**Indian Institute of Technology Kanpur**
**National Programme on Technology Enhanced Learning (NPTEL)**
**Course Title**
**Error Control Coding: An Introduction to Linear Block Codes**

**Lecture – 3B**
**Problem Solving Session – I**

**by**
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Welcome to the course on error control coding, an introduction to linear block codes, before we discuss decoding of linear block codes let us solve some problems today.

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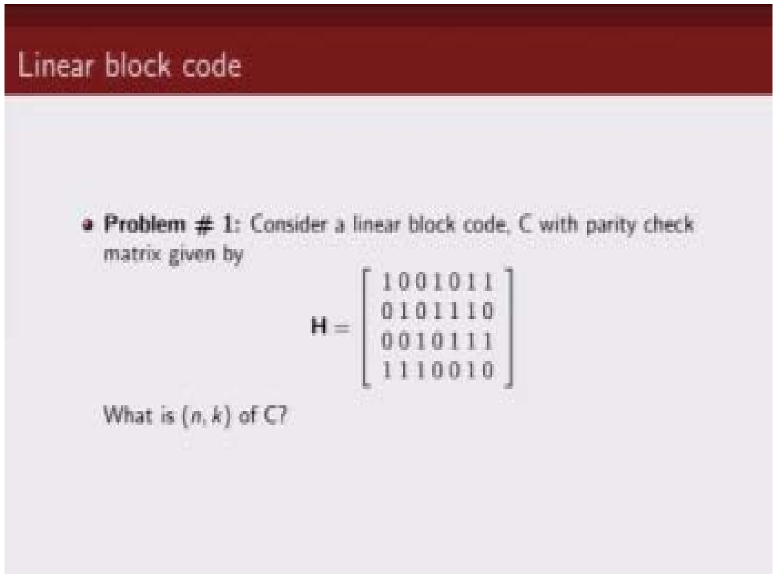


So first question that we are going to look at is, consider a linear block code c.

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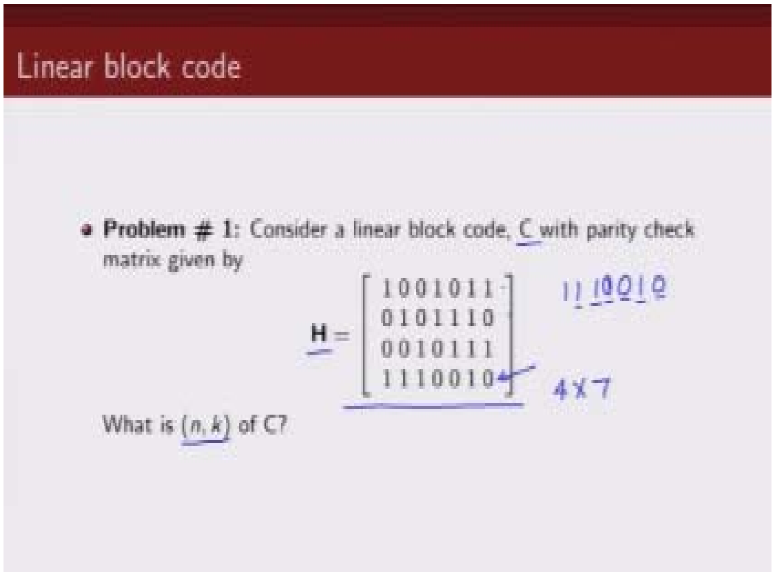


Whose parity check matrix is given by this and you are asked what are the code parameters n (n, k) n which is a block length a code word length and k is the size of, is the dimension of the basically information sequence length is k, now how do we solve it? We know, we will first find out what is the rank of this matrix H, now you can see this is a 4 x 7 matrix right?

So the maximum rank possible is four, let us see whether it has rank four, now if you add row one, two and three what do you get? 1111010 sorry 1110010 this is what you get, you can see this is 1, this is 1, this is 1, this is 0, this is 0, this is 1, and this is 0, and what is row number four, this is exactly same as this so you can see row 1, row 2, row 3 and row 4 add up to zero, that means it does not have rank four, so maximum rank possible is three.

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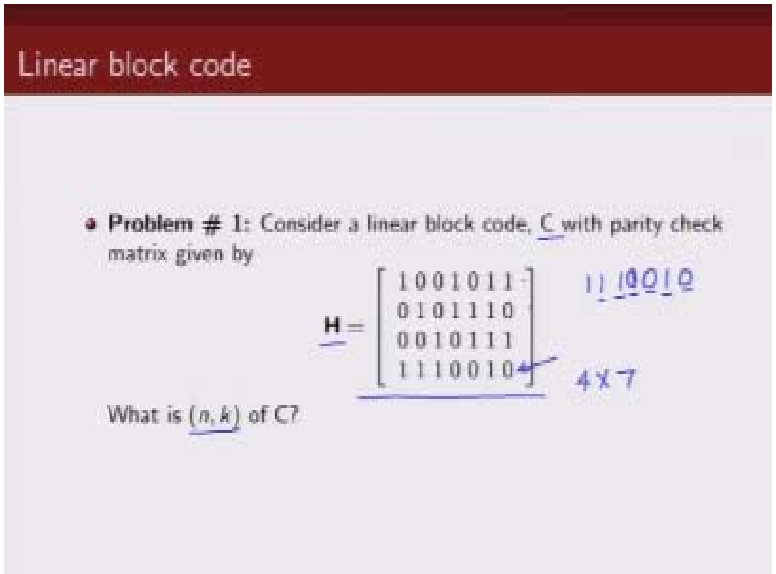


So let us see if any three rows combination add up to zero.

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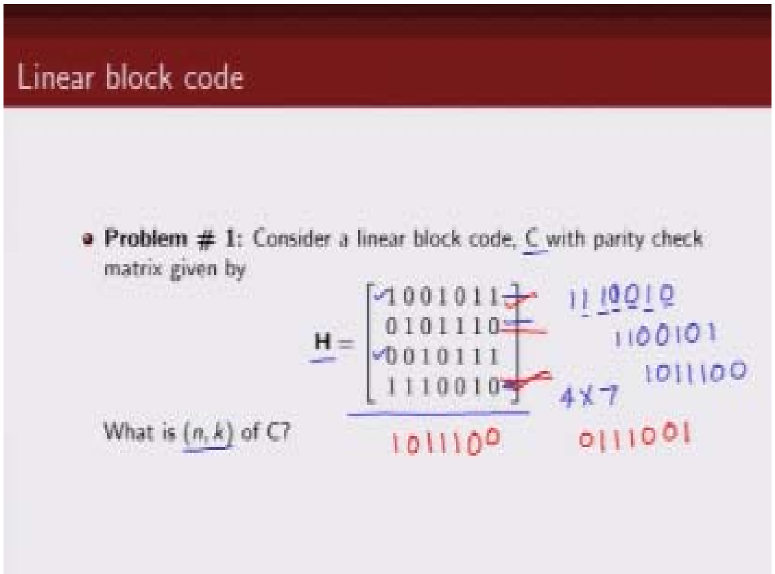


So let us see, let us see if we consider some of these two rows, this is what 1100101. Now none of the rows are equal to this you can see if we consider this row and this row we add these two rows let us see we what do we get is 1011100. Now note none of these rows are 2nr four is equal to this so these set of three rows basically they are independent, let us try adding up this and this, so if we add first row and fourth row what do we get 011100 and 1.

Now note row number three and two are not same as this, so like that we can check we can check for example row two and four we add up row two and four what do we get 1011100. Now note row number three and row number one are not same as this so we can see that any three rows do not add up to zero, so the rank of this matrix H is three.

