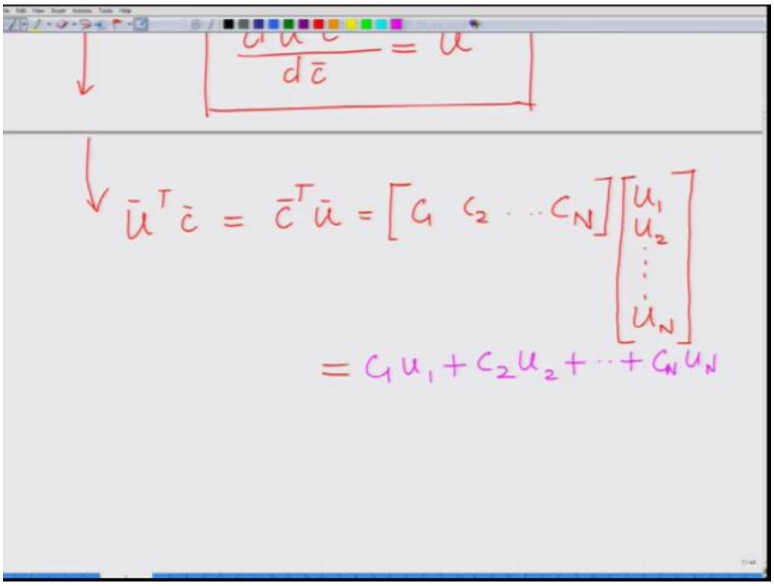

Therefore, this is equal to nothing but the vector u bar. We have this interesting result that the derivative of u bar transpose c bar with respect to c bar is u bar, we have this result over here. The previous result we have is with respect to the constant that is the derivative of this constant. So, we have this derivative when this f of c bar is the constant derivative with respect to c bar that is equal to zero.

Now, of course you will also see this u bar transpose c bar, this is also equals to c bar transpose u bar correct this is also equal to c bar transpose u bar which is basically your row vector c 1 c 2 c N times to the column vector u 1 u 2 u N and this is equal to c 1 u 1 plus c 2 u 2 plus c N u N. So, we have u bar transpose c bar is also the same thing as c bar transpose u bar. In fact, these are both scalar quantity. So, they are equal to the transpose of each other and c bar transpose u bar is basically nothing but the transpose of u bar transpose c bar.

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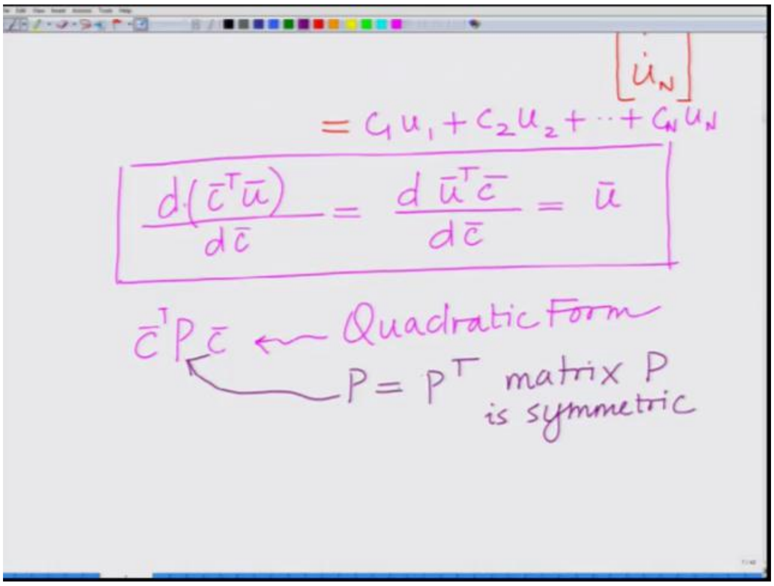

And therefore if we take the derivative we have the derivative of c bar transpose u bar with respect to c bar will be equal to the derivative of u bar transpose c bar with respect to c bar and we already know this derivative of u bar transpose c bar with respect to c bar, this is equal to u bar. So, we have this principle.

