Bayesian/MMSE Estimation for MIMO/OFDM Wireless Communications
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Lecture – 11
Reliability of Minimum Mean Squred Error (MMSE) Estimate – Part I

Hello, welcome to another module in this massive open online course on bayesian MMSE estimation for wireless communications. So, far we have looked at the MMSE estimate. And we also look at how to compute the varience or how to commute the mean squared error of the estimate of the MMSE estimate. Now let us look at a new frame work, to calculate the reliability of the MMSE estimate or basically calculate the number of samples or number of observations required to achieve a certain degree of reliability in the MMSE estimation, or in estimation process.

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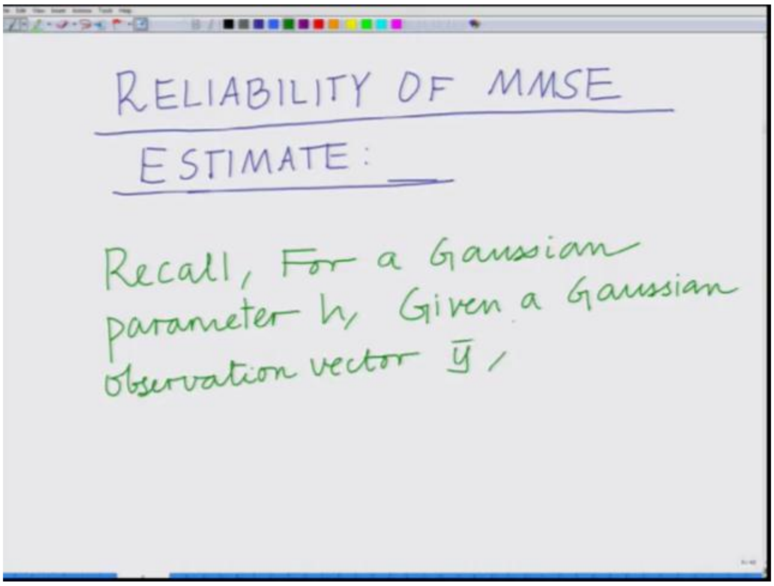

So, today we are going to focus on basically a frame work to characterise the reliability of the MMSE estimate that we have thus for computed. So, reliability of the MMSE estimate, and recall that; recall for a Gaussian parameter h, Gaussian parameter h given a Gaussian observation vector y bar, Gaussian says that h comma y bar are jointly Gaussian.

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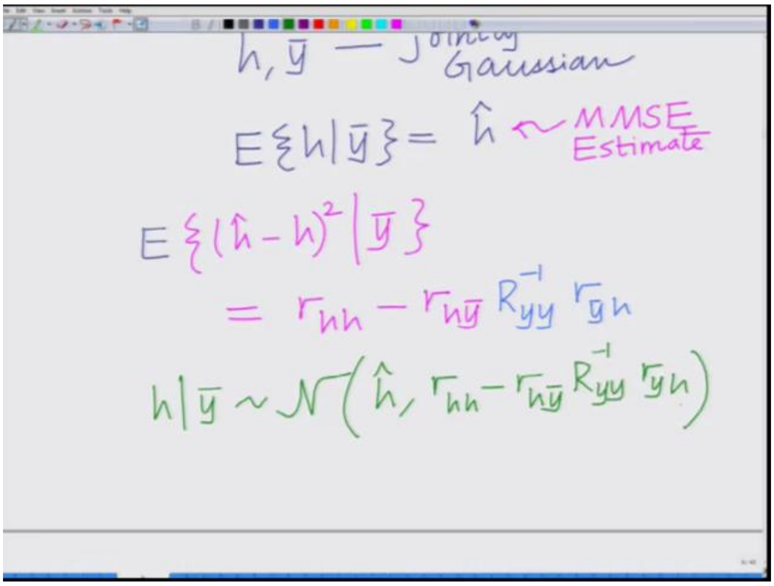

Then what we have is basically. So, also this also important we have h comma y bar this has to be jointly Gaussian. Then we have expected value of, h given y bar alright this is the MMSE estimate equals, h hat infact we have shown that this is the MMSE estimate infact in the very first few lectures, we have derived the expression for the MMSE estimate we have shown that the MMSE estimate is nothing, but the expected value of the parameter h given the observation factor y bar that is the expression for the MMSE estimate.

