Bayesian/MMSE Estimation for MIMO/OFDM Wireless Communications
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Lecture – 12
Reliability of Minimum Mean Squared Error (MMSE) Estimate – Part II

Hello. Welcome to another module in this massive open online course on Bayesian MMSE Estimation for Wireless Communications. In this mode we have started looking at developing a frame work to characterize the reliability of the MMSE estimate. And this can be characterized as basically the finding the number of samples such that with probability greater than 0.9999 the difference between the estimate h hat and the two parameter h should be less than or equal to sigma over 2.

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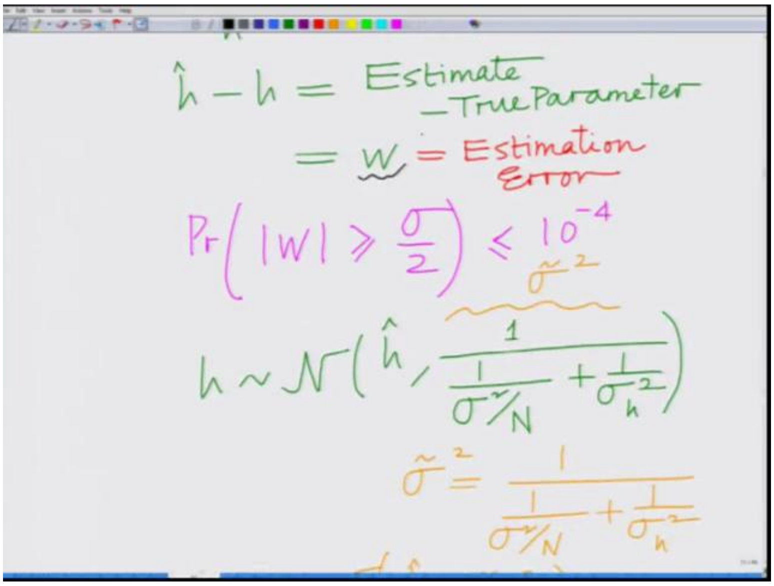

And we are said at this mean square estimation error that is if you look at this mean square estimation f you look at this estimation error that is W it is Gaussian with mean 0.

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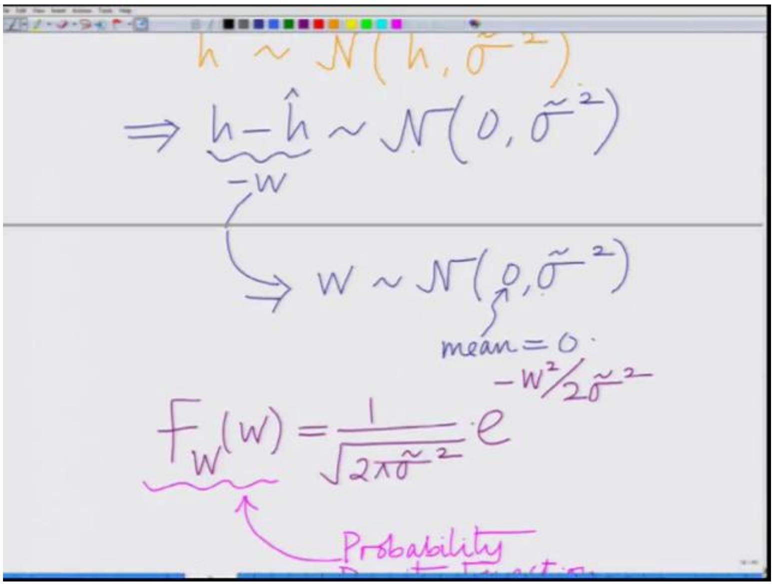

And variance sigma tilde square therefore its probability density function is 1 over 2 square root of 2 pi sigma tilde square e raise to minus W square by 2 sigma tilde square.

