Bayesian/MMSE Estimation for MIMO/OFDM Wireless Communications
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Lecture - 10
Minimum Mean Squared Error (MMSE) for Wireless Sensor Network (WSN) – Derivation and Example

Hello. Welcome to another module in this massive open online course on Bayesian Estimation for Wireless Communication Systems. So far we have looked at the principle of MMSE estimation when the observation is Gaussian and the parameter is Gaussian. And we looked at two applications of this principle in the context of one for a wireless sensor network, two for the wireless channel estimation problem.

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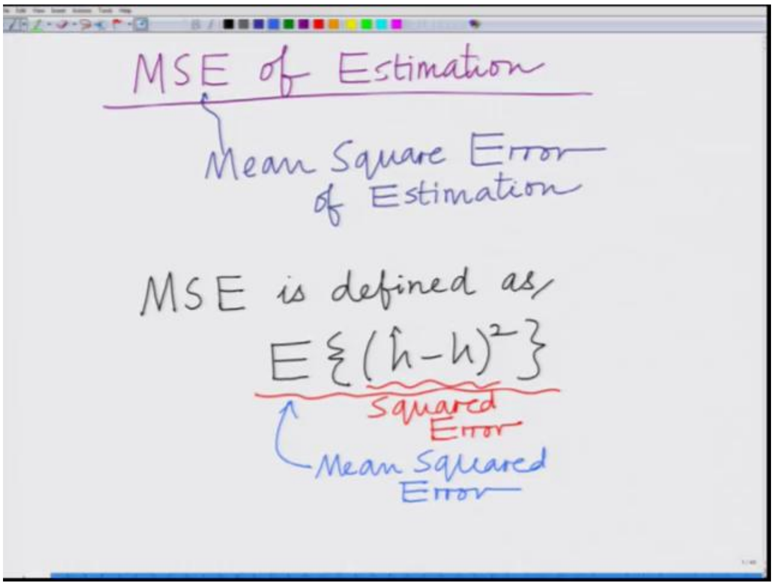

Now, what we are going to do let us complete this analysis by also looking at the mean squared error of estimation. So, what we are going to do we have not discuss this so far we are going to look at the MSE of estimation or this is the mean square error; MSE stands for the mean square error. The mean square error of estimation for the parameter h we know that the MSE is defined as the MSE the mean square error is defined as the expected value this is the mean of h hat minus h whole square; h hat minus h whole square this is naturally this the squared error.

Where, remember h hat is the estimate correct h is the underline 2 parameter so we are looking at the error which is h hat minus h we are looking at the squared error which is h hat minus h square and then getting the expected value. The expectation operator which gives us the average value of the squared error which is nothing but the means squared error.

So, this is basically your squared error. And once you take the expected value to whether with the expected value this becomes your mean squared error. And this one important parameter, when we talk about the MSE estimation or we will talk about any estimation when you want to characterize the quality of the estimate or how accurate the estimate is, so we have to characterize it in terms of the means squared error of estimation.

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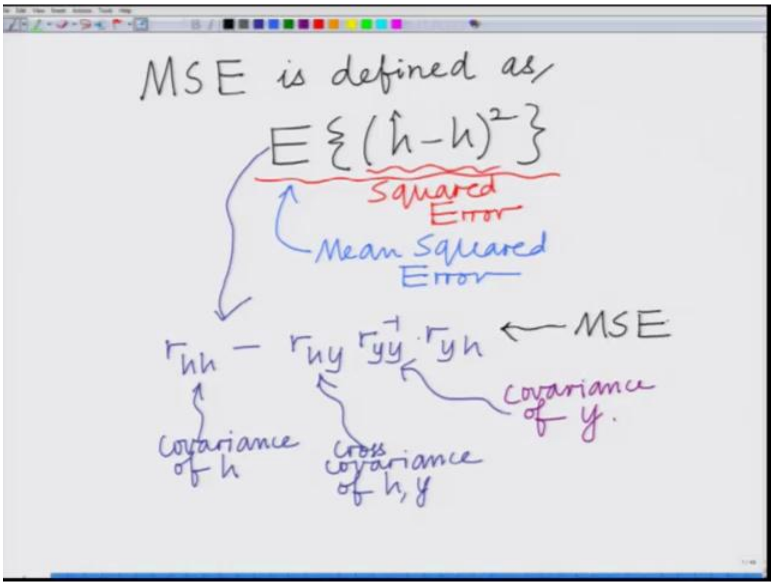

And we have also derived an expression of this the mean squared error of estimation for the MMSE estimate we have shown that the mean squared error of estimation this is equal to r hh minus r hy into r yy inverse into r hy. In our case this will also be given as alright, so this is the general expression for the mean squared error of estimation for instance let us consist of for instance this is the covariance of h, cross covariance of h comma y and this is the covariance of y or covariance matrix depends on y is the vector or a or a single scalar quantity; for our case this is the covariance of y.

So, depends on the nature of y, if y is a vector then this is going to be a covariance matrix otherwise y is scalar this is simply the covariance. This is the expression for the means squared error; so this is the expression for your MSE we have derived this expression. In fact, you have derived this expression for the scalar scenario the vector scenario and so on.