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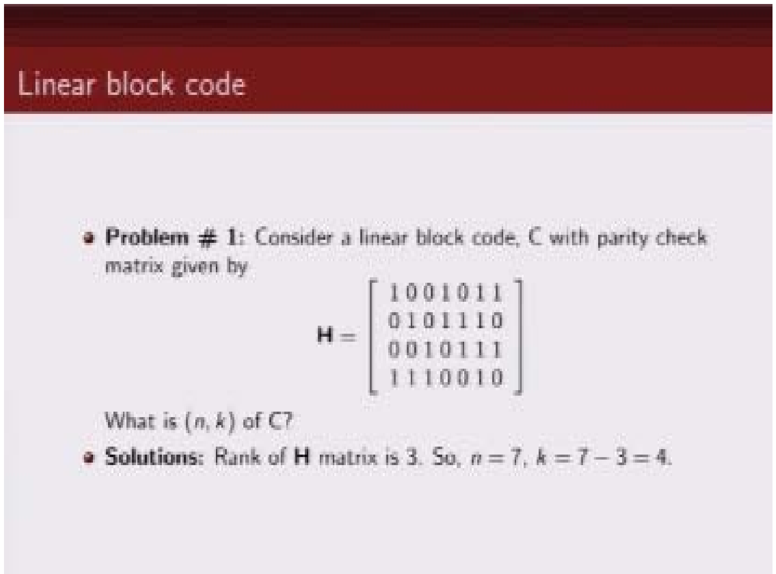


So if the rank of this matrix is three now we know parity check matrix is n-k x n.

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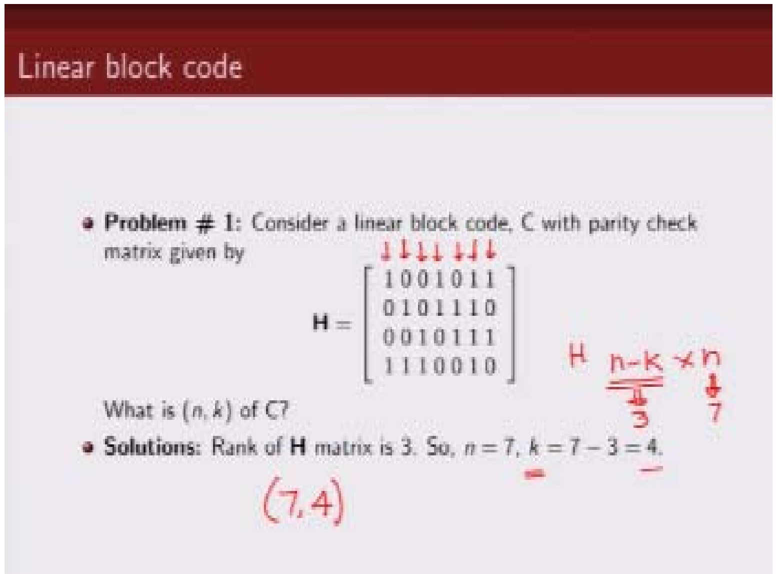


So n-k is in our case equal to three and what is n, n is number of columns of this so that is one, two, three, four, five, six, seven so n is 7, so that would then give us k=4, so this is an example parity check matrix for a (7, 4) linear block code, okay.

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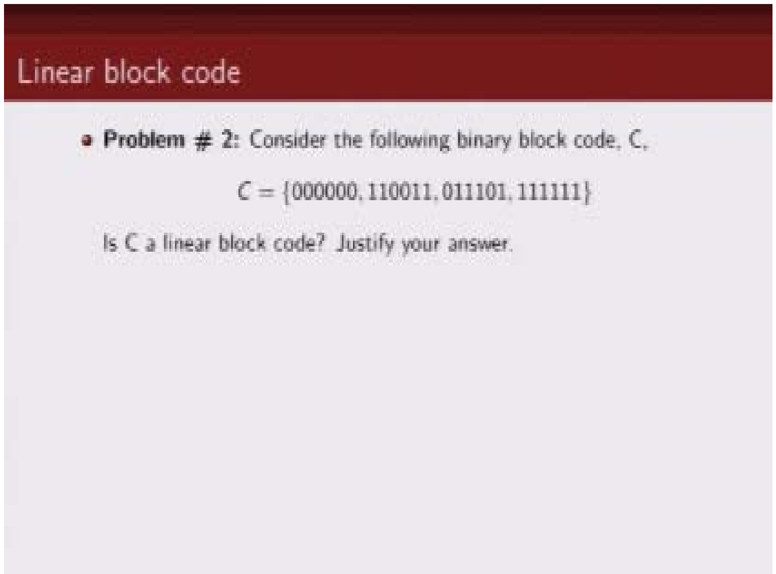


Now let us look at another problem, you are given a set of code words and what are these code words these are binary code words.

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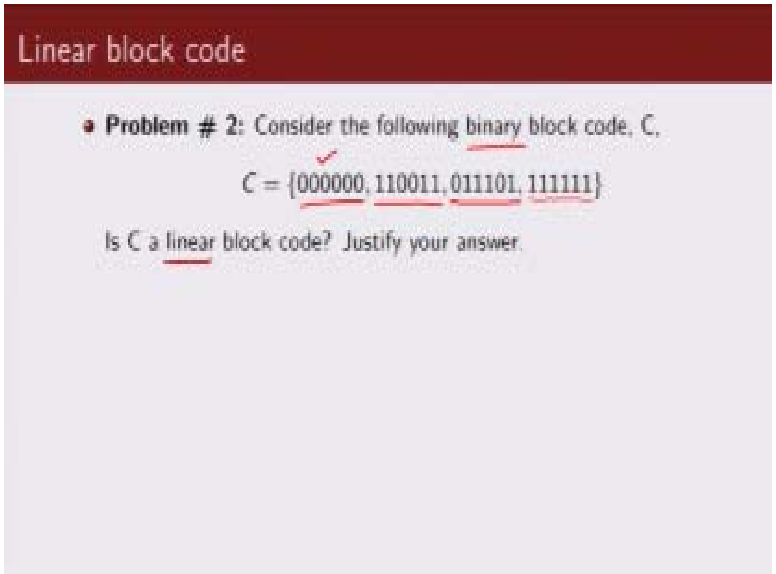


So this is all zero, 110011, 011101 and 11 all 1 sequence and the question that has been asked is, is this a linear code, is this a linear code? Now what do we know about linear code? A linear code should have all zero code word which this code word has and sum of any two code words is also a valid code word.