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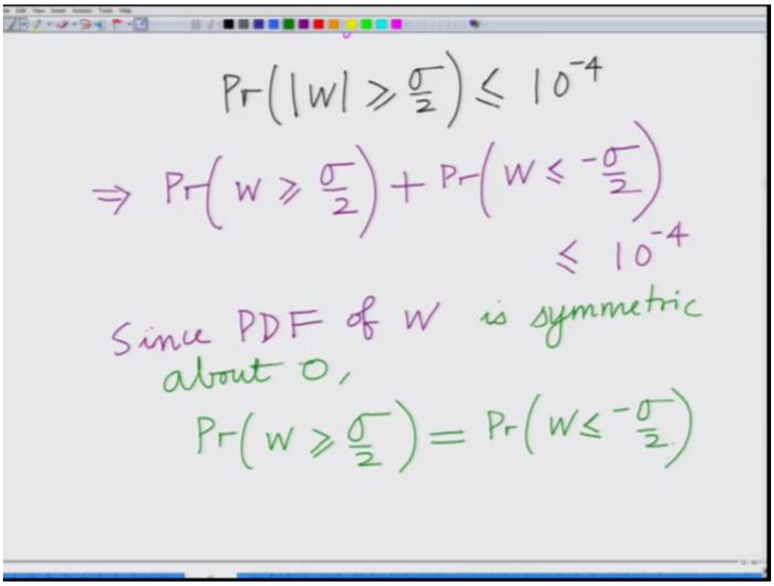

And we need to have the condition probability magnitude W greater than or equal to sigma by 2 must be less than or equal to 10 to the power of 4 this is what we already seen. Now the probability magnitude W greater than or equal to sigma by 2 this implies either W is greater than or equal to sigma by 2 plus W is less than or equal to minus sigma by 2. And the sum of the probability is corresponding to both this sinuous must be less than or equal to 10 to the power of minus 4.

Now, if you can see basically this W is a Gaussian random variable so it is probability density function is symmetric about 0, therefore the probability that W is greater than or equal to sigma by 2 is the same as the probability that W is less than or equal to minus sigma by 2. Now probability this therefore we have since PDF of W is symmetric about 0 therefore, the probability that W is greater than or equal to sigma by 2 equals the probability that W is less than or equal to minus sigma by 2.

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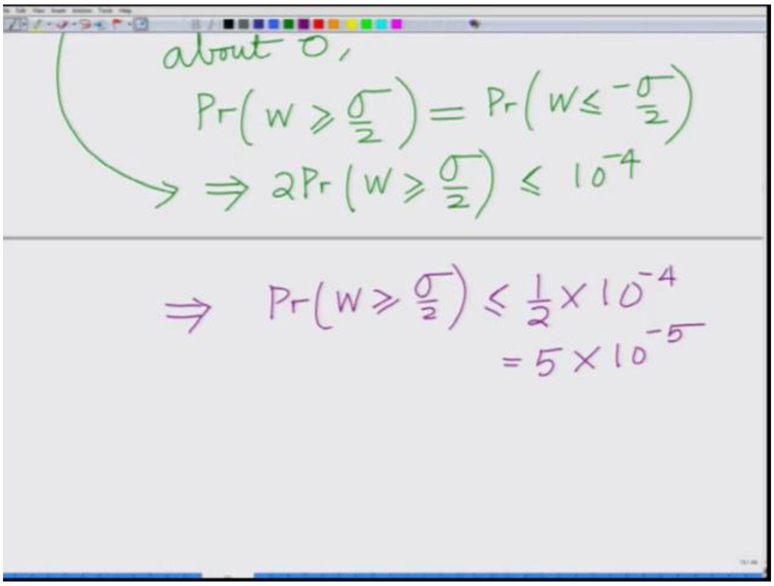

And therefore, the probability this basically you can simply during this property the twice the probability W is greater than or equal to sigma by 2 has to be less than or equal to 10 to the power minus 4 taking this factor of 2 to the other side, this implies that the probability W is greater than or equal to sigma by 2 has to be less than or equal to half into 10 to the power of minus 4 equals 5 into 10 to the power of minus 5.

