Bayesian/MMSE Estimation for MIMO/OFDM Wireless Communications
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Lecture – 14
Minimum Mean Squared Error (MMSE) for Wireless Fading Channel Estimation - Example and Properties of Complex Channel Coefficient Estimate

Hello. Welcome to another module in this massive open online course on the Bayesian MMSE Estimation for Wireless Communications. We are looking at the MMSE or the minimum means squared error of wireless channel estimation.

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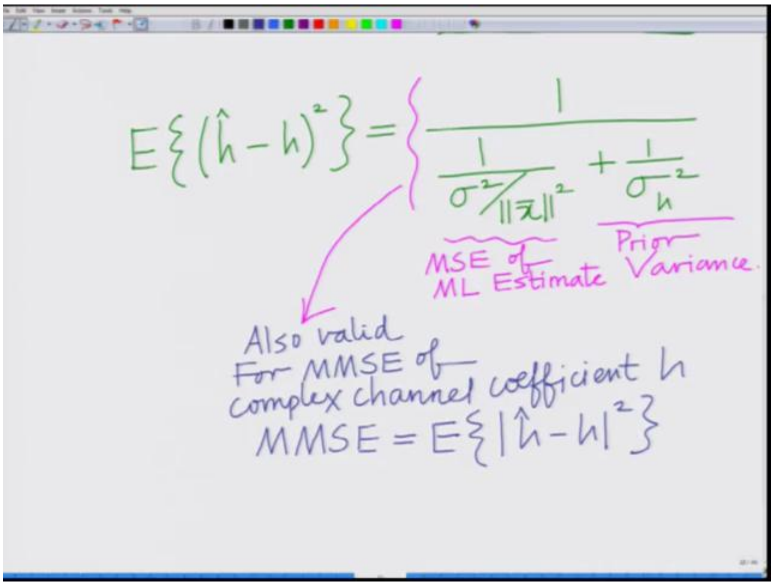

So, we are looking at the MMSE that is the minimum mean squared error for your wireless channel estimation; that is what we have been looking in the previous module. And we have derived the expression where h is the channel coefficient of the MSE the minimum mean squared expected h hat minus h whole square this is equal to the harmonic mean of the ML or mean squared, but that is 1 over sigma square divided by norm x bar square plus 1 over sigma h square which is the prior variance.

So, this we said this is the MSE of the maximum likelihood estimate and this is the prior variance. This is the MSE of the maximum likelihood estimate and the prior variance.

And we said this also valued by the way we said this is also expression also valid for a complex parameter; that is when the channel coefficient h is the complex quantity. So, this is also valid for MMSE of a complex channel coefficient. Except in that case the MMSE will be the minimum mean squared error will be we cannot simply use expected value of h hat minus h square it will be expected value of magnitude h hat minus h whole square. Since it is a complex quantity naturally we have to considered the magnitude, expected value not of the quantities square expected value on the magnitude h hat minus h whole square and that will be given by in this expression.

We are going to look at the estimation or the properties of the estimation of a complex channel coefficient shortly that is we are going to look at more properties of the nature of the estimate of this complex channel coefficient shortly. But mean while let us do an example, first let us do an example to understand just look at how this is used in principle and then will proceed to the other aspects.

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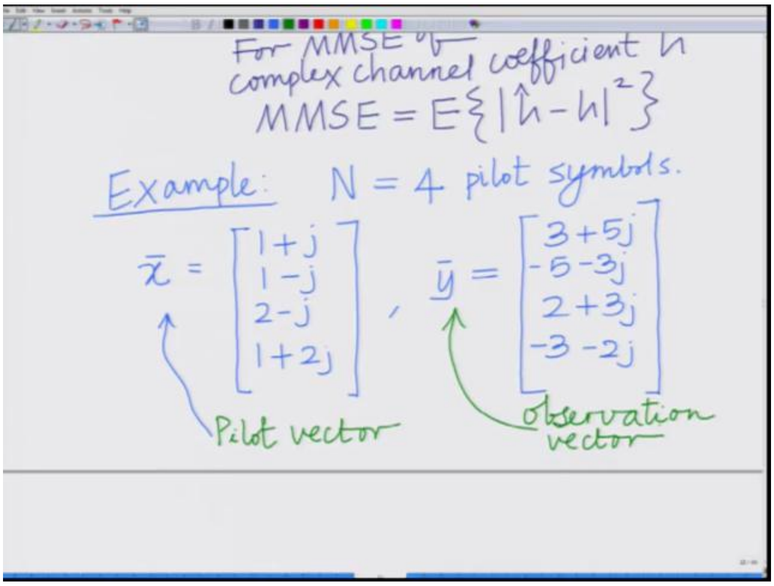