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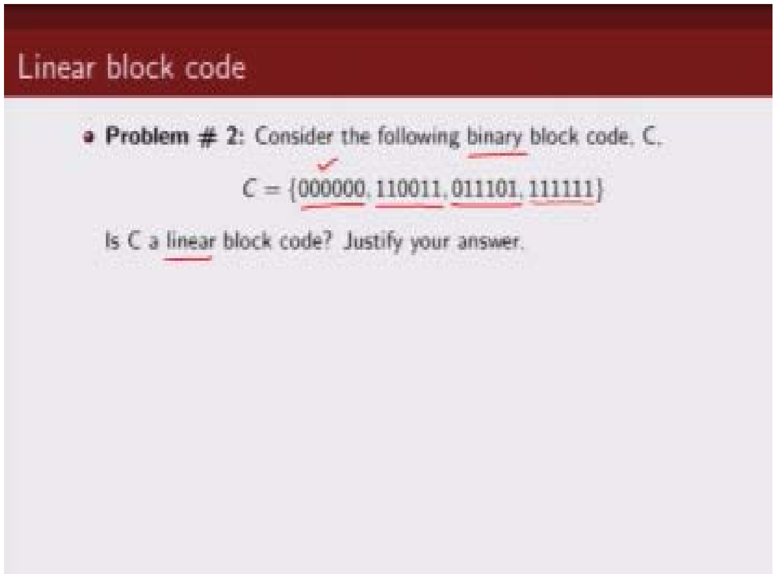

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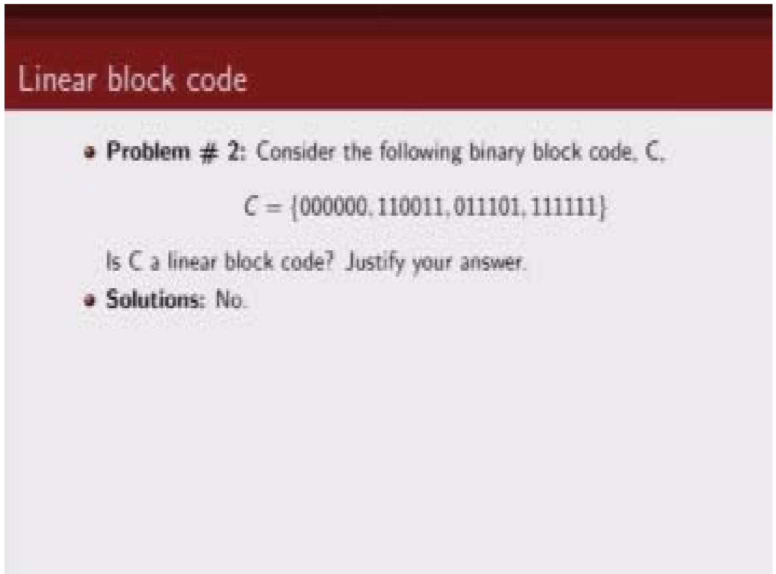

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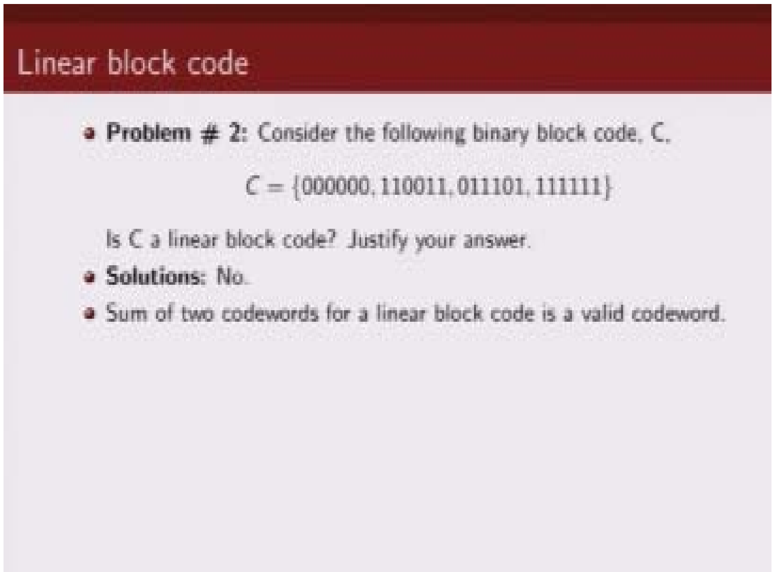


So let us see so let us see if sum of all code words is already a valid code word.

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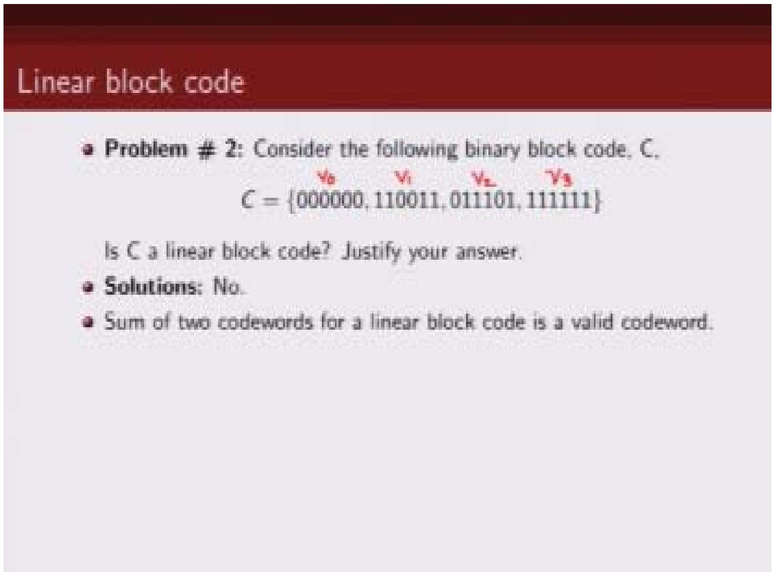


So let us call this $v_0$, $v_1$, $v_2$ and $v_3$ so what we want is all possible combinations of $v_0$, $v_1$, $v_2$, $v_3$ should also be a valid code word, they should be in c.

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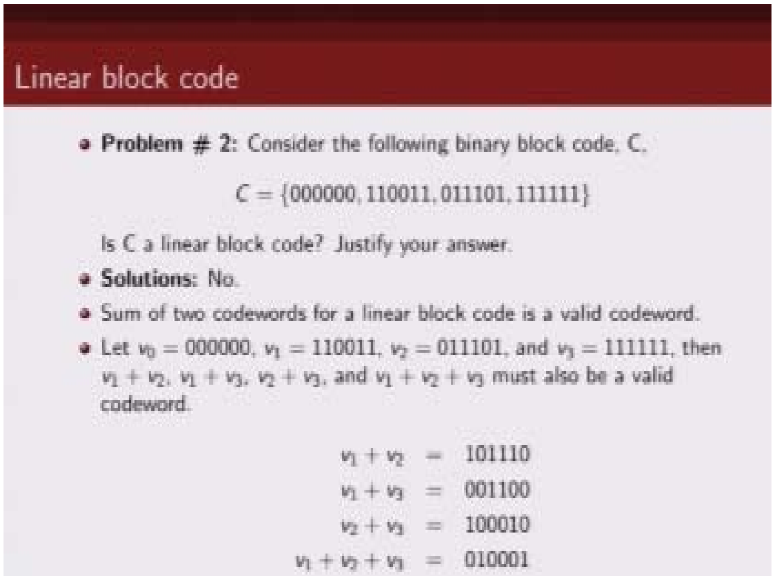

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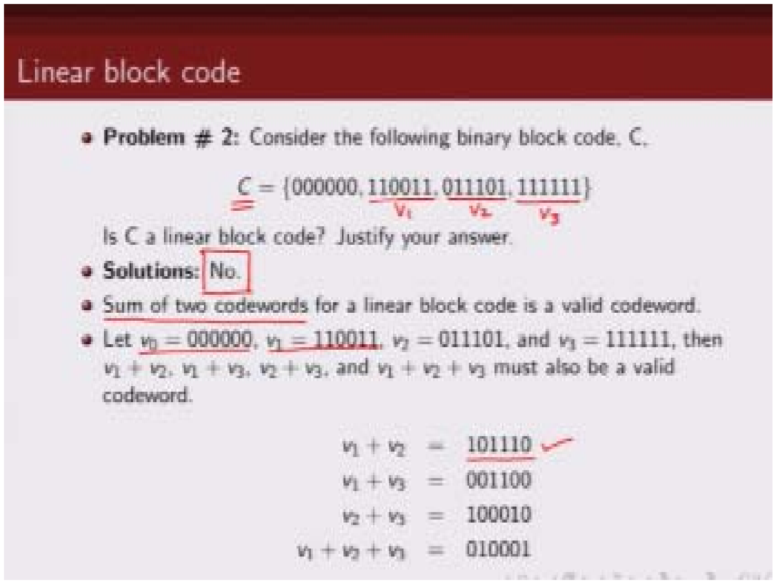


So let us see, so as I said we take $v_0$ to be all zero code word, $v_1$ is given by this, this is $v_1$, this is $v_2$ and this is $v_3$, now let us see all possible combinations of $v_1$, $v_2$, $v_3$ the non zero code words. So we consider $v_1+v_2$ what is $v_1+v_2$, so $v_1+v_2$ is we can see this is 101110 this given by this, now is this code word in c, we do not see any code word which is 101110 listed here that means the c is not a linear code, why it is not a linear code?