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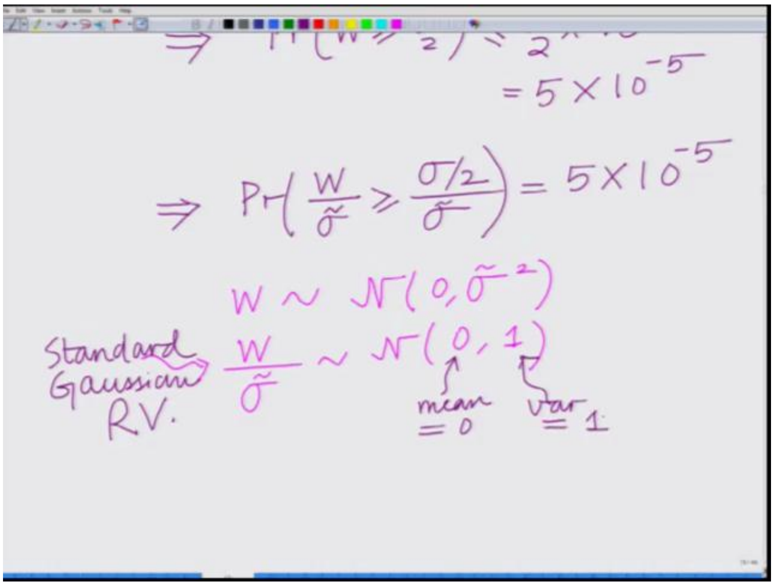

Now, what I am going to do is I am going to do something interesting, and I am going to divide this by probability of W divided by if W is greater than sigma by 2 W divided by sigma tilde is greater than or equal to sigma by 2 divided by sigma tilde which is equal to 5 into 10 to the power of minus 5. Now observe this quantity W is a Gaussian random variable with mean 0 and variance sigma delta square, therefore W divided by sigma tilde is a Gaussian random variable with mean 0 and variance 1, and this is the principles that we are going. W is Gaussian random variable with mean 0 variance sigma tilde square implies W divided by sigma tilde is Gaussian random variable with mean 0 and variance 1; this is also known as a standard Gaussian random variable. This is also known as a standard Gaussian random variable because mean equals 0, variance equals 1.

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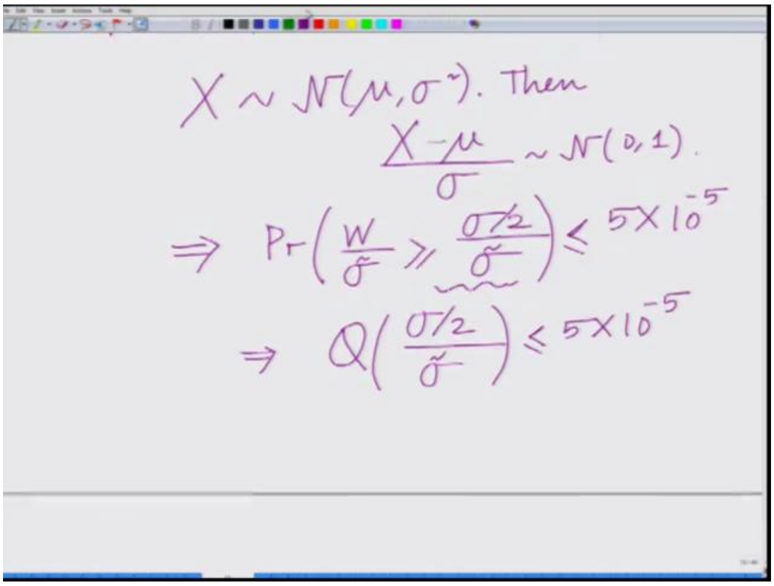

For any Gaussian random variable X is X has mean mu variance sigma square, then X minus mu divided by sigma is also Gaussian and this has mean 0, variance 1; that is a standard Gaussian random variable. So, W already has a mean of 0, so W divided by sigma tilde has mean 0 and variance 1. Now we have what is the probability, so we have implies the probability W divided by sigma tilde greater than equal to sigma by 2 divided by sigma tilde that has to be less than or equal to 5 into 10 to the power of minus 5.