Now, let us consider the second order derivative that is a quadratic function of c bar that is we have considered the constant, we have considered a linear function of c bar where it is waited by the combining vector u bar that is u 1 c 1 plus u 2 c 2 so on up to u N c N that is the u bar transpose c bar or c bar transpose u bar. Let us now look at a quadratic function of c bar that is function which involves squares of the terms of this vector, squares of the components of this vector c bar and that function is basically your c bar transpose p c bar, this is known as a quadratic form and further we are considering interesting scenario where p is equal to p transpose that is matrix p is symmetric. Further, we are considering a scenario where this matrix p is symmetric that is p is equal to p transpose and in this scenario it can be shown that the derivative of c bar transpose p c bar with respect to c bar, this is equal to twice p c bar.

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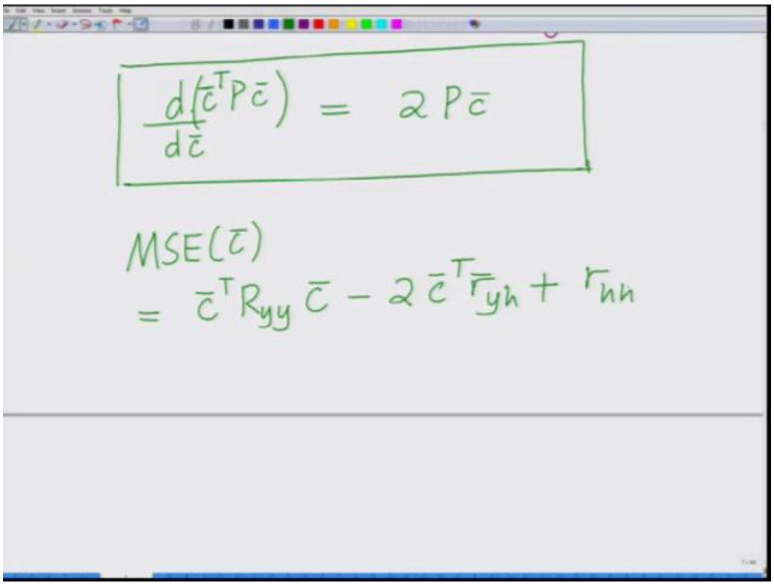

This can be shown alright we will not be show this explicitly over here. The derivative of this quadratic form, the derivative of c bar transpose p c bar with respect to c bar is equal to twice the matrix p into c bar where this matrix p is a symmetric matrix that is p is equal to p transpose. Now, we have all the basic results related to the vector derivative, derivative of this function of this vector c bar with respect to c bar. Let us now go back to our MSE the mean square error cost function, as a function of this vector c bar combining vector c bar differentiate it, set it equal to zero to find the optimal vector c bar which minimizes the mean squared error that will give us the MMSE estimate. This is linear in nature that is the linear MMSE estimate or the LMMSE estimate. Now, if you look at our MSE as a function of c bar recall we already derived this previously, the MSE as a function of c bar, this is equal to your c bar transpose r y y c bar minus twice c bar transpose r bar y h plus r h h.

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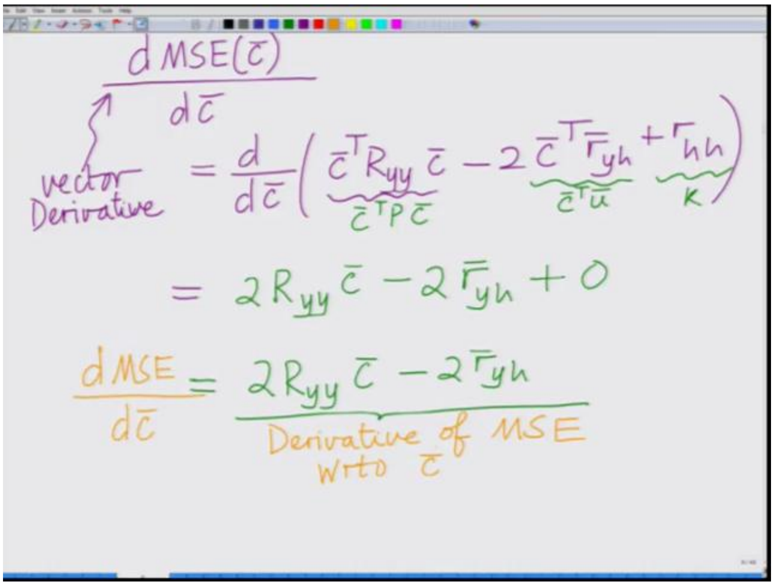