Further the MMSE the mean squared error is, expected value of your h hat minus h whole square given y bar I am not going to write this explicitly because the estimate is calculated only a conditioned on the oberservation factor y bar. This is basically your Rhh minus Rhy bar expected value the cross co-varience expected value of h times y bar co-varience inverse of the co-varience matrix times r, y bar h. So, what we have therefore, is basically that your h given y bar, this is Gaussian correct this has a Gaussian distribution with mean h hat given by the MMSE estimate and varience given by nothing, but the mean squared error that is your Rhh, Rhy bar, Ryy inverse Ryh this is also known as the Posterior Probability Density Function.

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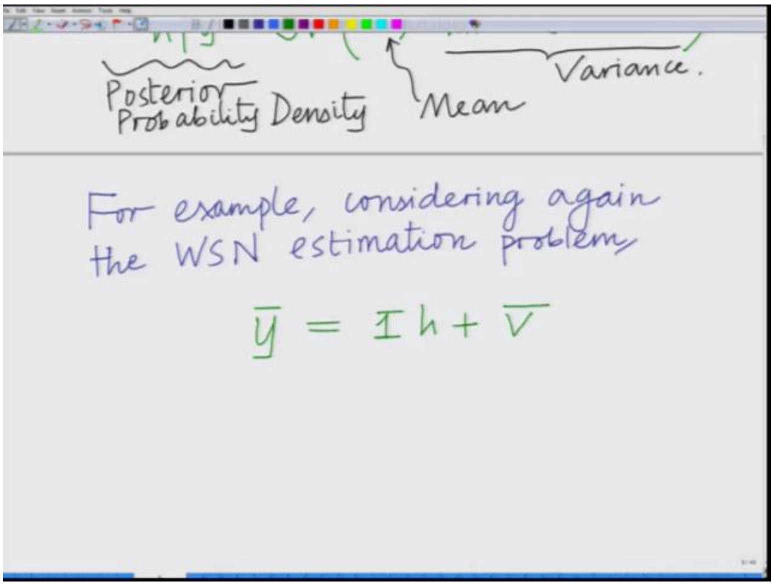

We have the prior this is also known as the posterior probability density the mean of the posterior probability density, is basically youRhhat and this is basically the varience. So, basically what this say is given the observation vector y bar the true parameter that is a given observation vector y bar the true parameter h correct it lies it distributed as a Gaussian conditioned on the observation vector y bar it is distributed as a Gaussian, with the mean centered at h hat that is expected h given y bar is h hat and the varience is given by infact the mean squared error that is Rhh minus Rhy bar types Ryy inverse into Ry bar h correct.

For example, consider or wireless sensor network estimation problem. For example, cosidering, again our w s in the WSN senario, correct? The WSN estimation or the wireless sensor network, the wireless sensor network problem we have that is what is our wireless sensor network the WSN estimation problem we call that the WSN estimation problem is y bar equals 1 bar that is the indimentional vector of all ones 1 times h plus v bar where v bar additive might Gaussian noise.

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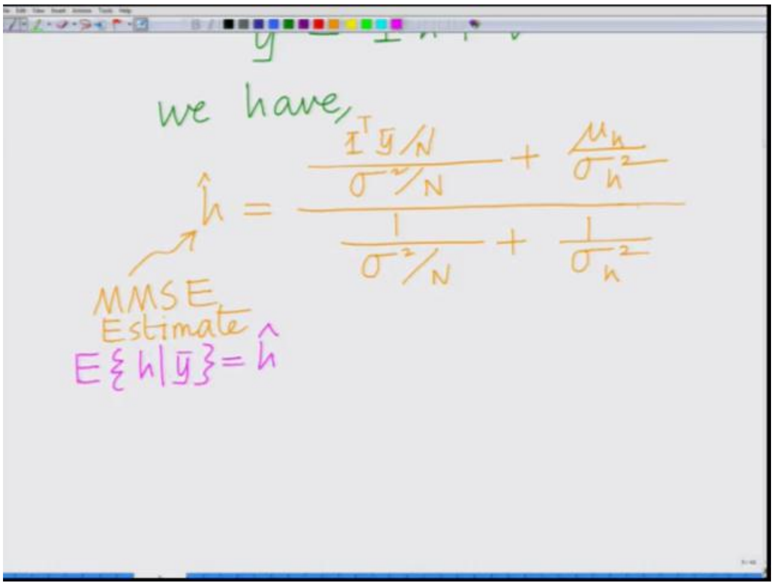