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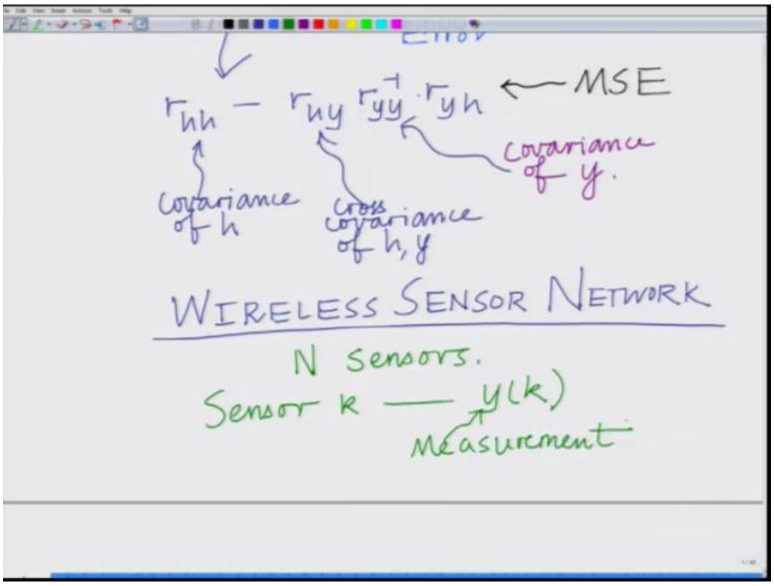

Now, let us look at it in the context of our wireless sensor network. Let us go back to our example of the wireless sensor network. Let us look at it in the context of wireless sensor network, remember in the wireless sensor network we are considering a scenario with N sensors sensor i or sensor k since the measurement y of k, this is the measurement; this is the measurement y of k correct.

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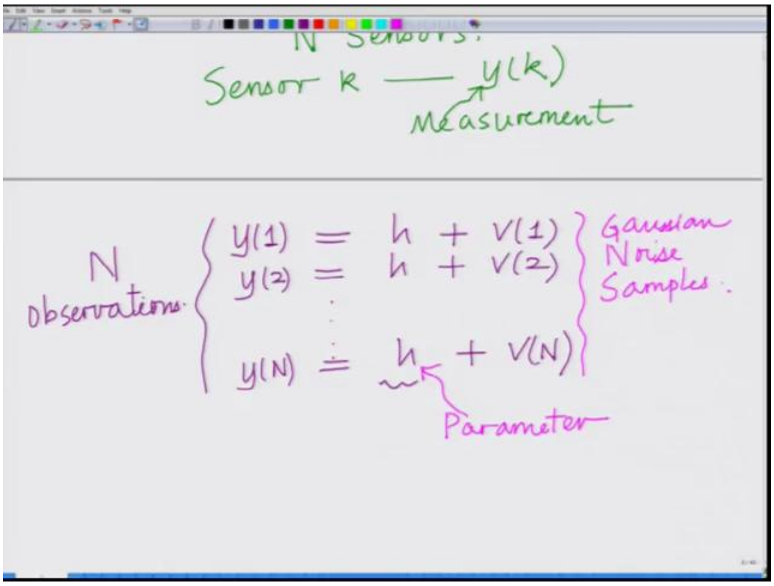

And what we have is basically we have N such measurement. Therefore what we have is we have y 1 equals h plus V 1, y 2 equals h plus V 2 so on y N equals h plus V N these are N observations. This is the parameter h is the parameter of interest, these V 1 V 2 V N these are the Gaussian noise samples. And we have also shown that this system model this setup can be represented as y bar equals 1 bar the vector 1 bar times h the parameter h plus V bar where V bar is the noise vector.

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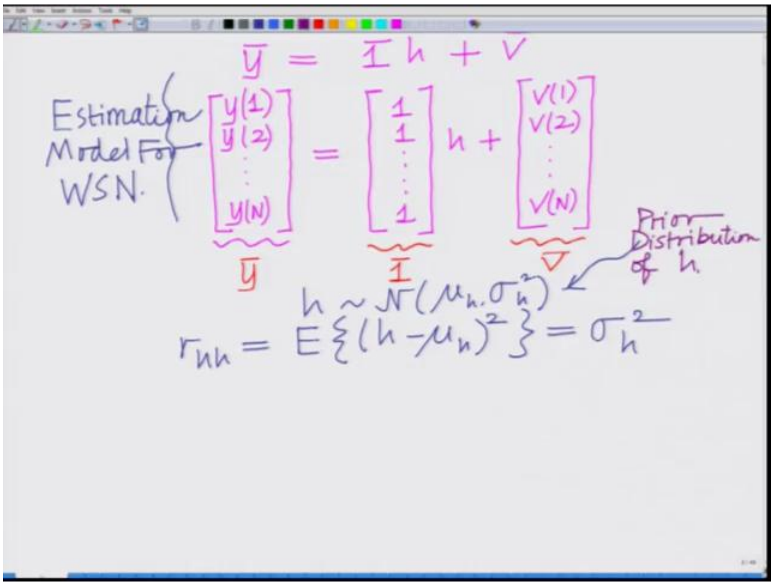

So, I can write this as y bar equals vector 1 bar times h plus V bar, where y bar equals y 1 y 2 so on up to y N. This is the observation vector, 1 bar is the vector of all once, h plus V bar is V 1 V 2 up to V N. So, this is your vector y bar this is your vector 1 bar this is your vector V bar which is the noise vector. This is your estimation models for WSN. This is our estimation model this is our wireless sensor network model; the estimation model for the wireless sensor network.