Because sum of any two code words is also a valid code word, now $v_1$ and $v_2$ are valid code words in c, so sum of $v_1+v_2$ should also be in c but we noticed that 101110 which is sum of $v_1+v_2$ is not there in c and that is why we said that c is not a linear block code. Now my next question is can we add additional code words here such that c becomes a linear block code, now how do we do that?

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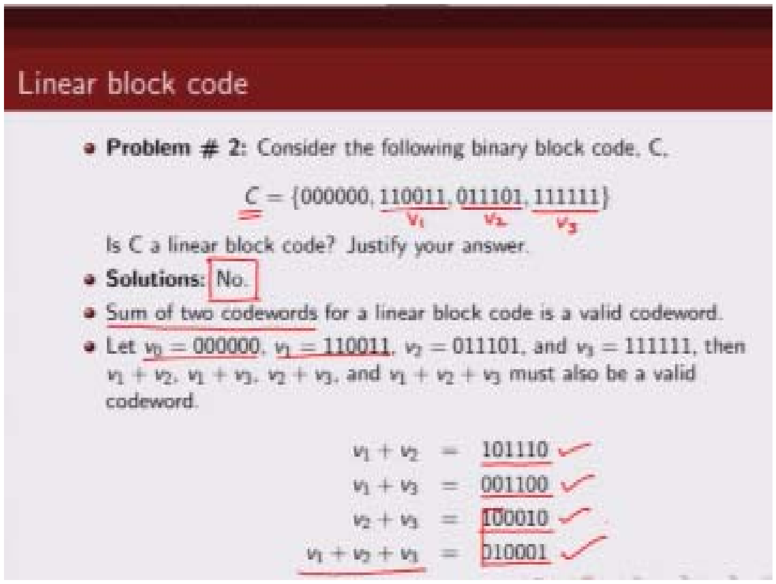


To do that we will have to ensure all possible combinations of these code words is also there in c, so let us then compute $v_1+v_3$ which is basically given by 001100. Let us look at $v_2+v_3$ which is given by 10010 and let us look at $v_1+v_2+v_3$ is basically given by 010001. So note that I have listed all possible combinations of these code words here. Now none of these sums are there in this linear block code so if we add them in this set of c, set of code words then we, our block code c will become a linear block code.

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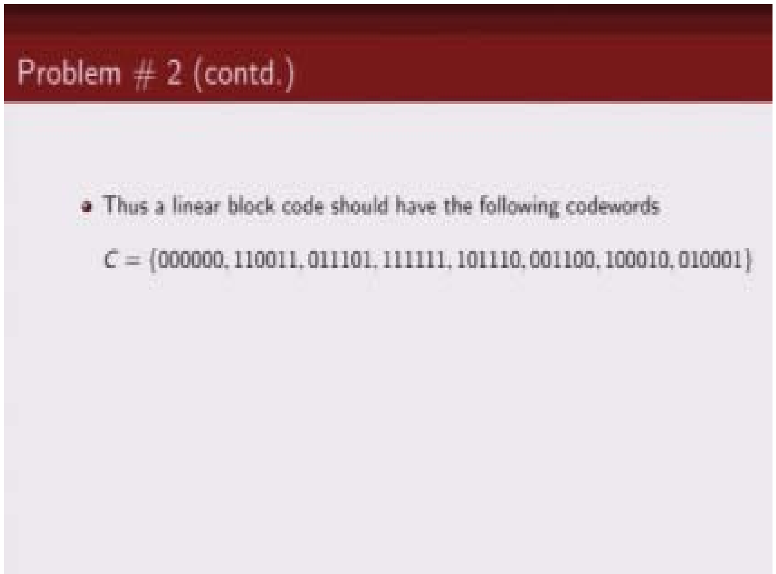


So if we want to make it a linear block code what do we need to do?

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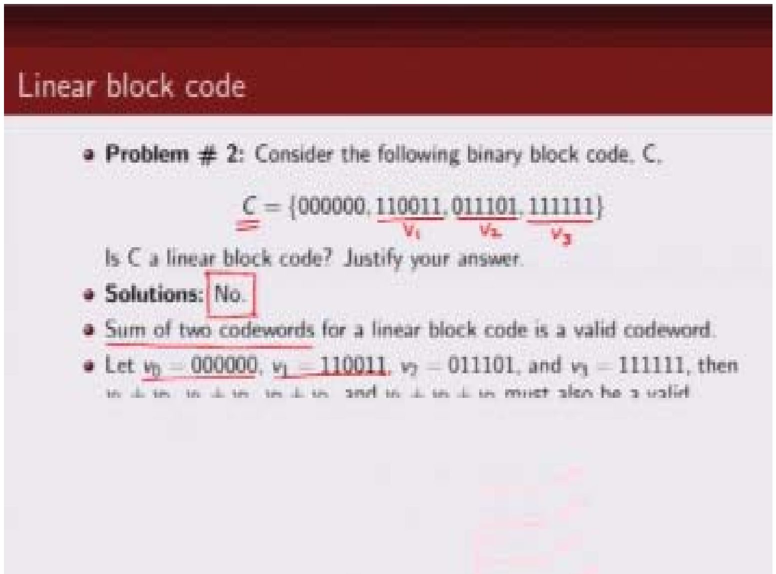


In this set of four code words $v_0$, $v_1$, $v_2$ and $v_3$ we need to add these set of code words which was basically $v_1+v_2$.

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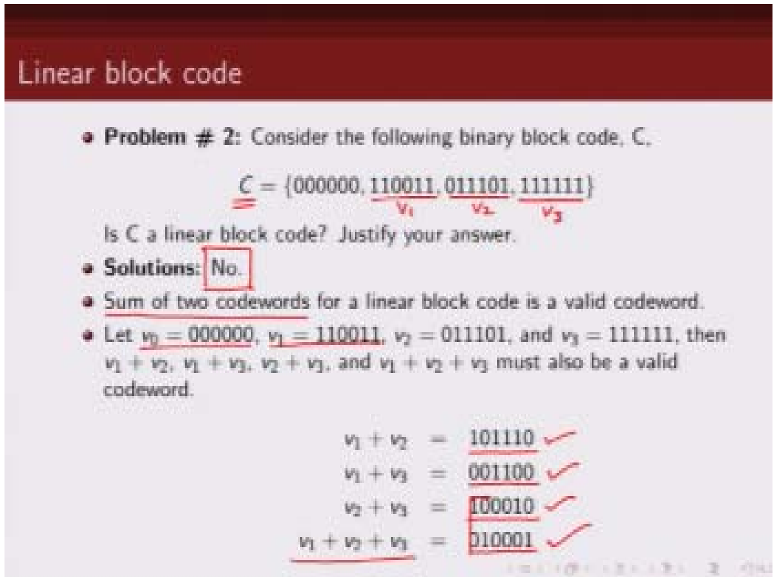