We have the estimate recall that the estimate h hat is 1 bar transpose y bar devided by n by sigma square devided by n plus the prior mean mu h devided by sigma square h square devided by 1 devided by sigma square n plus one devided by sigma h square. This is the estimate this is the MMSE estimate, this is also the mean this is the your MMSE recall that this is the MMSE estimate and this is also the mean this is also your posterior mean that is expected of the expected value of h given y bar this is also equal to h hat.

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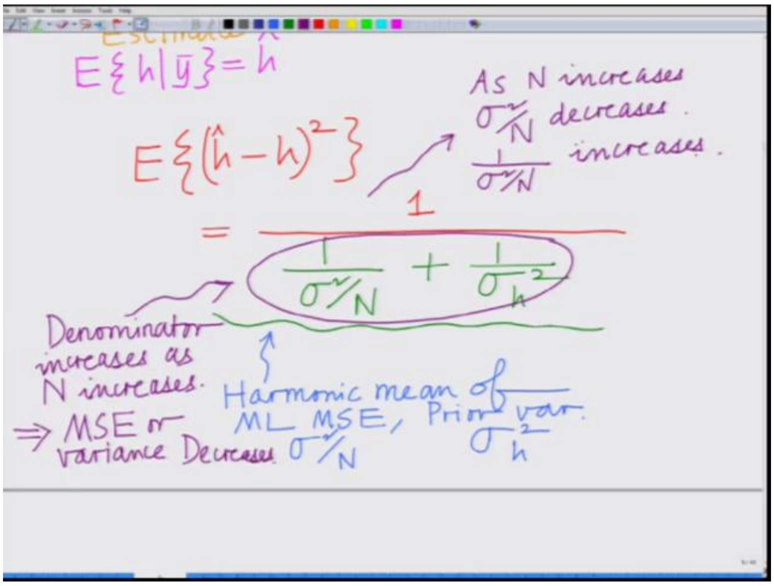

And further the varience noise if you look at, the varience that is the mean squared error in this case is nothing, but the variance of the posterior that is expected value of h hat minus h whole square this is equal to and we have derived to expression of this that is one over sigma square devided by n, plus 1 over sigma h square and you can see this is you have derived in the previous module this is the harmonic mean, harmonic mean of the M1 MSE that is M1 mean squared error comma prior varience, that is sigma h square. M1 mean squared error that is sigma square devided by n and the prior varience that is sigma h square. This is the harmonic mean of the mean squared error of the M1 estimates and prior varience and therefore, now you can also observe an intresting thing that has n increase. Now let us look at the behaviour of this, as n increases observed that sigma square over n decreases. Therefore, 1 over sigma square over n increases. Therefore, the denominator increases as n decreases implies MSE or varience decreases.

So, easy to keep it is important to keep that in mind what we are saying is as n increases this mean squared error decreases alright and remember the mean squared error is also the varience of the posterior density function, which means remember it is an posterior probability density function is a Gaussian . So, as the varience as the n is increasing this varience is decreasing which means a the true parameter h correct a comes closer and closer in a probabilistic sence to the mean which is the estmate because, you can see that the varieance is decreasing as a function and. So, even we cannot say that it is always going be close the true parameter h is always going to be closed to the estimate, MMSE estimate h hat with the very high probability that is to a very high degree of certainity the true parameter of h hat is the true parameter of h is very close to the MMSE estimate h hat.