And this function this quantity the probability that the unit norm the standard Gaussian random variable with mean 0 (Refer Time: 06:33) is greater than or equal to any quantity X is given by Q of X, so this implies this Q of (Refer Time: 06:45) of sigma by 2 divided by sigma tilde is less than or equal to 5 into 10 to the power of minus 5.

Where what is the Q function, let me define the Q function.

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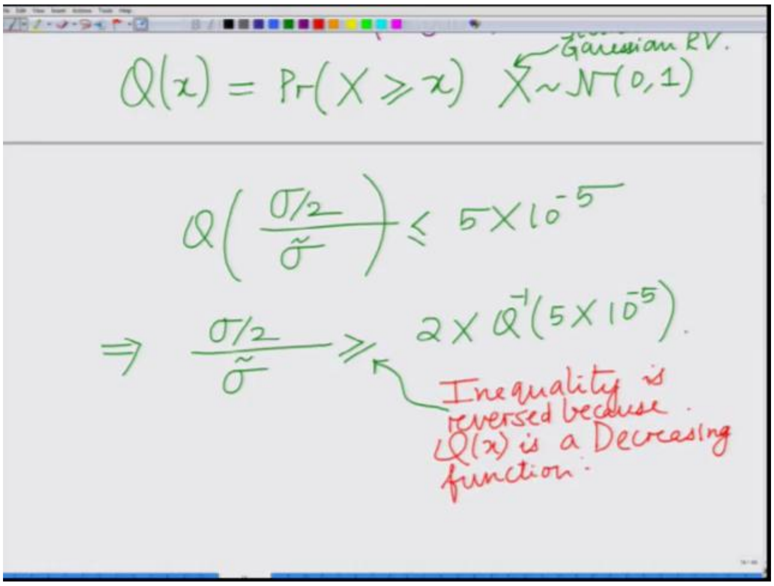

Q of x equals the probability that X greater than or equal to this quantity x, where X is the standard Gaussian random variable; that is x is a standard Gaussian random variable mean 0, variance 1. We are saying the probability that W divided by sigma tilde is the standard Gaussian random variable is greater than or equal to sigma by 2 divided by sigma tilde that is Q of sigma by 2 divided by sigma tilde.

Therefore, we need this quantity Q of sigma by 2 divided by sigma tilde this quantity has to less than or equal to 5 into 10 to the power of minus 5 which implies sigma by 2 divided by sigma tilde this has to be greater than or equal to twice into Q inwards of 5 into 10 power minus 5. You can see inequality is reverse the lesser than becomes a greater than because Q function is a deceasing function, inequality is reversed inequality is reversed because Q of x is this is the decreasing function.

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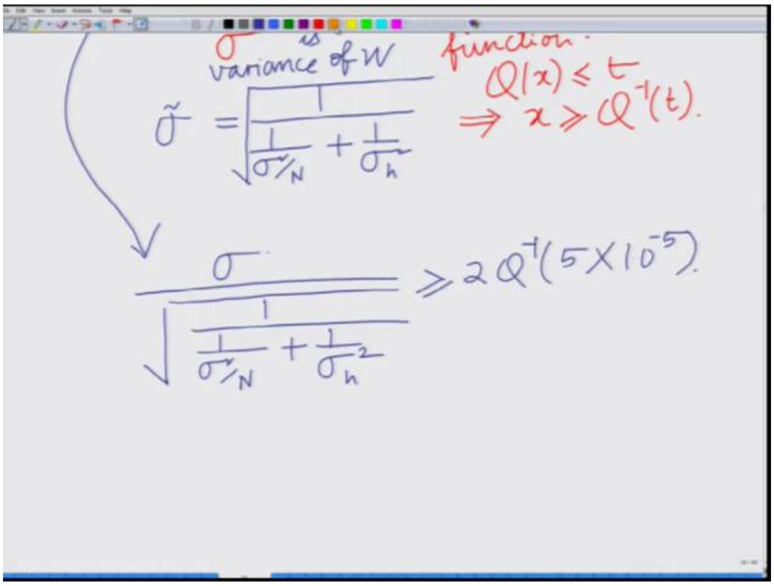