Now let us considered a simple example for instance, for a channel estimation scenario again we go back to the example that we have considered previously. I want to considered N equal to 4 pilot symbols or pilot vector x bat equals x bar will be 1 plus j, 1 minus j, 2 minus j, 1 plus 2 j this is something that we have already seen. And y bar this is the observation vector this is your 3 plus 5 j, minus 5 minus 3 j, 2 plus 3 j, minus 3 minus 2 j this is our observation vector, so this is our pilot vector. This is our pilot vector; this is your observation vector.

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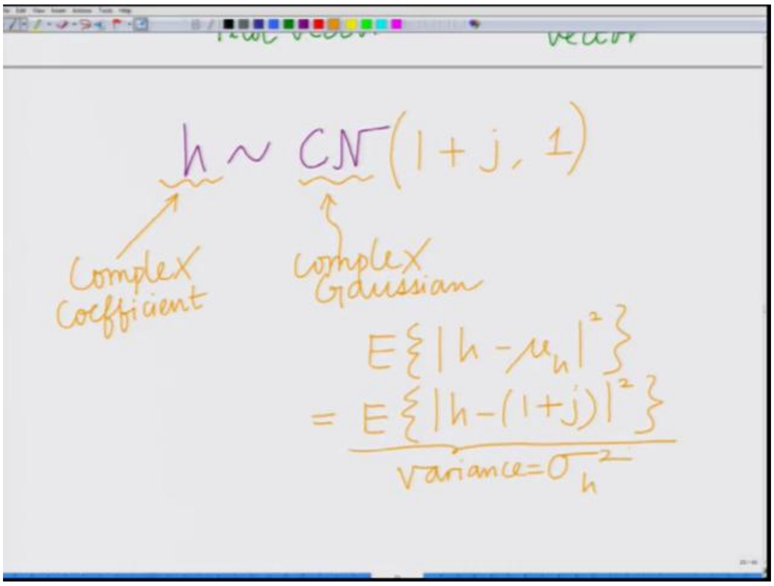

Now, let us say h is the complex channel coefficient, h is distributed as complex Gaussian this symbol C N we used N for Gaussian; this symbol this denotes complex Gaussian. We will talk about talk more about this shortly, so this is the complex Gaussian or complex normal random variables, so h is a complex channel coefficient. So, this is h is a complex is also known as a when its complex Gaussian this is also known as the Rayleigh fading channel coefficient.

This is a complex this is your complex coefficient let us say, complex normal distributed with mean 1 plus j it is a complex coefficient so the mean is also complex with the variance is always going to be a real quantity, because variance is defined as expected value of the magnitude of h minus mu h square. This is the expected value of the magnitude square which in this case is expected value of magnitude of h minus 1 plus j whole square. This is your variance that is sigma h square.

So, the variance is basically the expected value of the magnitude square. So, this is variance is expected value of the magnitude square the variance is always (Refer Time: 06:59) even for a complex normal distributed. Even for a complex Gaussian random variable the variance is real it is just that the mean can obey complex quantity there is the expected value of h. The mean is simply the expected value of h or the average value of the fading channel coefficient so this can still be a complex quantity. So, the variance always has to be a real quantity.