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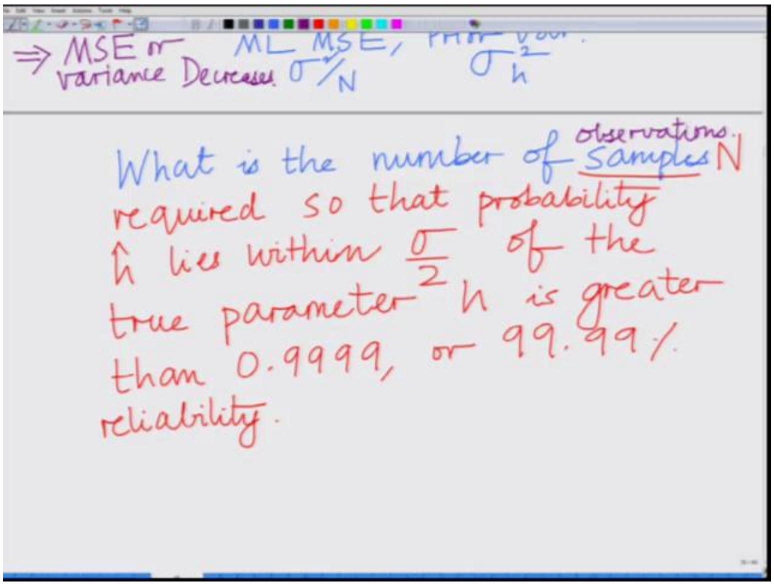

We are going to see that slightly later a using a pictorial discription of this right. So, the variences is decreasing. Now one can therefore, ask the question what is the number of samples required for certain degree of reliability and we are going to charaterises this. what is the number of samples required what is the number of samples required what is the number of samples required. So, that the probability that h hat lies within a sigma 2 of true parameter sigma by 2 of the lies within the distance of sigma 2 of the sigma by 2 of the true parameter h is greater than is greater than 0.9999 or basically we can say in a certain sence acheiving 99.99 percent reliability.

So, we asking this question what is the number of samples n remember we are charecterizing the number of samples or basically the number of samples is basically the same as your number of observations another word for that is basically your observations, what is the number of samples or observations n required. So, that the probability that the true parameter h lies within a sigma by 2 radious of the estimate h hat or the estimate h hat lies within a sigma by 2 radious of h is greater than 0.9999 or basically we can say that we have achieved 99.99 percent reliability.

And. So, if you want to represent it pictorially if you represent it pictorially this our gaust poterior Gaussian and distribution and the mean is basically of this posterior is h hat correct.

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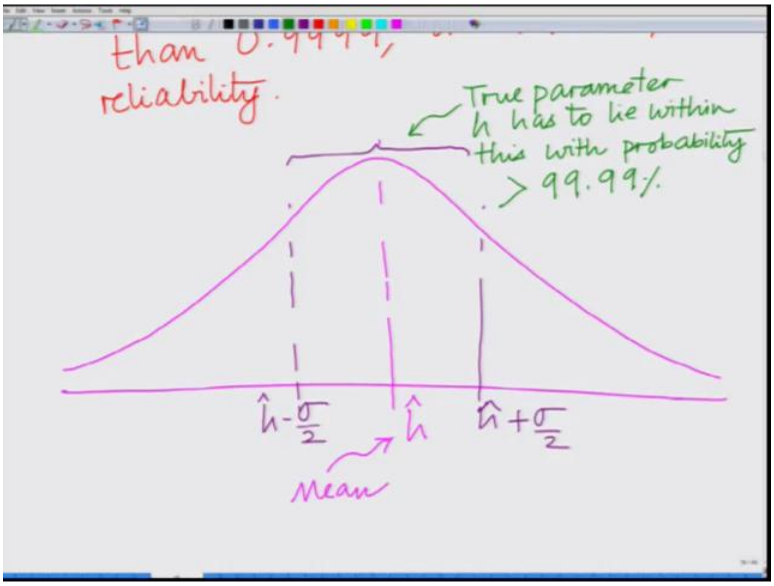

Therefore, what you can see is basically if you draw a sigma by 2 radious if we consider a sigma by 2 radious around this things for instance this is lets say your. So, this is youRhh hat plus sigma by 2 and this is h hat minus sigma by 2. So, the true the true paramete Rhhas to lie within this sigma by 2 radius. So, true parameter h has to lie within this sort of box with probability greater than with probability greater than 99.99 percent.