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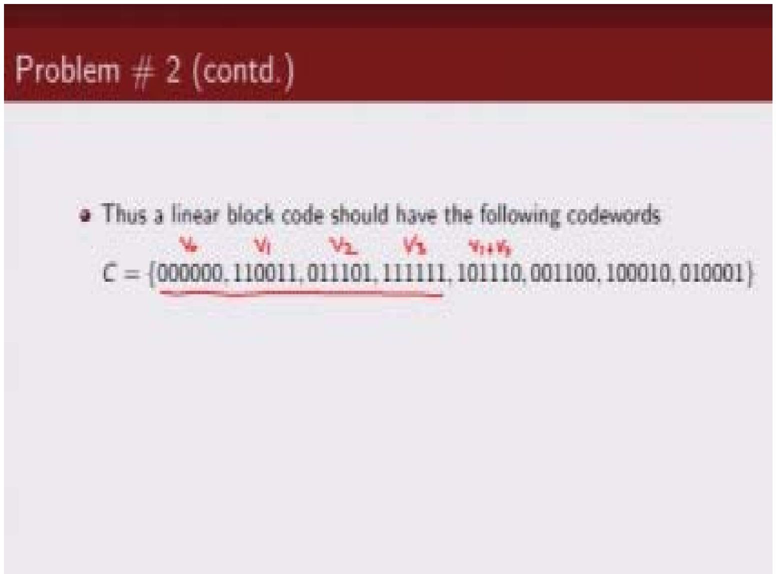


This is $v_1+v_2$.

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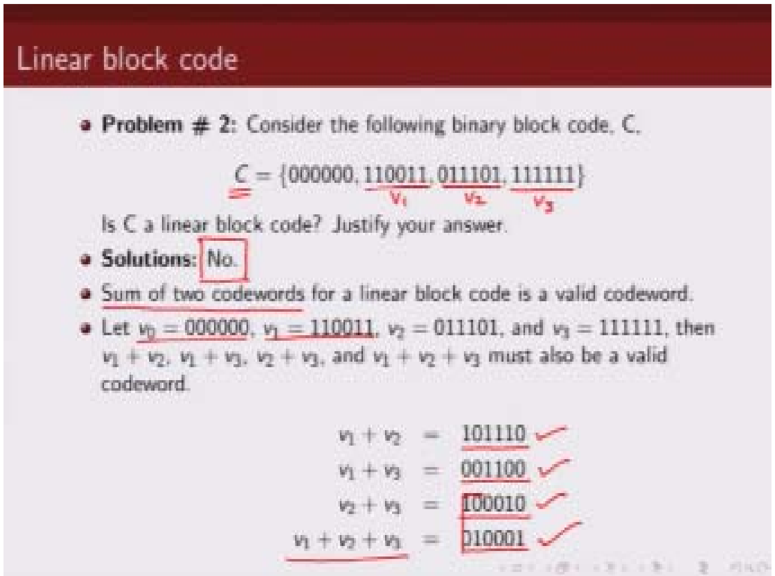


This is $v_1+v_3$.

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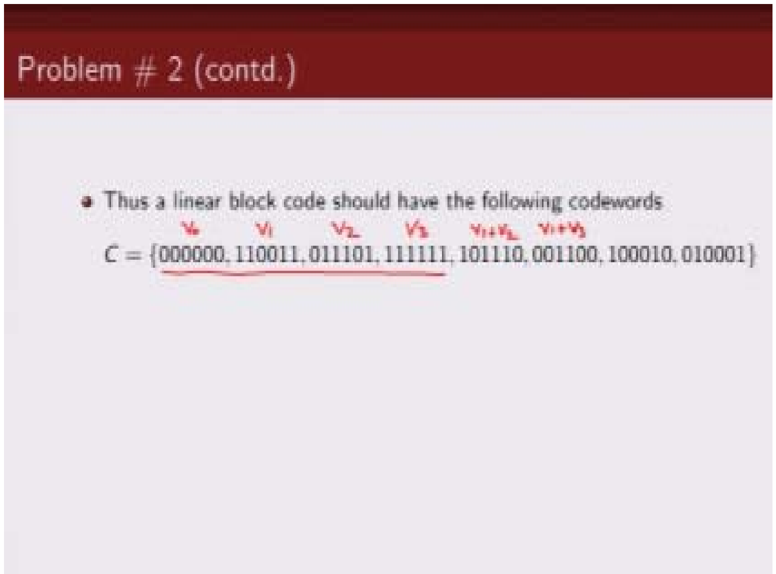


$v_1+v_3$

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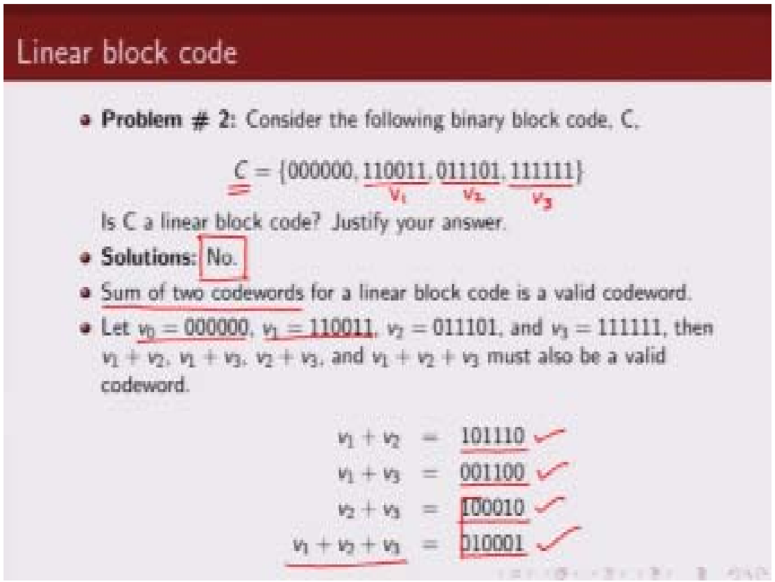


Then this one is $v_2+v_3$.

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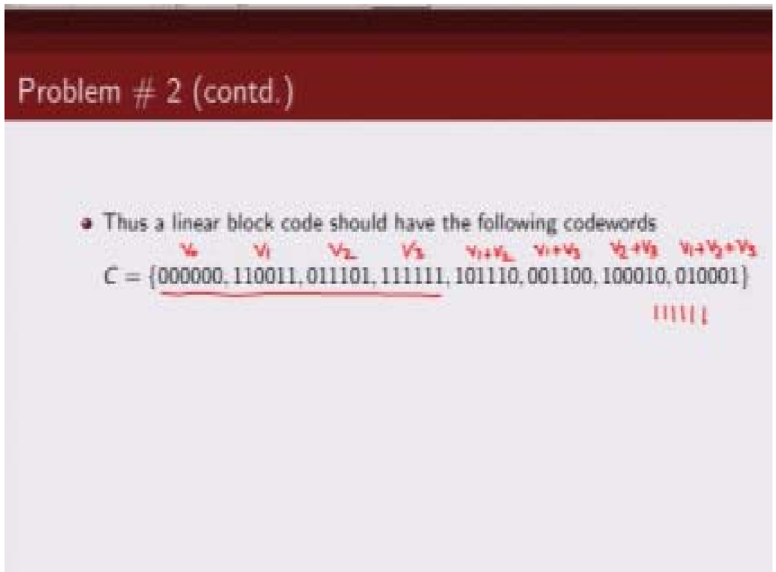


$v_2+v_3$ and this one was $v_1+v_2+v_3$, so let us look at these two code words, this is $v_1+v_2$ and this is $v_1+v_2+v_3$, so if we add these two what we will get is $v_3$, we can double check so if we consider add these two the first bit will be 1, this 0+1 will be 1, then 1+0 will be 1, then 1+0 will be 1, then 1+0 will be 1 and 0 +1 will be 1.