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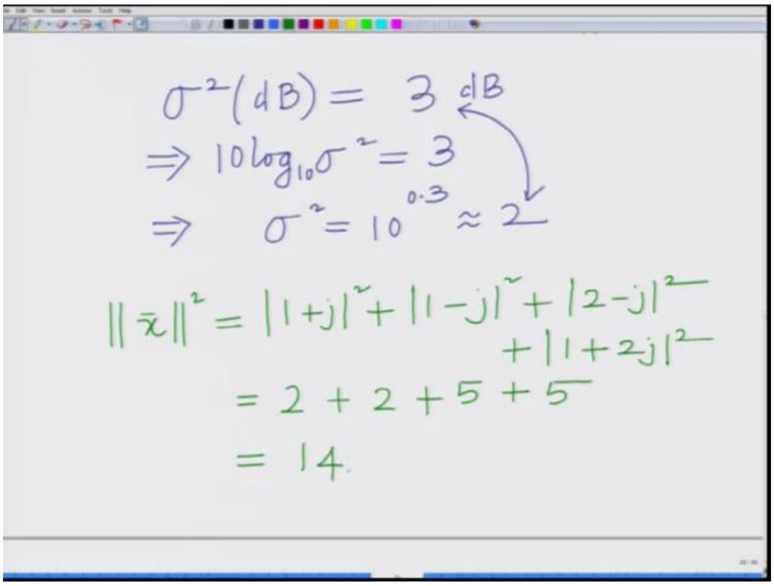

Now what we need, we need also let us say the noise variance sigma square sigma square dB the dB noise variance this is again something that we already have that is 3 dB this implies 10 log 10 sigma square equal to 3 this implies that is your dB noise variance as basically 3 dB.

This implies sigma square equals 10 to the power of 0.3 which is approximately for 2 3 dB corresponds to a noise variance of basically this is standard result standard result in communication or a standard sort of rules of thumb which is often used that is 3 dB corresponds to 2 seen therefore 6 dB corresponds to 4 minus 3 dB corresponds to half and so on or let us say you should be familiar with these kinds of simple rules or simple kinds of rules of thumb rather that are used in analysis of examples and problems related to communication systems.

And what we need is norm x bar we have the pilot vector. Now we need to calculate the MMSE you know we need norm x bar square this will be magnitude of some squares of the magnitude of all the pilot symbols magnitude 1 plus j square plus magnitude 1 minus j square plus magnitude 2 minus j square plus magnitude 1 plus 2 j square which is equal to 2 plus 2 plus 5 plus 5 this is equal to 14.

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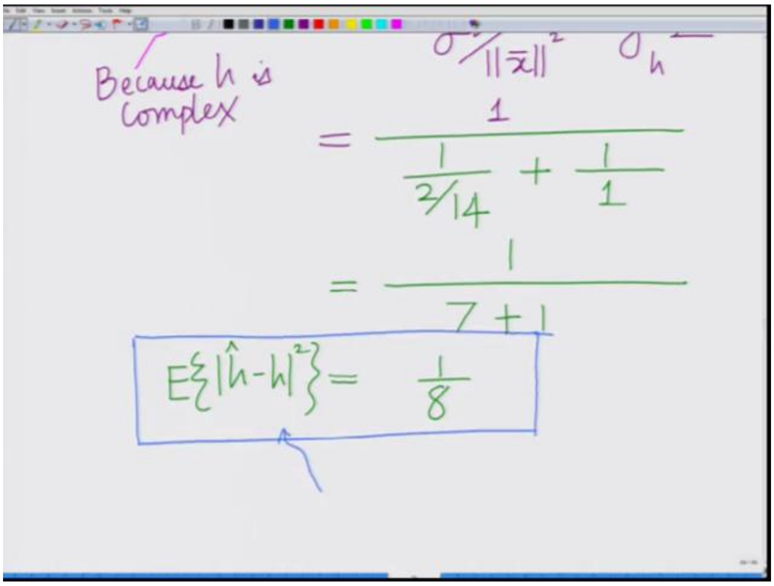

Now, we have the expression for the MSE and expression for the MSE is expected value of because this is a complex coefficient we are using the magnitude remember this is because you have to use the magnitude because h is the complex coefficient. And this will be equal to 1 over 1 over sigma square divided by norm x bar square plus 1 over sigma h square, a very simple expression therefore what we have is; in fact we had spent quite some time to simply the complex expression we had to arrive at this 6 simple expression. And this simple expression we already also set is very insight full because it yields several insights it is a linear it is a harmonic mean of the ML the mean square of the ML estimate and also the prior variance.