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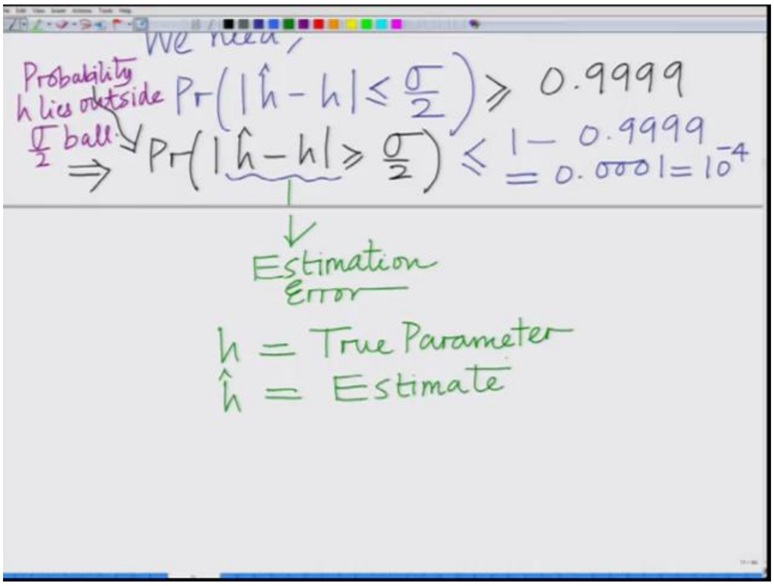

What we are saying is that therefore, a this age has to lie within this sigma by 2 that is h hat plus sigma by 2 to h hat minus sigma by 2 with probability greater than a 0.9999 or 99.99 percentage percent reliability and you can now see that as n is increasing this varience is decreasing alright.

So, basically the Gaussian become will become increasingly more and more concentrated around h hat which means the probability that it is closer to h hat keeps increasing as n is increasing alright. As n increases the probability that h hat lies within this sigma by 2 ball the probability that h lies within this sigma by 2 ball around h hat progressively keeps increasing and what is and we want to find that n such that this probability is atleast a 0.9999 or greater than basically 99.99 percent. So, that is what we have to find and we can proceed with that as followes and this is an intresting problem. What we need is basically we need for all given degree of reliability, we need that is magnitude h hat minus h less than or equal to the probability that magnitude h hat minus h is less than or equal to sigma by 2 has to be greater than 0.9999 this is what we need fine.

Now, this implies that the probability that this it lies outside this ball that is probality of magnitude h hat minus h greater than or equal to sigma. Where 2 probability lies this is the probability it lies h lies outside the ball probability h lies outside the sigma by 2 ball naturally, if h hat minus h is less than equal to sigma by 2 it lies inside the sigma by 2 ball and magnitude h hat minus h is greater than sigma by 2 it lies outside. So, the probability it lies inside is greater than or equal to 0.9999 it means that the probability it lies outside has to be less than or equal to one minus 0.9999 which is equal to 0.0001 that is basically your 10 to the power of minus 4 and that is also very simple to undserstand because, what we are saying is the probability that it lies outside the sigma by 2 ball has to be has to be greater than or equal to point a probability that lies it lies inside this sigma by 2 ball has to be greater than or equal to 0.9999. Which means the probability that it lies outside this ball has to be lessthan or equal to one minus 0.9999 which is basically 10 to the power of minus 4 that has to be the case because sum of both these probability lies inside the ball and plus probability lies ourside the ball has to be equal to 1.

So, 1 is greater than or equal to 0.9999 which means the other has to less than 1 minus 0.9999 which is 10 to the power of minus 4. So, it very simple and now look at this we have this h hat minus h now this is basically you have to realise this is nothing, but your estimation error.

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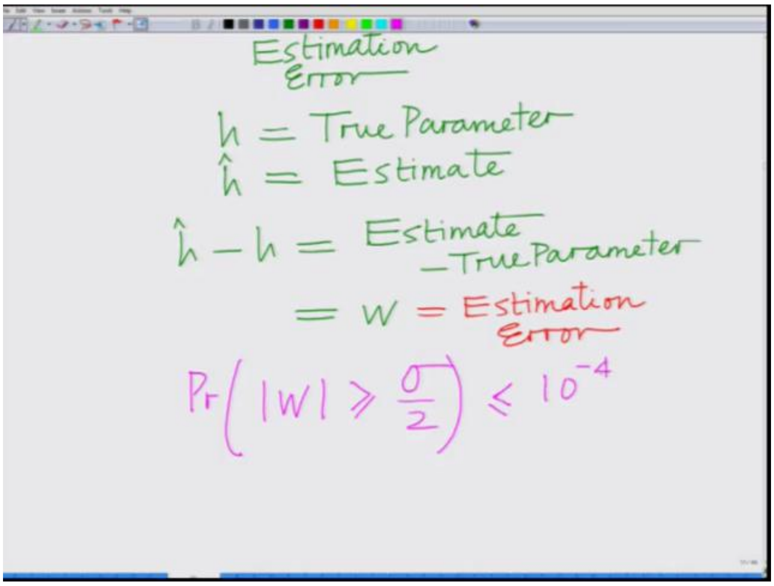