Therefore, Q of x is a less than or equal to t implies x must be greater than or equal to Q inverse of t that is a principle that we have used because the Q of x is decreasing function that is why we need to take the Q inverse to the other side the inequality gets reversed. Basically that implies sorry, this 2 will not be here because you taking this 2 to the other side.

Now let us substitute this explanation for sigma tilde sigma tilde is the square root of the variance that is the standard deviation of the estimation (Refer Time: 09:39) remember sigma tilde square remember where we got the sigma sigma tilde square is variance of W and sigma tilde square equals 1 over 1 over sigma square by N plus 1 over sigma h square. So, what this implies, basically this implies your sigma divided by sigma tilde which is square root, so sigma tilde square equals this which means sigma naturally equals to square root of this 1 over square root of 1 over 1 over sigma square by N plus 1 over sigma h square this has to be well greater than or equal to twice Q inverse of 5 into 10 to the power of minus 5.

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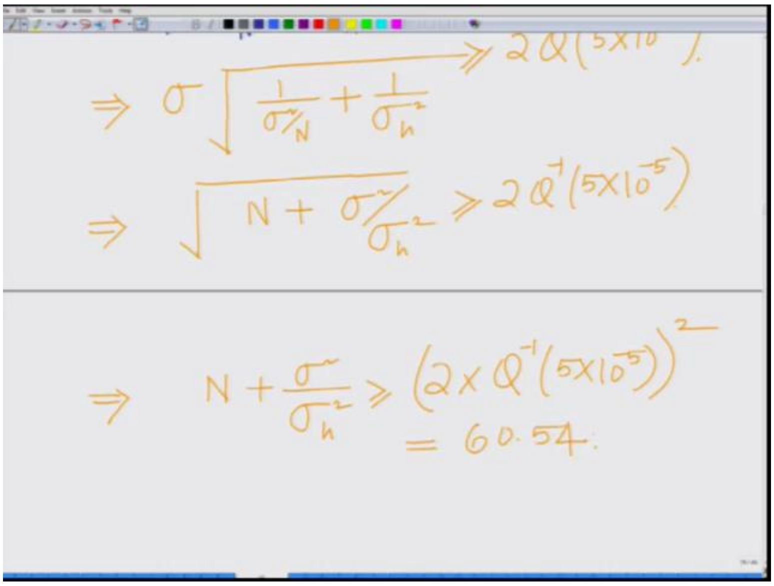

This implies that sigma times square root of 1 over sigma square by N plus 1 over sigma h square should be greater than or equal to twice Q inverse 5 into 10 to the power of minus 5 now take the sigma inside this implies square root of, well you take the sigma inside what you have is N sigma square divided by sigma square divided by N that is N plus sigma square N plus sigma square divided by sigma h square must be greater than or equal to 2 twice Q inverse 5 into 10 power minus 5.

This implies N plus sigma square divided by sigma h square greater than or equal to twice into Q inverse 5 into 10 power minus 5 whole square. you can evaluate this using either a computer or using the Q function tables you can evaluate this, so this let say you evaluate this I am leaving this you.