And this is already there in this set of code words this is $v_3$, okay. Similarly take this two, this one is $v_1+v_2$ and this is $v_2+v_3$ if we add them what we get is $v_1+v_3$ we will get this. If we consider these two, we will get $v_2$ we consider this we will get $v_3$ if we consider these two sum of these two we will get v3, we consider sum of these 3 what we will get, we will get v3. So you can see basically linear combinations of all the code words of already there in the sea, so this sea which contain.

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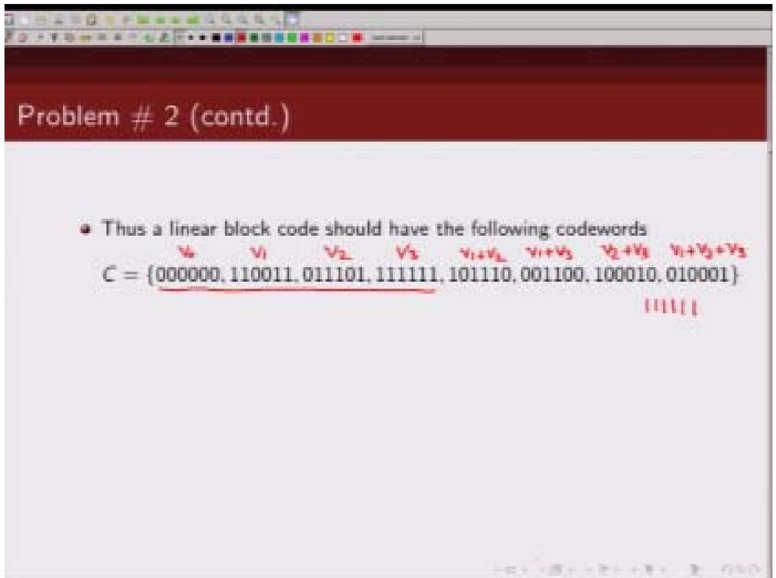


This set of 8 code words is a linear code.

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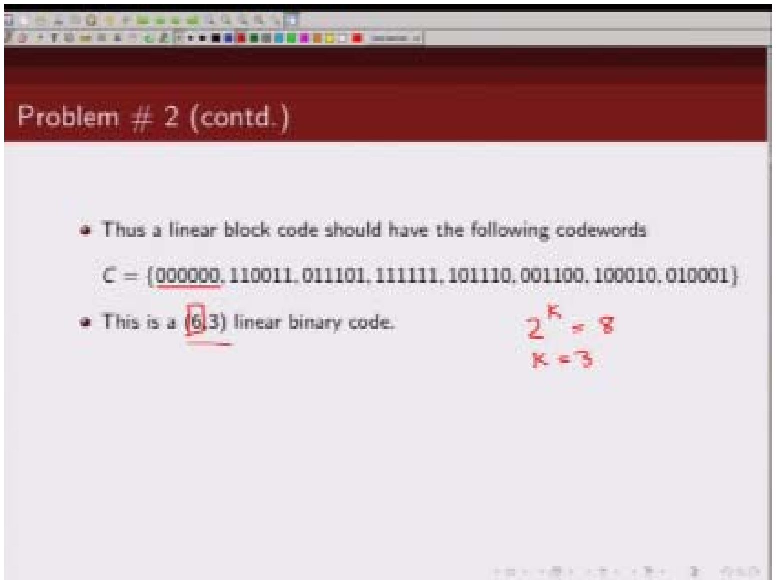


And what are the parameters n and k, now the length of this code words is 6 each, each of these code words are 6 bits so that is why n is 6, and there are total 2k code words and in our case 2 K is basically 8 so k is 3, so this is basically a 6, 3 linear binary code.

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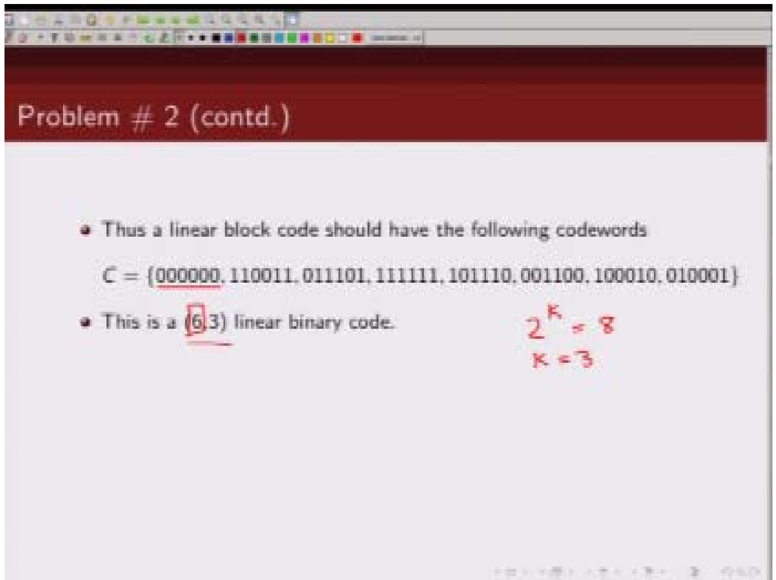


Now if I ask you tell me what is a generator matrix that will generate this set of code words, now how can you do that?

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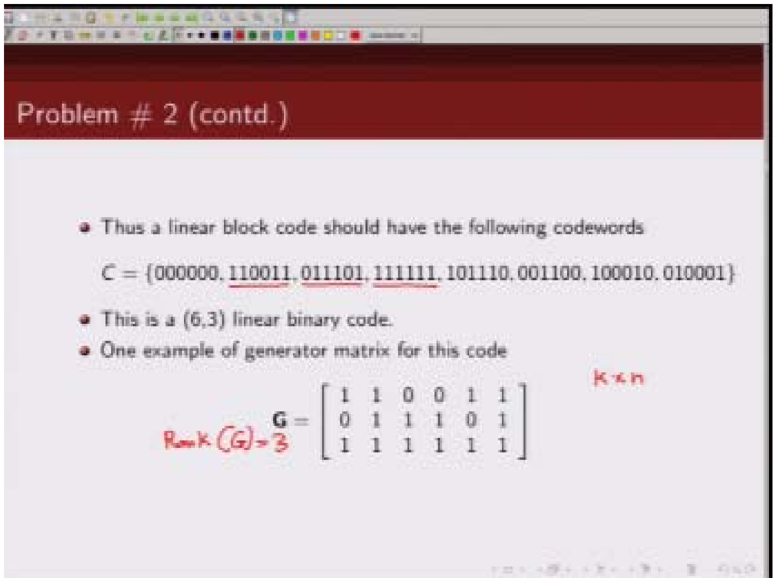


So we know the generator matrix is basically a k x n matrix right, so if you take basically 3 k in this case is 3 if you take 3 code words which are linearly independent basically if you take them and form them as rows of your generator matrix then you get your generator matrix. So I just took this v1, v2 and v3 and you can verify that rank of this matrix G is 3 so it full rank okay. So then this G will be able to, this generator matrix will be able to generate this set of code words.