So, now I substitute the quantities sigma square is 2 divided by norm x bar square this is 14 plus 1 over sigma h square this is the prior variance which is 1 this is equal to 1 divided by 7 plus 1 which is equal to 1 divided by 8. And this is your expected value of magnitude h hat minus h whole square this is the minimum mean squared error. And this is basically your minimum mean squared error or the MMSE estimate of the channel coefficient h of the fading channel coefficient h for this given scenario.

Now we have shown an example we have completed this example or we have illustrated how to compute this minimum mean squared error for the estimation of the complex channel coefficient h. Now let us look at some other properties of this complex estimate on the complex fading channel coefficient h. Now we now that h is complex channel coefficient therefore it has a real part and an imaginary part.

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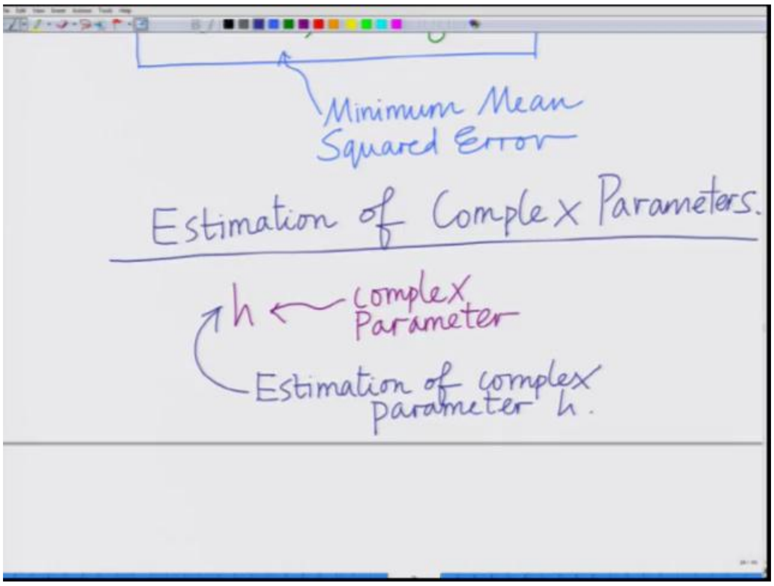

So, what we want to see is going to characterize the properties of this complex estimate all right try to understand a better since so estimation. So, let us title this section as estimation of complex parameter. So, what we want to do is going to consider estimation of a complex parameter. Remember we said this coefficient this h we have trying to estimate is a parameter. Now, because it is complex in nature it becomes complex parameters, so considered the estimation of a complex parameter h.

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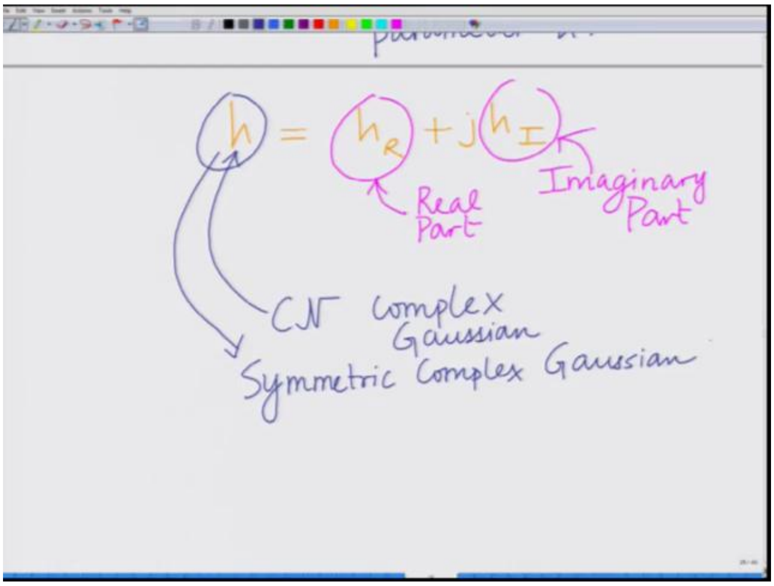

Now since h is complex correct h will have naturally since this is complex this is going to have real part and also on imaginary part. This has both real and imaginary part. So, I can represent h has h equals h R plus j times h I, where h R this is the real part of course everyone this is fairly obvious this is your real part and this is your imaginary part.