Now, what we want to do we have already computed the MMSE estimate, so the MMSE estimate h hat for this scenario that is what we have already seen. We have already seen the MMSE estimate and in fact we also seen an example of how to calculate this MMSE estimate for a simple example. Now what we have going to do we are going to calculate the MSE that is the mean squared error for this wireless sensor network estimation scenario. So, now we know for this scenario the mean square error will be calculated as follows r hh we know this quantity is basically expected value of h minus mu h square this is equal to sigma square.

More importantly you remember we said we assumed h to be Gaussian with mean mu h variance, so h the prior distribution what is this this is the prior distribution of h; this is the prior distribution of prior of the parameter h, where the parameter h is assumed to be Gaussian with mean mu h and variance sigma h square. Therefore, r hh is a expected value of h minus mu h square which is sigma h square.

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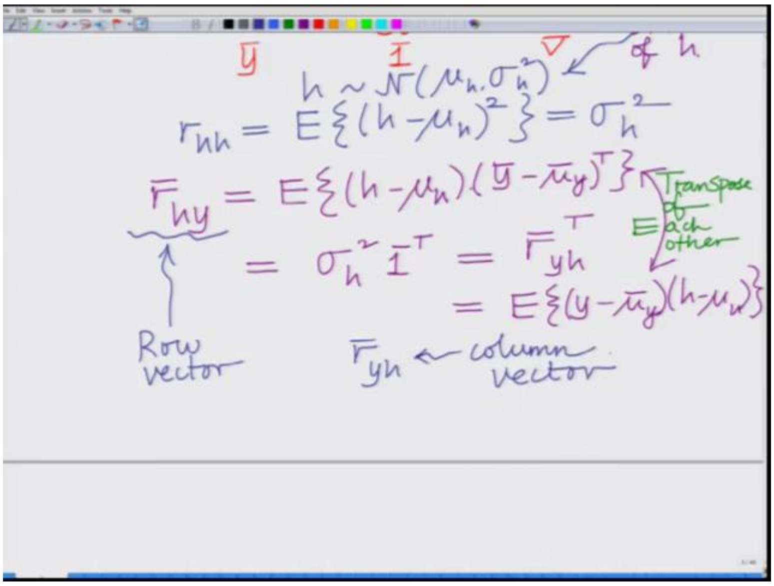

Further, we have r bar hy this is going to be a vector, this is going to be expected value of h minus mu h into y bar minus mu bar y transpose. We have already evaluated this quantity r bar hy this is going to be sigma h square into 1 bar transpose sigma h square into 1 bar 1 bar, this is going to be sigma h square into 1 bar transpose that is it this r hy is expected value of h minus mu h into y bar minus mu y bar this is sigma h square and 1 bar transpose. You can also see this is r bar yh transpose, because yh transpose is expected value of y minus mu bar y into h minus mu h and you can see this quantity r bar h this quantity r bar hy is the nothing but the transpose of this quantity r bar yh.

So, these two quantities are basically transpose of each other; that is what you can see these two quantities are transpose of these two quantities are basically the transpose r bar hy is the transpose of r bar yh that is the r bar hy is a r bar yh transpose. And in fact, r bar yh is a column vector because it is expected value of y bar minus mu y bar which is column time scalar h minus mu h is a column. Why r by hy which is expected value of h minus mu h times y bar minus mu bar y transpose where y bar minus mu bar y transpose is a row vector, so r bar hy is going to be a row vector. You can see again you can check all these things, this is going to be a row vector and r bar yh this is a column vector.

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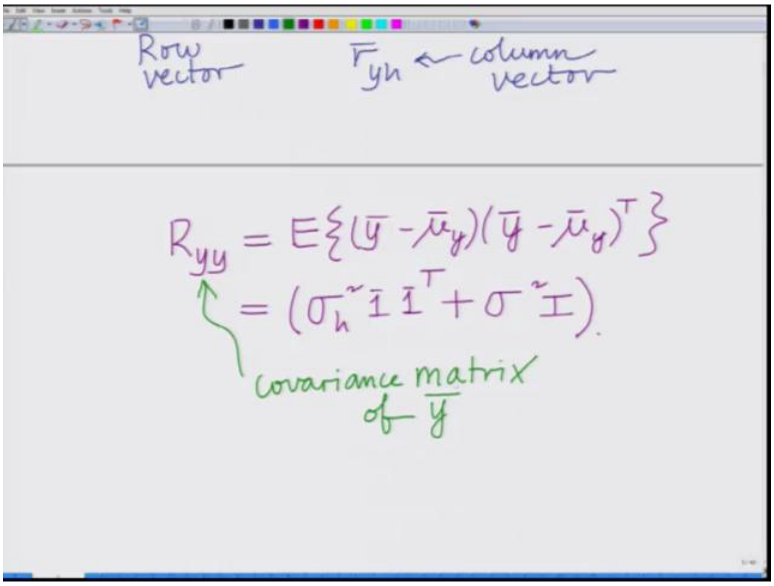

Now, what we are going to do and we have one more quantity remember. The covariance of y r yy, this is going to be a expected value of y bar minus mu bar y into y bar minus mu bar y transpose which we have already derived as sigma h square 1 bar 1 bar transpose plus sigma square and identity. What is this quantity? This quantity is the covariance matrix of the observation vector y bar. Now, we have all the ingredients of this, we have the variance of h we have the cross covariance of h comma y and we have the covariance matrix of y bar. So, now we are going to substitute this and derive the expression for the MSE.