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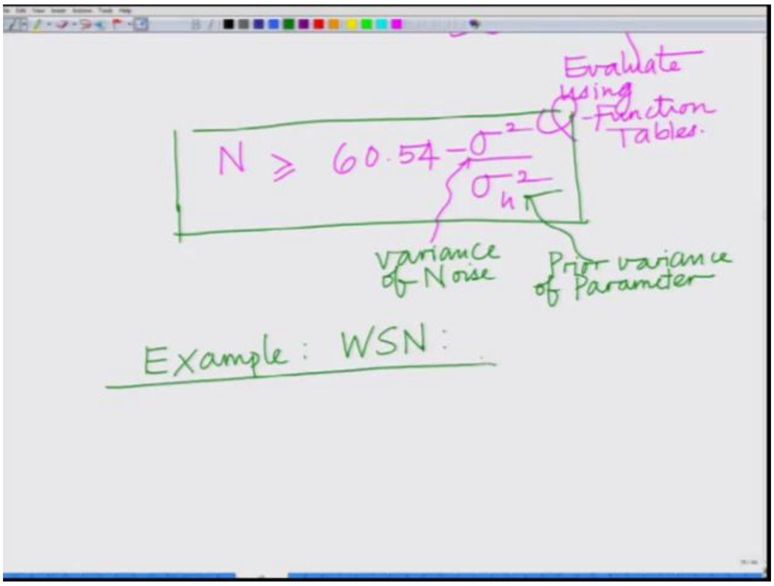

You can this evaluate using the for instance you can use MATLAB or Q function tables etcetera. So you can either use MATLAB or a Q function table that is printed by use of the values of the Q function for various values of the argument you can use it evaluate this constant and that will be 60.54. And therefore, what you have is N must be greater than or equal to 60.54 minus sigma square divided by sigma h square. And what is sigma square sigma square equals that is remind you once again sigma square equals variance of noise and sigma h square this is the prior variance of the parameter; sigma square divided by sigma h square and this is the prior variance of the parameter and now we have derived or number of samples that is required.

And remember let me explained you this number of samples that is required is basically the number of samples that is required let if you go all the way back and you recall what we were deriving this is basically the number of samples N that is required such that the probability h hat lies within sigma by 2 that is it lies between that is h lies between h hat minus sigma by 2 2 h hat plus sigma by 2; that is h lies between the sigma by 2 ball of the true parameter h probability with which it lies in the sigma by 2 ball around h has to be greater than or equal to 0.9999 or with 99.99 percent reliability that is how we have derived. And now you derived that this number of samples N that is required for this purpose is N equal to 60.54 minus sigma square divided sigma h square. Let us look at the simple example to understand this.

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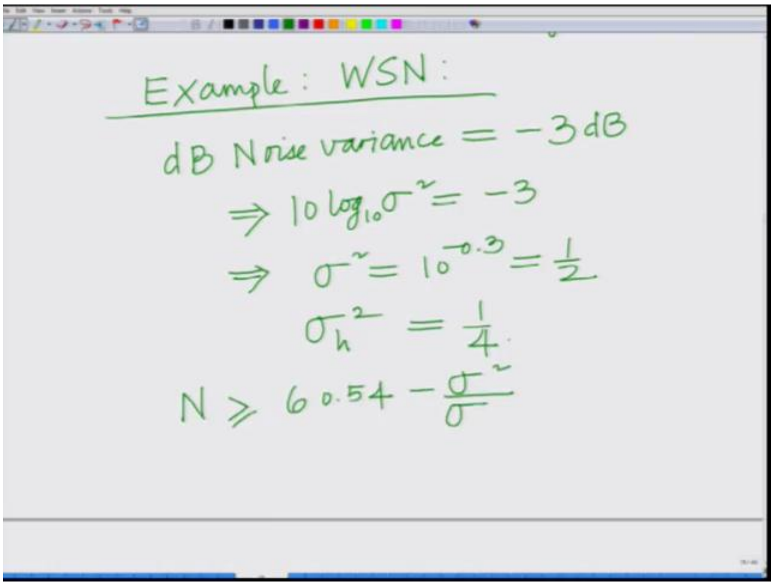

Again let us go back to WSN; let us go back to our wireless sensor network example. Remember we had the dB noise variance equals minus 3 dB right, implies 10 log 10 sigma square equals minus 3 implies sigma square equals 10 to the power of minus 0.3 that is equal to half. We also have the prior variance sigma square h square equal to 1 by 4 this is the prior variance. Therefore, we derive that N is greater than or equal to 60.54 minus sigma square by sigma h square as we derived above.