And naturally now we are going to assume this h is complex Gaussian that is what we are saying, so h is complex Gaussian. In addition we are also going to assume that zero mean h is the symmetric complex Gaussian quantity I am going to explain it; h is a symmetric is not just complex Gaussian symmetric complex.

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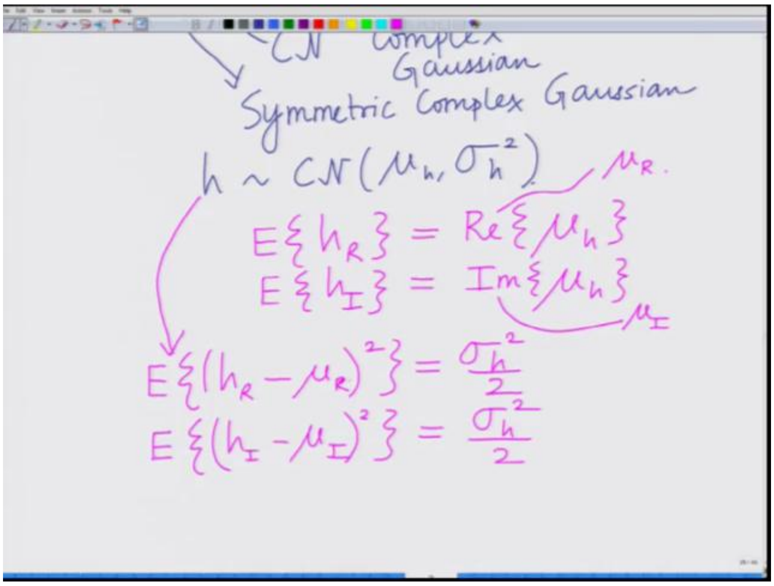

Symmetric complex Gaussian in the sense that if h is complex Gaussian with mean mu h and variance sigma h square then it satisfies the pro following properties, the real part of course it is symmetric, therefore the real part the mean of the real part equals the mean of the imaginary part correct mean of the real part equals mean of the imaginary part which is basically equal to half of the complex which is basically they mean of sorry, so we have the mean of the real part which is equal to the real part of the mean for instance h expected value of h R equals the real part of the mean. This is of course obvious expected values of the imaginary part equals the imaginary part of the mean.

Further, if you look at the variances of the real part and imaginary part that is if you look at expected value of h R let us call this has real part of the mean let us called this as mu R and this let us called this as mu I they imaginary mean, so h R minus mu R whole square equals sigma h square by 2 that is it has half the variance and imaginary part also has half the variance. So, it is symmetric complex Gaussian, therefore the mean of the real part of h R is equal to the real part of mean real of part of mu h the mean of the imaginary part is basically the imaginary part of the mean and the variances of the real part and imaginary part are both they are both equal and they are both equal to sigma h square divided by 2. We are going to assume further the real and the imaginary part are uncorrelated. In this case since they are both Gaussian they are also independent.

So expected value of h R minus mu R times h I minus mu I equal to 0, which means these h R comma h I these quantities h R comma h I these are both Gaussian and each has variance equals half sigma h square. Since they are Gaussian these are uncorrelated, but since they are Gaussian uncorrelated also implies that they are independent. Now normally for any two random variables which are uncorrelated does not mean that there independent, but in this case since h R and h I the real part is Gaussian the imaginary part is Gaussian they are both also uncorrelated. So, naturally means that in this case they are both independent, so that is known as a symmetric complex Gaussian random variable in which the real part and imaginary part have basically identical variance and they are independent.