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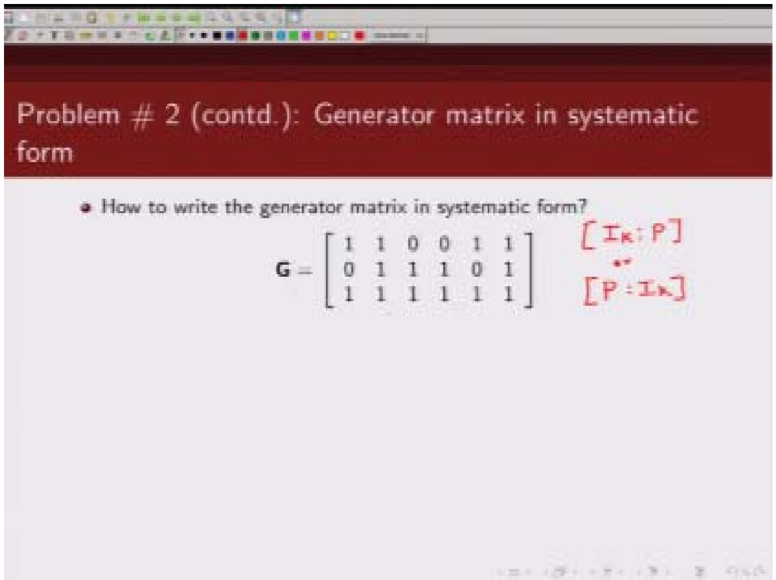


Now can we put this, is this generator matrix in systematic form, the answer is no, because if to get it in systematic form what we need is our generator matrix should be of the form like this or something like this okay, but this is not in this particular form so we will have to get some identity matrix and some matrix p. Now by doing elementary row operation we can put this in systematic form so let us do that.

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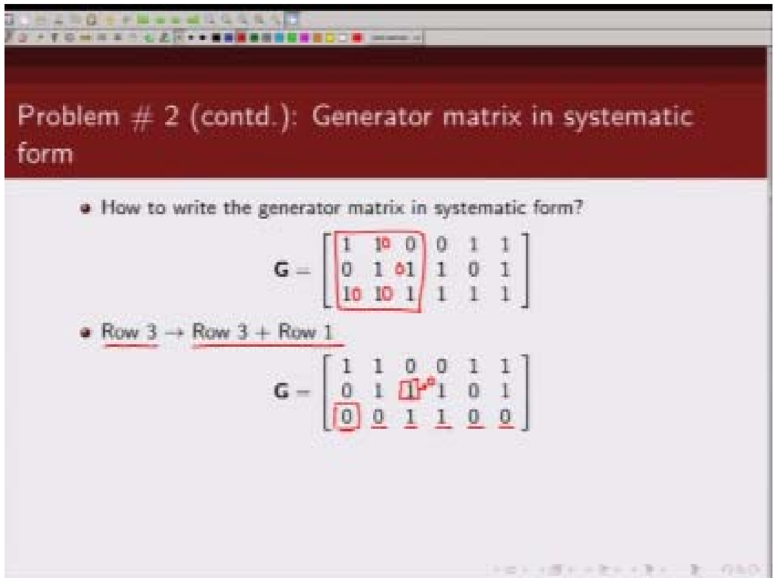


So note if we want to get let us say this in the form of identity what do we need, we would need basically here we would need a 0, here we would need a 0, here we would need a 0, here we would need a 0 right. So let us first try to get this 1 to 0, now how can we make this 0? So if do this transformation that row 3 is row 3 + row 1, so row 3 is row 3+ row 1,  if we do that then 1+1 this will be 0, 1+1 this is 0, 0+1 this is 1, 0+1 this is 1, 1+1 this is 0, and 1+1 this is 0, okay.

So we got a 0 here right, next we want a 0 here, we want this you want to make this 0 so how can we do that?

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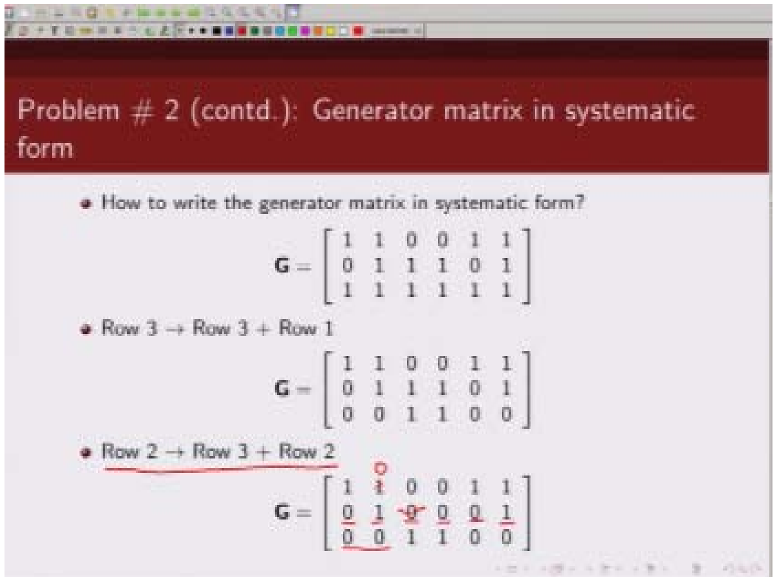


We do this transformation that row 2 is row 3+ row 1 row 2, so if we row 2 is row 2+row 3 then what is going to happen this will remain 0, this will remain 1, but this 1 will become 0, so let us do that. So this is 0+ 0 is 0, 1+ 0 is 1, 1+ 1 is 0, 1+1 is 0, 0+ 0 is 0 and 1+ 0 is 1 okay, so we got these zeros, we got this 0 okay, now what do we have to do, we will have to get this a here, we have to get a 0. So how can we get a 0 here?

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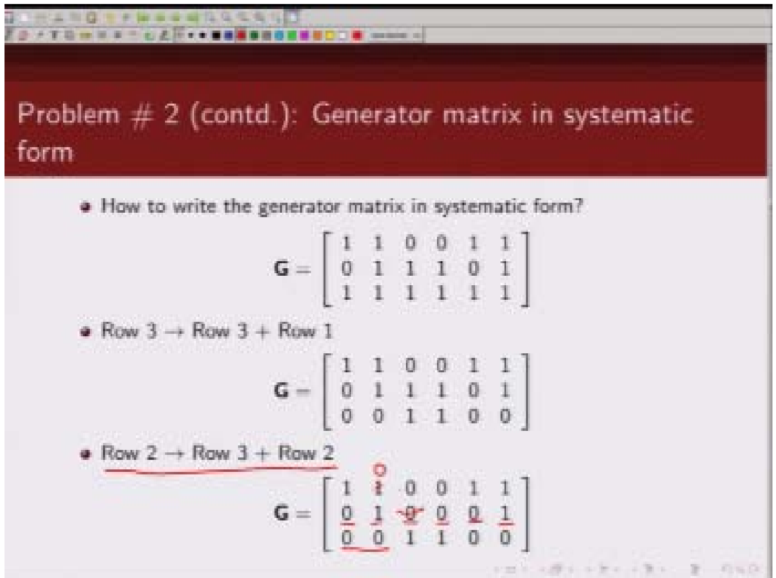


We will do this transformation we will add row 1 and row 2 and replace row1 by this, so we are going to add these 2 rows, if we add these 2 rows what is going to happen, this 1 will remain 1, 1+1 this will become 0, and this will remain 0, this will be 0, this will be 1, and this will be 0.