Now, we have to differentiate this MSE which is a function of c bar with respect to the vector c bar. So, this is a vector derivative or also basically known as a gradient and this is equal to the derivative with respect to c bar of c bar transpose r y y c bar minus twice c bar transpose r bar y h plus r h h. Now, you can see this will be equal to now look at this, this is of the form c bar transpose p c bar. So, it is derivative will be twice, p here is the matrix r y y, so it will be twice the matrix r y y times c bar minus this is of the form c bar transpose u bar. So, its derivative will be 2 c bar transpose u bar derivative is u bar that is r bar y comma h and r h h, this is a constant k. So, derivative of this with respect of c bar equals zero. This derivative with respect to c bar of the MMSE is twice r y y c bar minus twice r bar y h, this is the derivative of the MSE with respect to c bar. d MSE with respect to d c bar.

Now, remember to find the optimal c bar, I have to equate this derivative to zero. So, now, set this derivative to zero to find that c bar which minimized the MMSE that will give us the MMSE the minimum mean square error.

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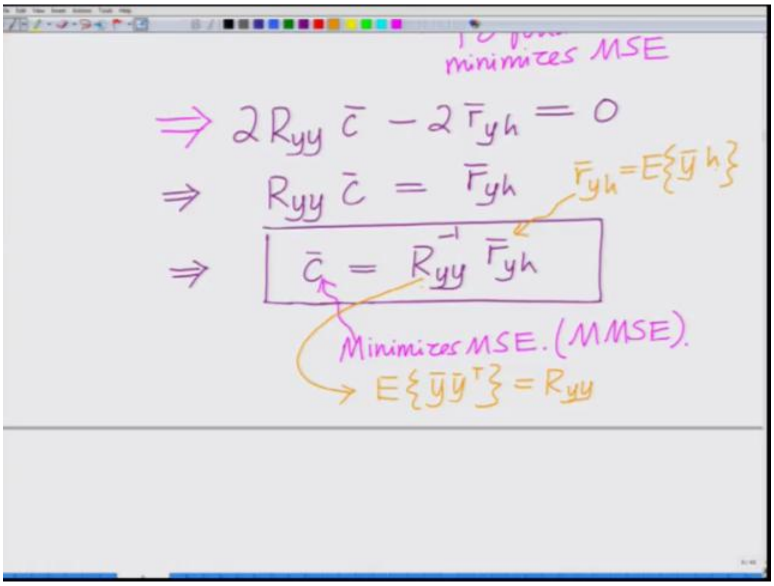

So, set equal to zero to find c bar which minimizes the MSE, this basically implies that your quantity, the derivative 2 r y y c bar minus two r bar y h is equal to 0, this implies that r y y c bar equals r bar y h, this implies c bar equals, this is the system of linear equation. Therefore, c bar equals r y y inverse r bar y h. So, this is now the vector which minimizes the MMSE that is the yields the minimum mean square error correct and this is the combining vector c bar that is c bar equals r y y inverse r bar y h, that is the expression now we have finally derived for the combining which remember what is c bar? c bar is nothing but contains the coefficients c 1 c 2 c N such that if you form c 1 y 1 plus c 2 y 2 so on up to c 1 c N y N, that yields the linear estimate which minimizes the mean squared error.

Also to recall what is r y y, just recall briefly in case it is not already clear r y y equals expected y bar y bar transpose and this r bar y h is the cross co-variance that is expected y bar into h that is the cross co-variance between the observation vector y bar and the parameter h.