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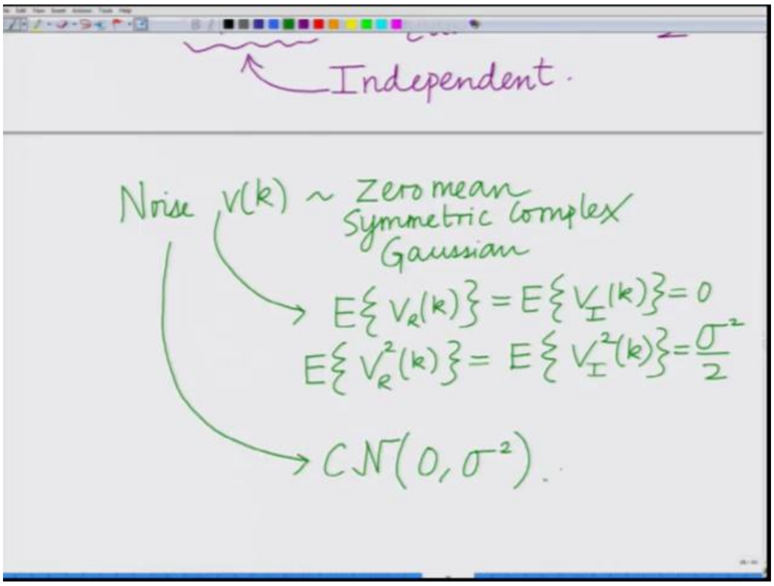

Now, let us also assume that the noise v k is zero mean symmetric complex Gaussian which also again implies that if you are the real part of the noise since it is zero mean equals imaginary part of the noise both have 0 mean and the variance more importantly expected v R square of k equals expected v I square of k equals sigma square by 2 that is both the real part and imaginary part have half the noise variance. So, the noises also zero mean. So, the noises also complex symmetric complex Gaussian in addition the noise we are saying is also zero mean. This can be represented as complex Gaussian zero means variance sigma square.

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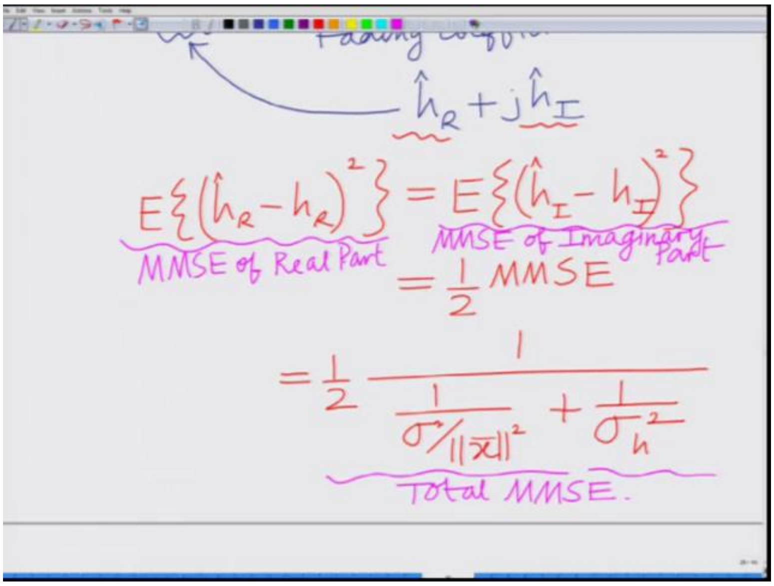

Now in this case let h hat be the estimate of the complex fading coefficient. Now, naturally since h hat is the estimate of the complex quantity h hat is also going to be a complex quantity therefore, h hat will also have a real part and an imaginary part; that is real part of the estimate and imaginary part of the estimate. So, I can represent this has h hat of R plus j h hat of I h hat of R plus j h hat of I.

And now we can show that the MSE or the minimum mean squared error of each of this part the real part and imaginary part that is if you calculate these MMSE is h hat of R minus h of R whole square, this will be equal to the MMSE of the imaginary part that is h hat of I minus h of I whole square and these will be equal to half the total minimum mean squared error. And we know what is the minimum mean squared error, so this will be a half of 1 over when 1 over 1 over sigma square divided by norm x bar square plus 1 over sigma x square. So, this is the MMSE of the real part, this is the MMSE of the imaginary part and this is your actual MMSE or you can also call this as the total MMSE.

So, what we are saying is that the MMSE of the real part and the MMSE of the imaginary part that is we treat the real estimate of the real part separately estimate the imaginary part separately we consider the real estimate and the imaginary estimate separately; that each of the MMSE of each of them will be half the total MMSE that we have derived before.