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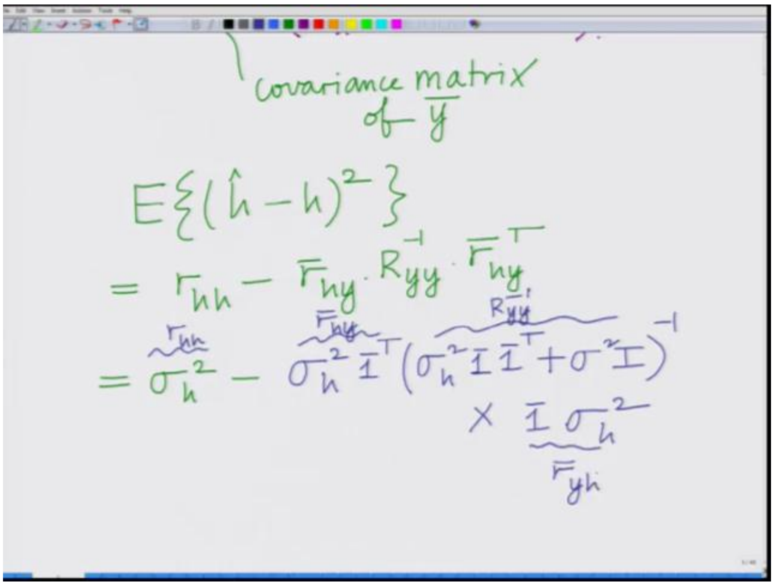

So, the MSE is basically given as expected value of h hat minus h whole square equals, what is that we have already said that is r hh minus r bar hy into the covariance matrix r yy inverse times r bar r bar yh which is r bar hy transpose. So, this I can write this as sigma h square minus r bar hy which is basically sigma h square 1 bar transpose times sigma h square 1 bar 1 bar transpose plus sigma square identity inverse times 1 bar into sigma h square. And now this is your r hh that is your sigma h square, this is your r bar hy, this is your r yy inverse. And this quantity is basically your r bar y into h.

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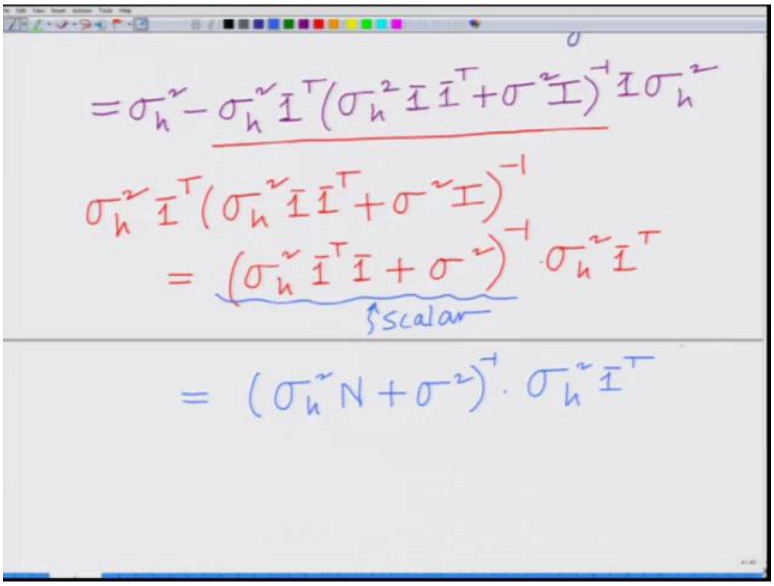

So, now we have this expression which is basically your sigma h square minus sigma h square 1 bar transpose into sigma x square 1 bar 1 bar transpose plus sigma square identity inverse into 1 bar times sigma h square. Now what we want to do? We want to simplify this expression for the MSC further. And to simply the expression further we going to use a result that we already seen before that is this quantity remember we have seen already before and you can refer to the lecture on estimation in the wireless sensor network that is sigma h square 1 bar transpose sigma h square 1 bar 1 bar transpose plus sigma square identity inverse.

We have seen that this quantity is nothing but well this quantity is equal to sigma h square 1 bar transpose 1 bar plus sigma square inverse times sigma h square 1 bar transpose. In fact, we also know that this quantity the sigma h square 1 bar transpose 1 bar plus sigma square this is scalar. Therefore, this quantity is a scalar, because 1 bar transpose 1 bar 1 bar is a transpose is a row vector 1 bar is a column vector 1 bar transpose 1 bar is N. So, this quantity is a scalar in fact its inverse is also simply going to be it is a reciprocal because it is a scalar.