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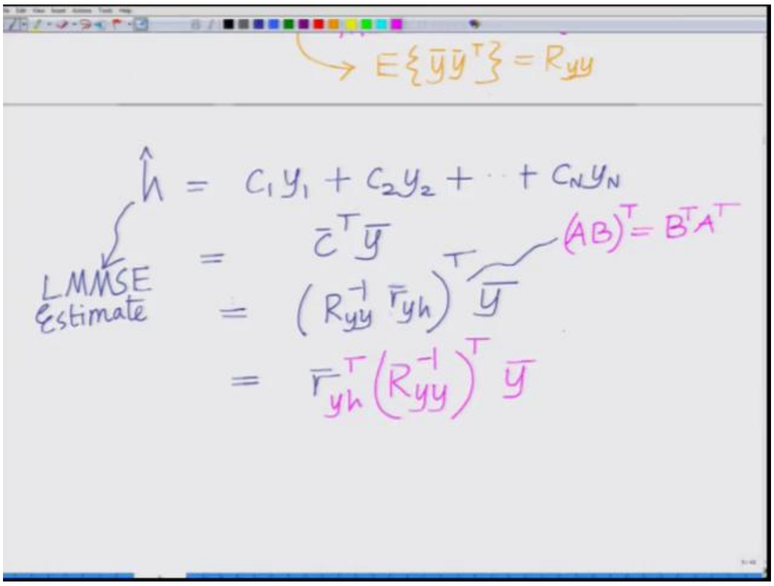

Now, what do we have? The optimal MMSE estimate h hat this is c 1 y 1 plus c 2 y 2 plus so on c N y N, this is the LMMSE estimator, remember now we are in the LMMSE domain, this is the LMMSE which is equal to c bar transpose y bar and we have already derived the expression for c bar, c bar is r y y inverse r bar y h, this transpose times y bar which is equal to now look at this, this is a b transpose we will use the property that a b transpose equals b transpose a transpose to write this as r bar y h transpose r y y inverse transpose into y bar.

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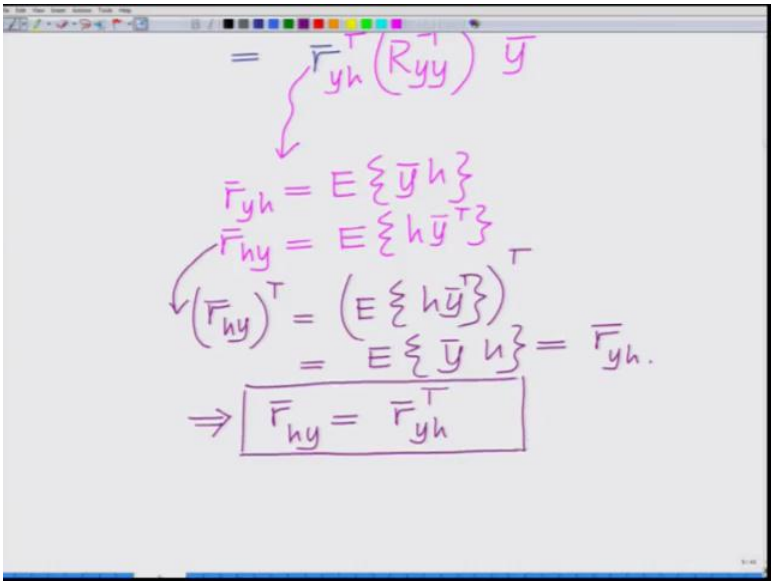

Now, observe two things one is this r bar y h is expected value of y bar into h and r bar h y is expected value of h y bar transpose and therefore, if you take the transpose of r bar h y, now if you look at this r bar h y transpose that is equal to expected h y bar transpose which is equal to expected value of y bar transpose into h because h is a scalar quantity, h transpose equal to h.

So, this is equal to r bar r bar h this is equal to r bar of course, this has to be r bar h y h h y value h y. This will be y bar into h. So, this will be r bar into y h. So, r bar h y transpose is r bar y h which means r bar h y is equal to r bar y h transpose. This is the relation between these two quantities they are transposes of each other r bar r bar h y is the transpose of r bar y h.