Because we have h equals equal to the true paramete Rhh at equals the equals the

0065stimate that is your MMSE estimate. So, we have h hat minus h that is the estimate we have h hat minus h equals estimate minus true parameter minus your true parameter and this is equal to w which is the estimation error. So, this is w which is the estimation error. So, basically now we are asking this question. So, basically now I can replace this h hat minus h by basically this quantity w and now basically we are asking this question that what is the probability, right? We have probability that this estimation error magnitude w greater than or equal to sigma by 2 has to be lessthan or equal to 10 to the power of minus 4.

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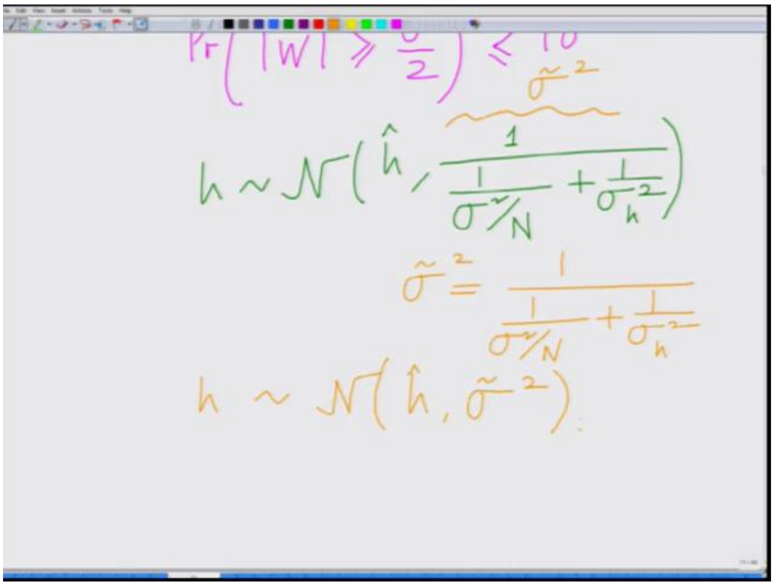

So, that is the question that we are asking now recall that your h is Gaussian distributed with mean h hat and varience one over we have already seen this for the wireless sensor network h is distributed this is the mean posterior mean h hat a posterior mean which is given by the h hat and the varience which is given by the MSE that is 1 over sigma sqaure by n plus 1 over sigma h square and therefore, what we have is we look at this now. So, now, let us denote this quantity by sigma tilde square this quantity that is sigma tilde square equals 1 over 1 over sigma square by n by 1 over sigma h square. So, I can say h is distributed the posterior distribution infact I have to say h given y bar this is the posterior distribution is Gaussian with mean h varience sigma tilde square and therefore.

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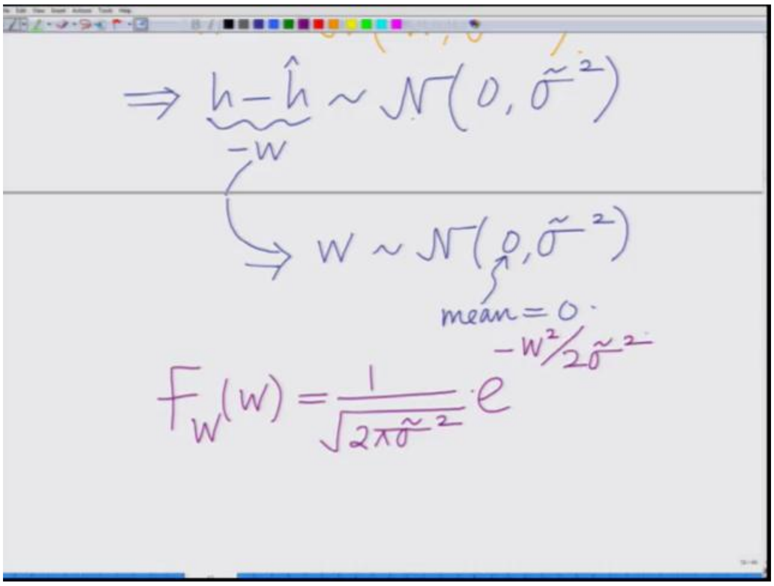