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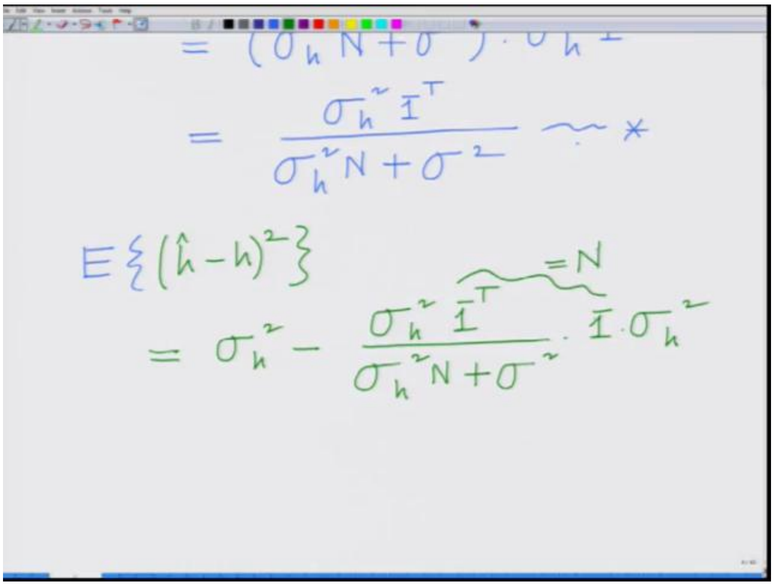

So, this is going to be sigma h square 1 bar transpose 1 bar is N plus sigma square inverse times sigma h square into 1 bar transpose which is sigma h square sigma h square transpose divided by when sigma h square times N plus sigma square, so this is the simplification. Therefore, now we are going to substitute this quantity let us call this as your star we are going substitute these two are equivalent. So, we are going to take this star here and substitute it above and therefore what we have is expected value of the MSE that is expected value of h hat. H hat minus h whole square equals sigma h square minus sigma h square 1 bar transpose divided by sigma h square N plus sigma square into when 1 bar times sigma h square. Now look at this 1 bar transpose into 1 bar this is basically equal to N.

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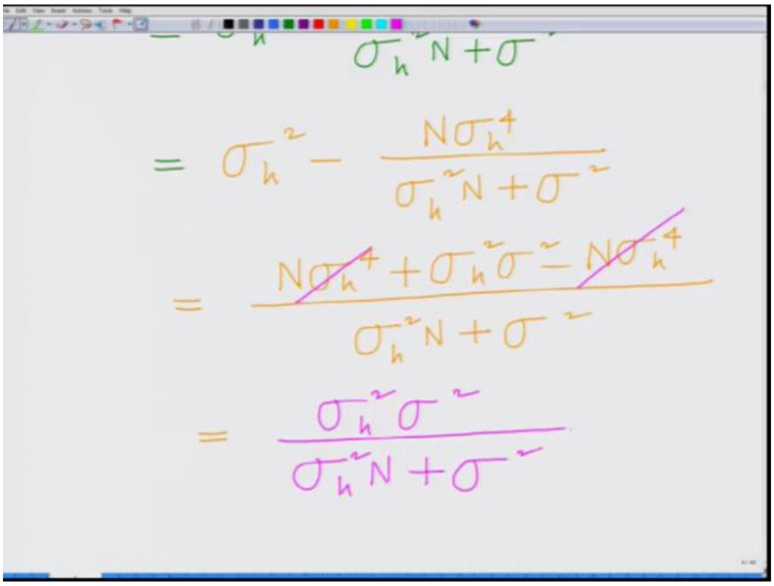

Therefore, what I am going to have is finally sigma h square minus N into sigma h raise to the power of 4 divided by sigma h square N plus sigma square. This is further equal to now look at this sigma h square into sigma h square N, so this is N. Once you simplify N sigma h 4 plus sigma h square sigmas square minus N sigma h raise to the power of 4 divide by sigma h square N plus sigma square. Now this is equal to well these two obviously cancel so what you have is sigma h square time sigma square divided by sigma h square N plus sigma square.

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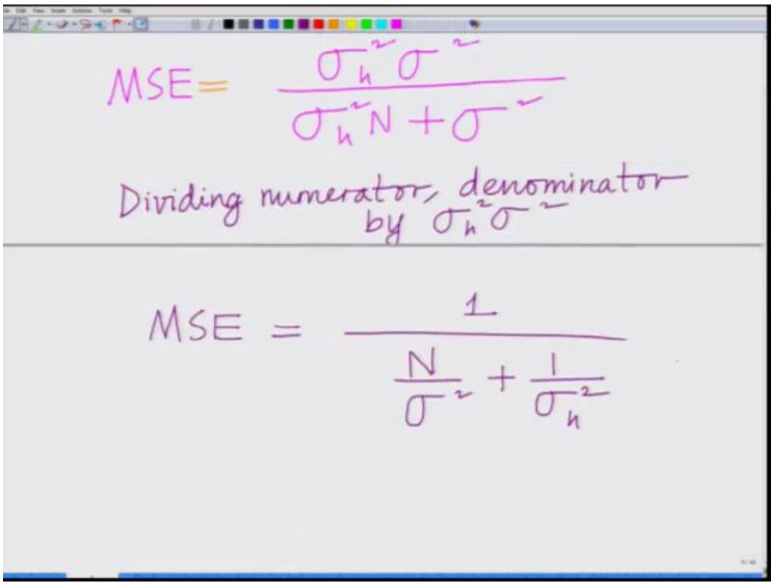

Now, if you divide both numerator and denominator by this quantity sigma h square sigma h square. We are going to divide to simplify further, we are going to divide the numerator and denominator by this quantity sigma h square times sigma square. So, this is your MSE. And now we have the MSE can be equivalent divide of course once you divide numerator by sigma h square sigma square you have 1 in the numerator divided by dividing sigma h square N by sigma h square sigma N square sigma square. So, you haves N divided by sigma square plus sigma square divided by sigma square sigma h square this is sigma h square. And therefore what you have is now this one final modification this is 1 divided by 1 over sigma square by N plus 1 over sigma h square. Now if you look at this what is this? It is basically the MSE.