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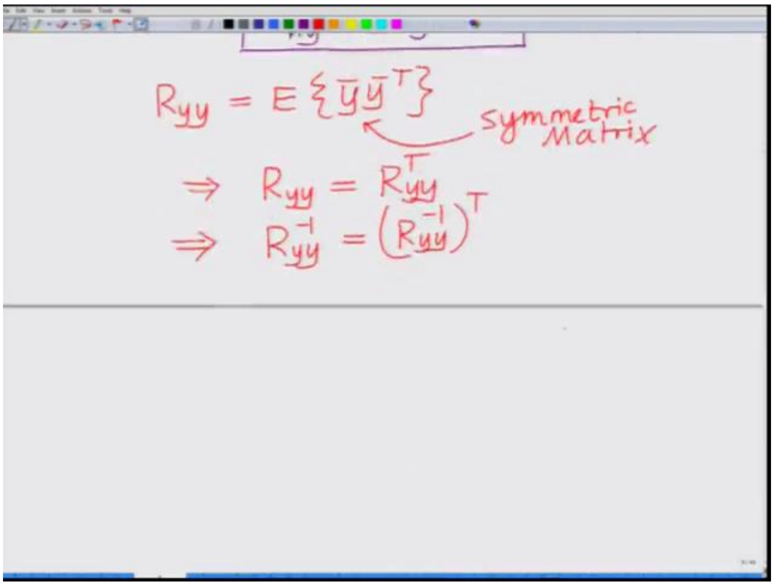

Now, if you look at this quantity r y y inverse transpose relies that r y y equals expected y bar y bar transpose, this is a symmetric quantity correct implies r y y equals r y y transpose since it is a symmetric and also naturally if the matrix is symmetric its inverse is also symmetric, this also implies r y y inverse is r y y inverse transpose and therefore, I can basically simplify this quantity here, first instead of r y h transpose, I can use r bar h y r y y inverse transpose this is the same as r y y inverse. So, I can write this as r y y inverse y bar and this is the expression for the LMMSE estimate and this is the succinct as well as very interesting expression.

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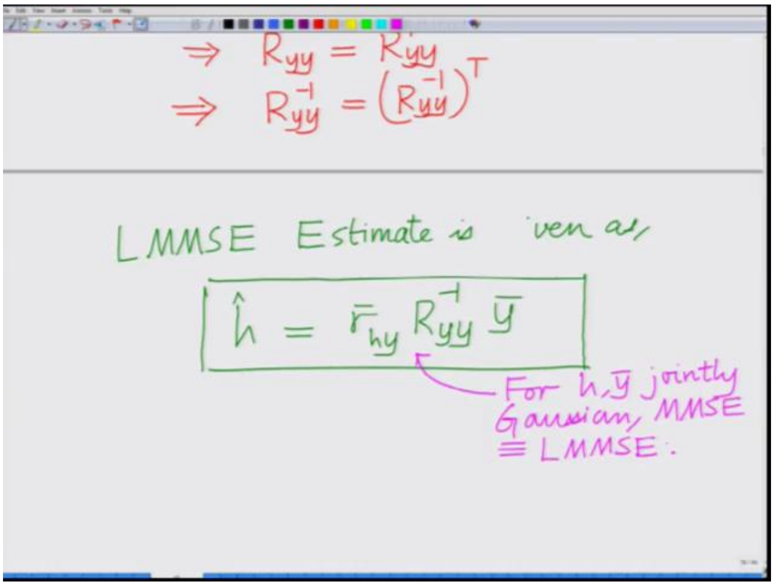

Let me rewrite it again, the LMMSE estimate is finally derived as, the LMMSE estimate is given as h hat equals r bar h y into r y y inverse into y bar, this is the expression for the LMMSE estimate and if you notice this, you will observe something very interesting, this looks exactly identical to the expression for the MMSE estimate that is minimum mean square error estimate, but realize something very important, this is not the MMSE estimate this is the LMMSE estimate, realize that the MMSE estimate was derived assuming that the observation y bar and the parameter h are jointly Gaussian.

So, what this means is something very interesting if h bar, if y bar and h are jointly Gaussian, then the MMSE and the LMMSE estimate are the same, otherwise this expression that is r bar h y times r y y inverse into y bar is only the LMMSE estimate that is it is only the optimal MMSE estimate in the class of linear estimators, but it is not the general MMSE estimate. The general MMSE estimate will still be given by the quantity expected value of h given y bar, that is the expected value of the posterior probability density function which may be difficult to find, that is why we embarked upon this LMMSE.

So, the point to remember here is for Gaussian or for the scenario for h comma y bar jointly Gaussian, your MMSE is identical to LMMSE.