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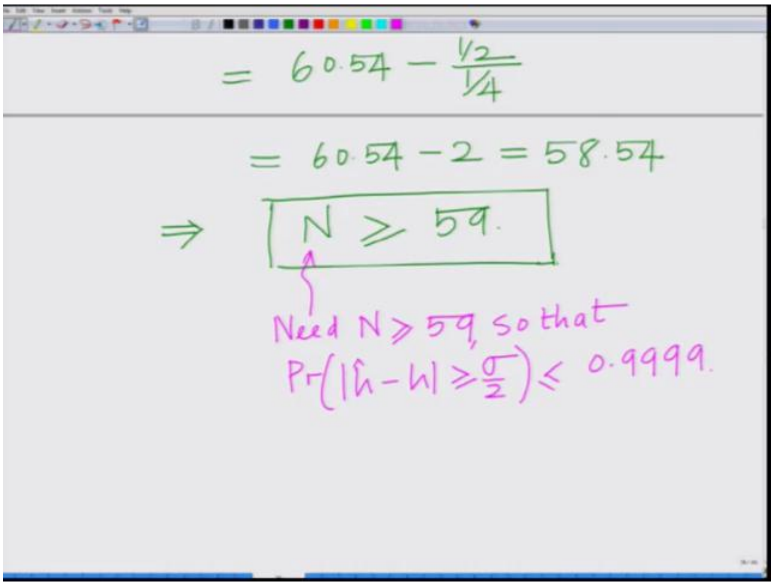

That is basically your 60.54 minus half divided by sigma h square which is 1 by 4 which is basically half divided by 1 by 4 is to so this is 60.54 minus 2 that is 59.0 that is basically your 58.54 which implies that N has to be greater than or equal to implies N has to be since we have an integer number of samples N has to be greater than or equal to 59 samples basically that is what we have derived, so need N greater than equal to 59.

So, that summarize this again we need N greater than or equal to 59 so that probability magnitude h hat minus h so that the probability h hat minus h greater than or equal to sigma by 2 this is less than or equal to basically 0.9999, which means it lies inside the sigma by 2 ball with probability greater than or equal to 1 minus 0.9999 that is as we saw 10 to the power of minus 4.

Now we have a neat framework and arrogate framework to basically characterize the reliability of the estimate. We are seen as the number of samples N increases the variance of the estimate the MSC mean squared error of the estimate decreases, means also the same the mean square error in this case the also same as the variance of the posturing density of the parameter h. Which means as the variance is decreasing remember the mean is h hat all right, the variance is the spread around the mean h hat which means as the variance is decreasing the spread around this mean h hat is decreasing. Which means in probabilistically speaking the true parameter h closer and closer (Refer Time: 18:37) that is with the very high probability with a very very high degree of certainty that two parameter h is very close to the estimate h hat.

Now to characterize that pressingly how many samples N that are required to achieve a certain degree of certainty or a certain degree of reliability that is what we have derived in this module so far. And we have shown that for all simple wireless sensor network example we have shown that if we required such reliability then we need N greater than or equal to 59 (Refer Time: 19:09).

And you can carry out this analysis for another examples with other values of sigma s square sigma h square and also more importantly with other desired reliability criteria. For instance, we have assumed a reliability criterion of 99.99 percent you can make it 99 percent or ninety nine 0.9999 percent and so on. You can explore how this answer varies with these varying degrees of reliability. So, basically this is the frame work which shows you how to choose the number of samples for MMSE estimate, number of samples in the MMSE estimation process so I as to achieve desired degree of accuracy and reliability.

So, we will stop this module here and continue with other aspects in the subsequent modules.

Thank you.