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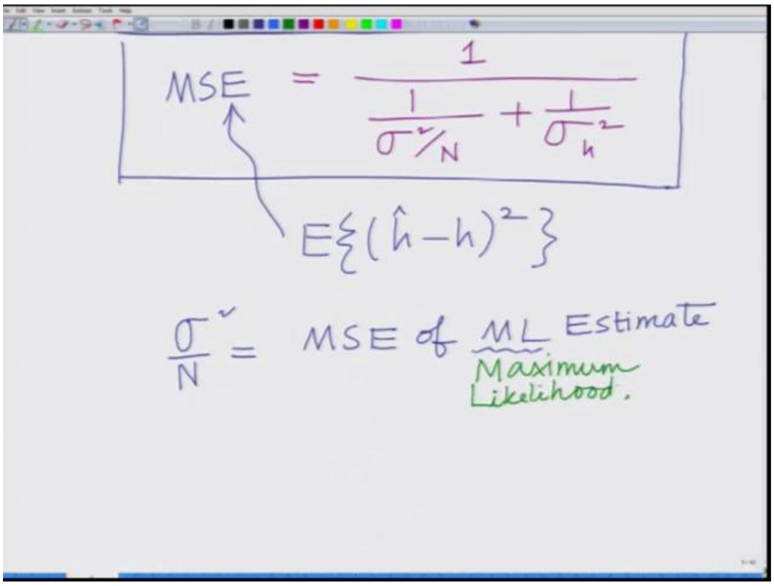

Let us first write this down this is the mean squared error expression for the mean squared error basically which is defined as expected value of h hat minus h whole squares. Now more importantly if you look at this is the mean squared error expected value of h hat minus h whole squares, and now we have an interesting interpretation this. Again look at this quantity you can recall that this quantity sigma h square divided by N this quantity is the mean squared error of the ML estimate.

We have seen that already during the estimation in the wireless sensor network that sigma square over N if there are N measurement and sigma square is the variance of the additive white Gaussian noise then sigma square over N is the MSE of the ML, where ML stands for the maximum likelihood.

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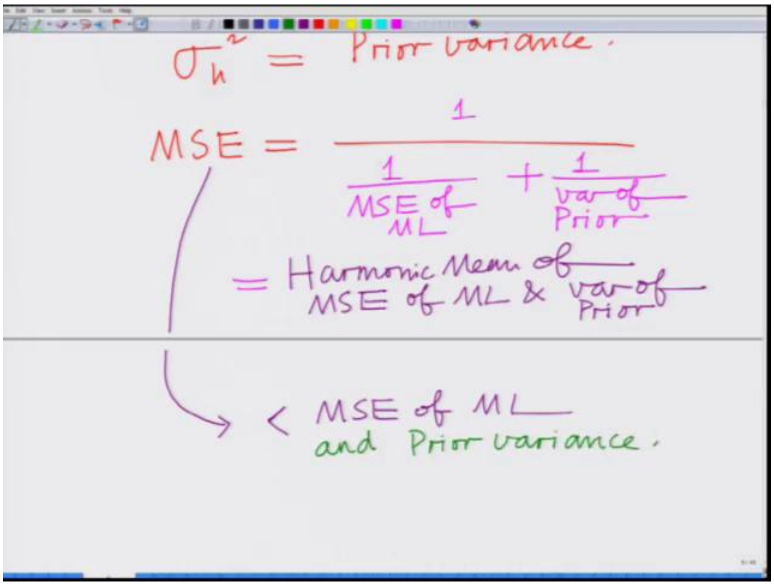

Similarly, sigma h square is the prior variance. Therefore, what we have is the MSE is basically you can now see that this is nothing but 1 over 1 over the MSE of ML mean square error of ML plus 1 over variance of the prior. This is an interesting result which basically means that look at this is nothing but the harmonic means, so this is basically harmonic mean of MSE of ML and variance of harmonic mean of MSE it mean squared error of the ML estimate harmonic mean of the MSE of the maximum likelihood estimate and the variance of the prior.

And you know from the property of the harmonic mean that it is less than both the harmonic mean of two quantities a and b is less than both a and less than b therefore this variance of the MSE of the MMSE estimate mean squared error of the MMSE estimate is less than both the MSE of the maximum likelihood estimate and also the prior variance. And that is naturally because you have some prior information you have the maximum likelihood estimate that you obtain from the samples and you are combining these two. So, you are using the information from both the observations and also the prior.

Therefore you expect the net error to be lower than the individual errors. That is an interesting observation that you have over here. So, this MSE remember is less than both

MSE; harmonic mean of both is less than MSE of ML and in fact you can say if MSE of ML is very small. Let us look at this, if now look at this the MSE.