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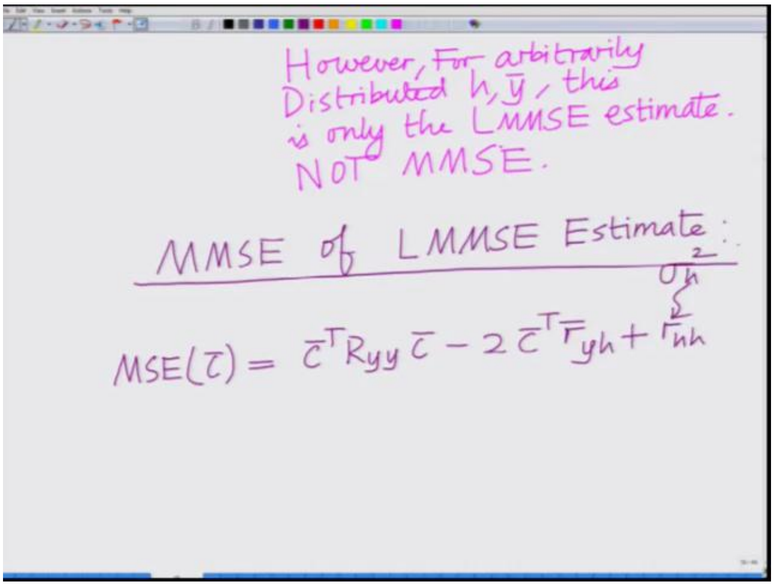

However, for arbitrarily distributed h comma y bar, this is only the LMMSE estimate. In fact, generally this will not be the MMSE estimate. So, generally speaking this will not be the MMSE estimate. So, I am just going to not the MMSE estimate. So, what you can see is if the joint probability density function of the observation vector y bar and the parameter h is Gaussian, then this expression r h y into r y y inverse into y bar is basically the MMSE estimate itself of course if the MMSE, then it is a MMSE and also it is linear.

So, it is naturally the LMMSE also, but however for arbitrarily distributed h bar and observation vector y bar this is simply the LMMSE estimator that it is only the lowest, that is it is only the estimate which is the lowest mean square error or the minimum mean square error amongst the class of linear estimators, but it might not be because there might be other non-linear estimators which might have a lower MM which might deal a lower mean square error, but we are not interested in finding them because finding them is very complex.

So, we are settling for the best estimator amongst the class of linear estimators and we are calling it the LMMSE estimate or the linear minimum mean square error estimator. So, that is the important thing to keep in mind that is for this scenario where this h and y y bar are not jointly Gaussian since evaluation of the MMSE estimate is complex, we have evaluated a simplistic LMMSE estimator right which is basically the optimal amongst the class of linear estimators alright. So, this is the expression for the LMMSE estimate which we are going to invoke again fairly of and an this an important result because the LMMSE estimate is widely used in both signal processing and communications and it is used in several applications in single processing and communications.

Now, let us also compute the resulting MMSE or MMSE of the LMMSE or MMSE of the LMMSE estimator and this can be derived as follows, again we have the MSE as a function of c bar recall we already derived that MSE as a function of c bar, this is c bar transpose r y y c bar minus twice c bar transpose r bar y h plus r h h, this is nothing but sigma h square.

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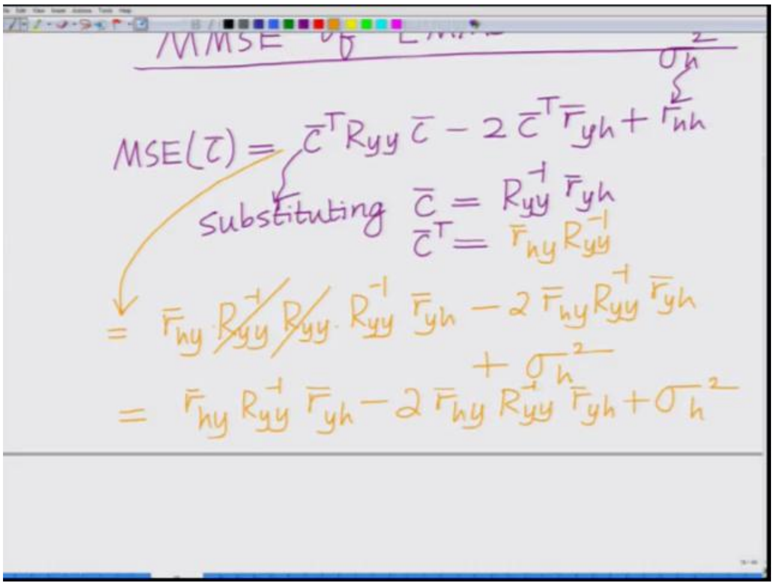