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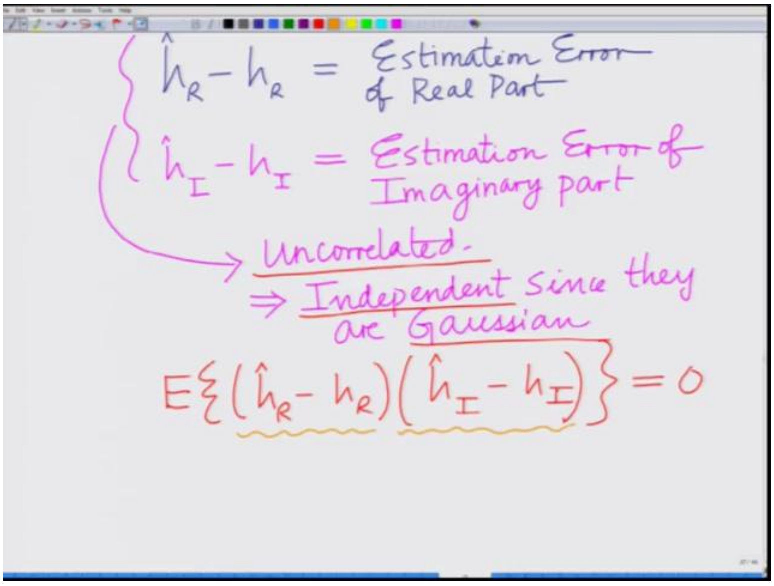

And more importantly the estimation errors are going to be uncorrelated that is if you look at the estimation error of the real part h hat R minus h R equals estimation error of real part h hat. And then we also have h hat I, this is equal to estimation error of the imaginary part. And these two things these are uncorrelated, for a general scenario these are uncorrelated.

However since both of these are Gaussian we have h R Gaussian h I is also Gaussian the estimates h hat R h hat I also Gaussian since this is the Gaussian estimation scenario, therefore the estimation errors are also Gaussian and these since they are uncorrelated they are also independent. Implies also in they are independent since I have to quality this statements since they are this is not always true because they are independent. So that means, that basically what we have is h hat that is if you look at this h hat R minus h R into h hat I minus h I this is going to be equal to 0.

Basically the estimation errors are uncorrelated or in this case also independent since they are Gaussian. So, this is the property that the estimation errors; this is the estimation error of the real part, this is the estimation h hat R minus R h R estimation error of the real part h hat I when is h I is the estimation of the estimation error of the imaginary part. And therefore, the excepted value of h hat R minus h R times h hat I minus h I is equal to 0. So, that is the property that we have. So, basically this gives you some more insight into the estimation of a complex channel coefficient what we have said is basically the same principle. And that is what we have already said even during the derivation of the MMSE estimate of the complex fading channel coefficient, but just by replacing the transports by the hermitian we can say that the principle or the formula of MMSE estimate for the fading channel coefficient h can be extended to that of estimation of a complex fading channel coefficient.

Further, we have said that this expression that we have derived for the MMSE the minimum mean squared error can also be extended in the straight forward fashion except realizing that the MMSE is now going to be not just expected value of h hat minus h whole square, but expected value of magnitude h hat minus h square. And what we are said is if we consider all the Gaussian random variables to be complex Gaussian symmetric complex Gaussian which means the real part and imaginary part are have equal variances the noise is zero mean further the real part and the imaginary part are uncorrelated or basically since they are Gaussian independent.

Then what we have seen is that the estimate the variance of the estimate of the real part is equal to the variance of the estimate of the or the MMSE of the estimate of the real is equal to the MMSE of the estimate of the imaginary part is equal to the half the total MMSE. And also the errors estimation errors of the real part and imaginary part are uncorrelated being Gaussian in this scenario they are also independent. So that gives you more insights. So, that is an (Refer Time: 28:27) slide modification we have can extend the principles and the results of estimation of real parameters to that for a complex parameter.

And therefore, we are not going to elaborate consider the specific kind of rule for the estimation of complex parameter or the MMSE for the estimation the complex parameter in future modules, but it is expected that you understand at based on this intuition that the variance of the real and imaginary parts are going to be half the total MMSE for a symmetric complex Gaussian scenario and also the estimation errors of the real and imaginary parts are independent for a Gaussian estimation scenario.

Thank you, thanks very much.