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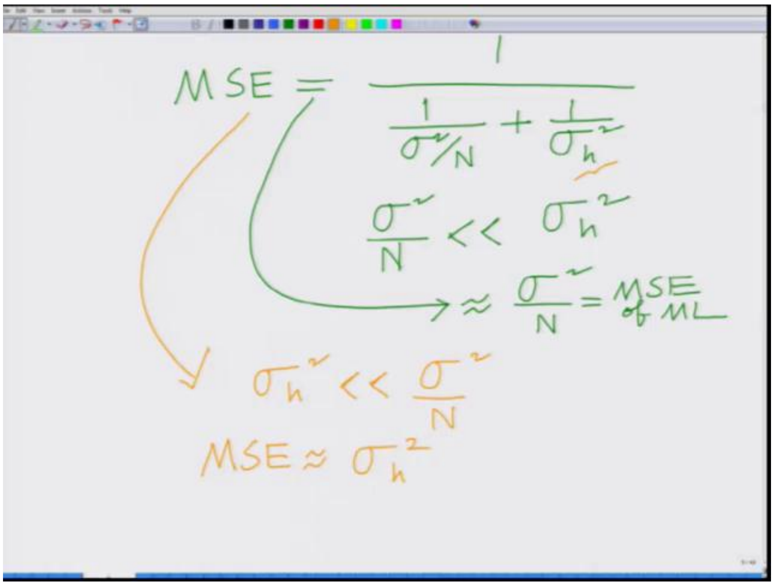

The MSE equals 1 over 1 over sigma square by N plus 1 over sigma h square. Now if sigma square over N is much smaller, let us say MSE of ML is much smaller than sigma h square. Then we can see that 1 over sigma square over N is much larger than 1 over sigma h square, therefore this will be approximately in this scenario this will be approximately equal to 1 over simply 1 over sigma square over N alright which is basically simply sigma square over N equals MSE of the maximum likelihood estimate.

On the other hand if for the other scenario if the prior variance sigma h square is much less than sigma square over N. then 1 over sigma h square is much greater than 1 over sigma square over N so this first term which will be this term will be dominating in this scenario (Refer Time: 26:48), so MSE is approximately simply equal to 1 over 1 over sigma h square which is equal to sigma h square. And this is also intuitive because what this means is basically the ML variance is much smaller and prior variance is larger, it means that the ML estimate is very accurate. So, you can simply look at the ML estimate and in this is scenario the net variance will be the ML variance.

On the other hand if the prior variance is very small compare to the ML variance that it means that the prior information that is the parameter is very close to the prior mean that is mu h and the net variance of the estimate is basically goes to the prior variance that is sigma h square. And for all the other combination in between basically it is the harmonic mean of the MSE of the ML estimate and the prior variance, this is a interesting observation that (Refer Time: 27:41).

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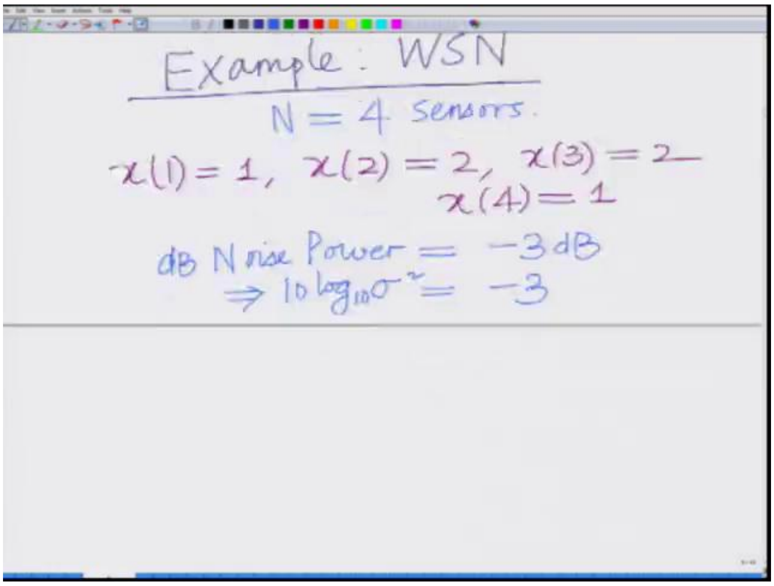

Let us do a simple example to understand this thing, it is similar to MSE estimate let us do a simple example to understand this. You have our WSN example that we have considered previously let us do the same example we have N is equal to 4 sensors and we have the measurements x 1 equal to 1, x 2 equal to 2, x 3 equal to 2, and x 4 equal to 1, we have the dB noise power equal to minus 3dB implies 10 log ten sigma square equals minus 3.

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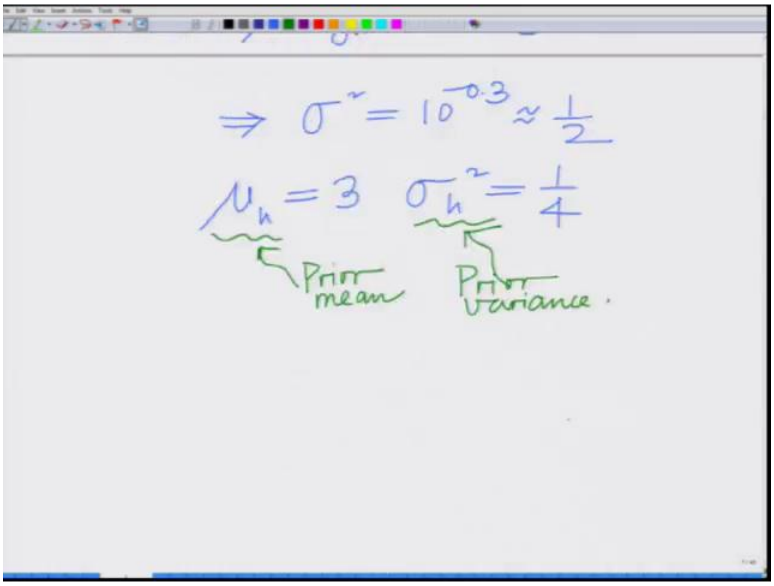

This implies sigma square equals 10 to the power of minus 0.3 which is approximately equal to half. Mu h, let us say the prior mean mu h this is equal to 3, prior variance sigma h square equals 1 by 4. So, this is the prior mean; mean of the prior. This is the prior variance sigma h square.