Now, this implies that if t look at this error that is if I look at h minus h hat. So, this is basically h minus h hat. So, this is from the parameter from the parameter I am subtracting the mean which means this makes this means 0. So, this is known as a Gaussian with mean 0 varience sigma tilde square. So, this is basically your h minus h hat which according to our definition is minus w.

So, what we are doing is. So, h is distributed as a Gaussian with mean h hat and variance sigma tilde square what we are doing is from now from h right we are substituting h hat we are substituting the mean. So, naturally what happens the mean becomes 0 and therefore, it is distributed as a Gaussian with mean 0 varience sigma tilde square from a Gaussian. If you substract the mean it becomes still a still remains a Gaussian, but it is a 0 mean Gaussian with the same varience. So, w minus w is Gaussian with mean 0 varience sigma tilde square now that also implies correct that also implies that w minus w is Gaussian w is also Gaussian with mean 0 variance sigma tilde square.

Since w is minus w is Gaussian with mean 0 variance sigma tilde square w is also Gaussian it means 0 variance sigma tilde square correct mean equal to 0 the mean equal 0. And therefore, now we have the probability density function of this estimation error w which is equal to basically 1 over this is Gaussian with variance sigma tilde square 1 over square route of minus 1 over square root of 2 pi sigma tilde square e raise to minus w square devided by 2 sigma tilde square this is the probability density function of w.

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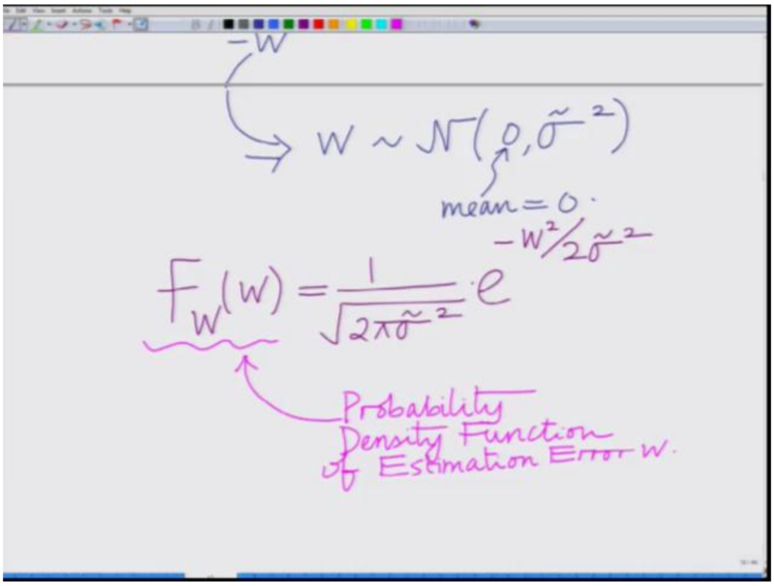

This is the probability density function of the estimation error w alright. We have the probability density function which is characterised by the Gaussian random variable with mean 0 and varience sigma tilde square. So, what we have done in this module. So, far is basically we have started devolop a frame work to characterise the reliability of a the MMSE estimate for the senario of Gaussian parameter estimation. Where h the parameter h and the observation vector y bar are jointly Gaussian, we said that the posterior probability density of h is characterised by the mean of h hat and the varience which is given by the MSE. So, what we have framed is we have framed an appropriate as a minimum number of samples required to acheive 99.99 percent reliability and now we have characterised the probability density function of the estimation error w as a Gaussian probability density function.

So, will stop this module here and complete this derivation in the subsiquent module.