Now, I will substitute the expression for c bar, substituting the expression for c bar, remember c bar equals r y y inverse r bar y h and c bar transpose, this is equal to r bar h y into r y y inverse. So, we can substitute that expression here and therefore, this becomes well c bar transpose is r bar h y into r y y inverse into r y y into r y y inverse into r bar y h minus twice c bar transpose that is minus twice r bar h y into r y y inverse into r bar y h plus r h h which is sigma h square and now you can see this r y y inverse and r y y cancels. So, we are left with r bar h y r y y inverse into r bar y h minus twice again the same quantity you can see interestingly we have the same quantity r bar h y r y y inverse into r bar y h plus sigma h square.

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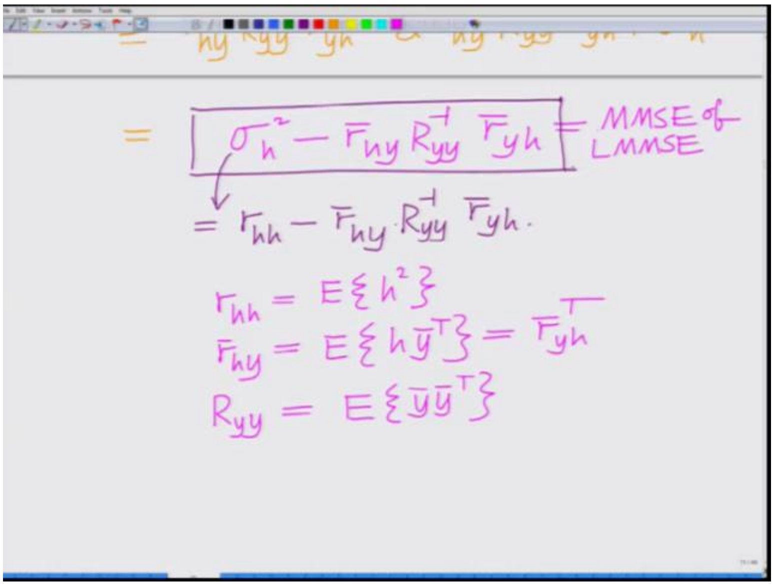

Therefore, finally this gives the sigma h square minus r bar h y r y y inverse r bar y h and this is equal to the MMSE of minimum mean square error of the LMMSE estimator, this is the expression for the minimum mean square error that is your sigma h square and you can also right this as this is also equal to your r h h minus r bar h y r y y inverse r bar y h.

Let us just write the various quantities again your r h h is expected value of h into h that is h square r bar h y equals expected value of h y bar transpose which is r bar y h transpose and r y y equals expected value of y bar y bar transpose and these are the different quantities and this is the resulting minimum mean square error of the LMMSE estimate. Again you see that it has the same expression as that of the minimum mean squared error for the Gaussian and that is naturally the case because for the Gaussian the MMSE and the LMMSE are the same ok.

So, both the LMMSE estimator and the MMSE estimator yield the same mean squared error or the same minimum mean squared error. However, for a Non-Gaussian probability density function of h and the observation vector y bar, this is simply the MMSE minimum mean square error yield which is given by the linear MMSE estimate that is the estimator which is best amongst the class of linear estimator, the actual MMSE of an optimal MMSE estimator if that can be found that the MSE of that will naturally be lower than this because this is simply the linear MMSE estimate.

So, that is something to keep in mind alright and of course, here we have considered a simple scenario where all these quantities are zero mean it is so important to keep in mind that this basically is based on the assumption that all the quantities, the parameter h, the observation vector y bar are basically zero mean alright and this can also be extended to scenarios where these are not necessary. So, that completes the derivation of the LMMSE estimator and we are going to look at applications of this LMMSE estimator in the subsequent modules.

Thank you very much.