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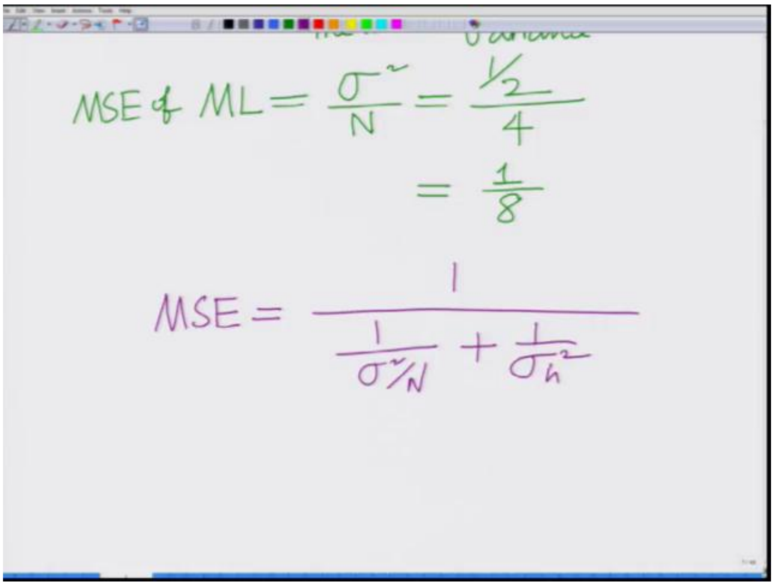

Therefore, the MSE now you have all the ingredients MSE and we have ML, MSE of ML equals sigma square divided by N that is ML sigma square is half divided by N is 4, so this is equal to 1 by 8. And therefore, the net MSE equals 1 divided by 1 over sigma square over N plus 1 over sigma h square.

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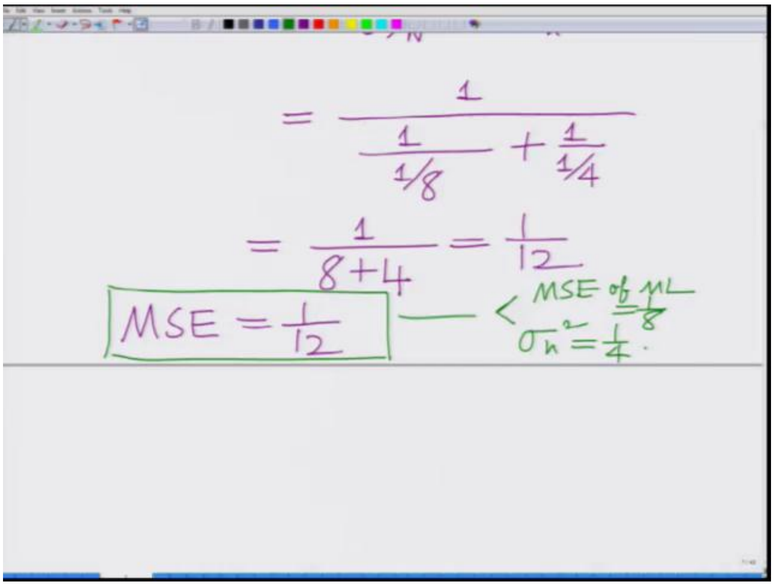

All we have to do is substitute the varies quantities, so this will become 1 over 1 over sigma square by N is 1 over 8 plus 1 over sigma h square is basically 1 over 4 at this will be 1 over 8 plus 4 equals 1 over 12, and therefore the net MSE equals 1 over 12. So, this is what we observe the net error of estimation mean square error.

And you can see clearly this is less than your MSE of ML which is equal to 1 over 8 and this is also less than sigma h square which is equal to 1 over 4. So, what you are seeing is because the MMSE estimated combines basically it optimally combines the maximum likelihood estimate and basically the prior mean mu h so its means squared error is lower than both the mean squared error of the ML estimate and also the prior variance. And what you see is in fact that is true what you have observed this is for a simple example this MSE its basically is given as 1 over 12 which less than the MSE of the ML which is 1 over 8 and also sigma h square which is 1 of the (Refer Time: 31:54).

So, basically what we have done in this module is we have basically simplified the expression for the mean squared error and we have illustrated how to compute this mean squared error in the context of a wireless sensor network and we have a given interested interesting very interesting interpretation for this. We have shown that the mean squared error of the MSE estimate is the harmonic mean of the mean squared error of the ML estimate and the prior variance therefore it is lower than both of them.

So, we will stop this module here and will continue with other aspect in subsequent modules.

Thank you very much.