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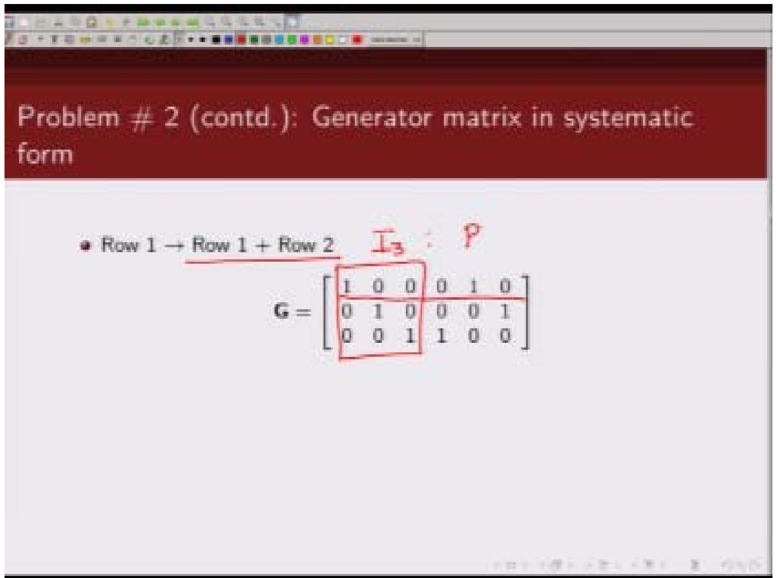


So if we do this transformation what we get is this. Now note that this is our NG matrix, it is a 3 cross, 3 identity matrix and then this is your another matrix speed okay, so by doing elementary row operation we are able to get our generator matrix in a systematic form.

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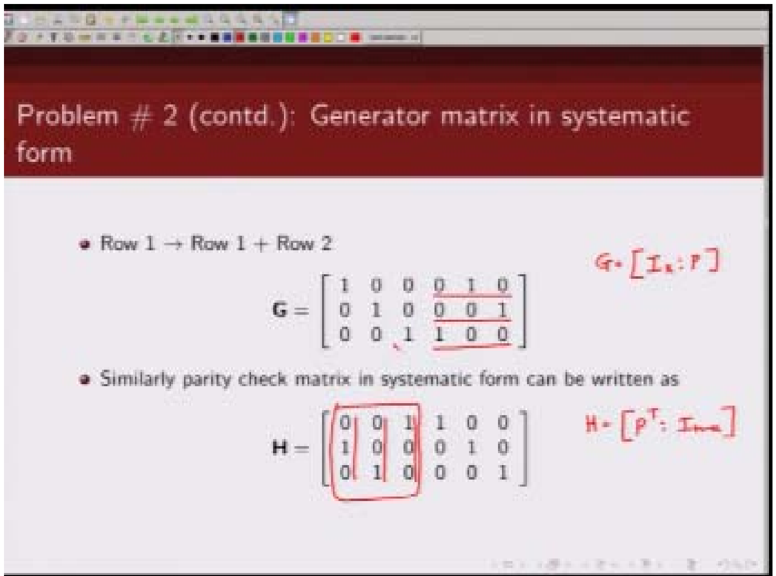


And if we have a generator matrix in a systematic form then we can very easily find out the parity gen matrix in systematic form, so this is like $I_k$ P then this H matrix will be P transpose IN - k so this is basically your p transpose, so this 010 this is will come here 010, 001 is, this is 001 and 100 is this, 100, and then you have this identity matrix which is here, okay.

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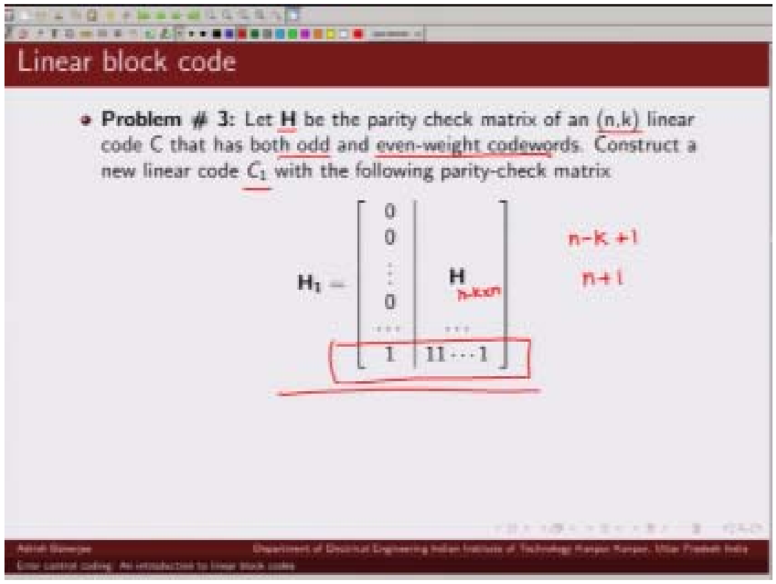


Next we are given a parity check matrix H of a linear block code with parameter nnk and it is given that this code C has both odd weight code words and even weight code words, in other words the number of ones in the code words, it contains both odd number of ones as well as even number of ones, and we are constructing a new code that we are calling as $C_1$ and the parity check matrix of the new code $C_1$ is given by this, so how do we find this new matrix parity check matrix H one we are adding a new column.

Which is 0 in the initial rows except in the last row where there is a 1, and here we are put our original n – k x n matrix and the last row is basically all ones okay, so the dimension of this matrix is so number of rows is n – k + 1 and number of columns are n+1.

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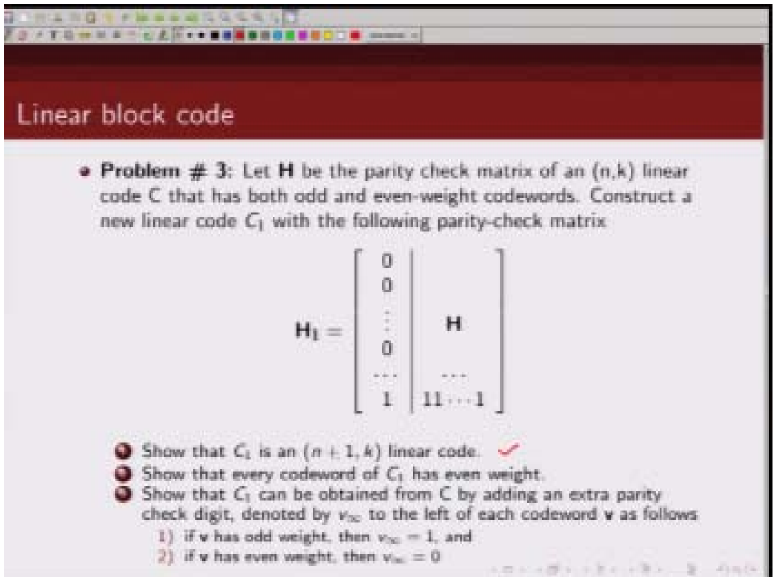


Now you are asked to show that the code generated by this parity check matrix each one is a linear code with parameters n+1 and k. Second thing you are asked to prove is that all the code words of this new code $C_1$ will have even weight, that means they will have even number of ones in them, the third thing that you have to prove is this new code $C_1$ is obtained from old code C by adding an additional parity bit which we are denoting by v infinity to the left of this code word and how do you select this parity bit v infinity? If the original code word has odd weight then you put v infinity as 1 otherwise if the original code word has even weight then you put this v infinity as 0. So let us prove one by one, let us first prove this, that code generated by this new parity check matrix is basically a new code with n given by n+1 and k given by k.

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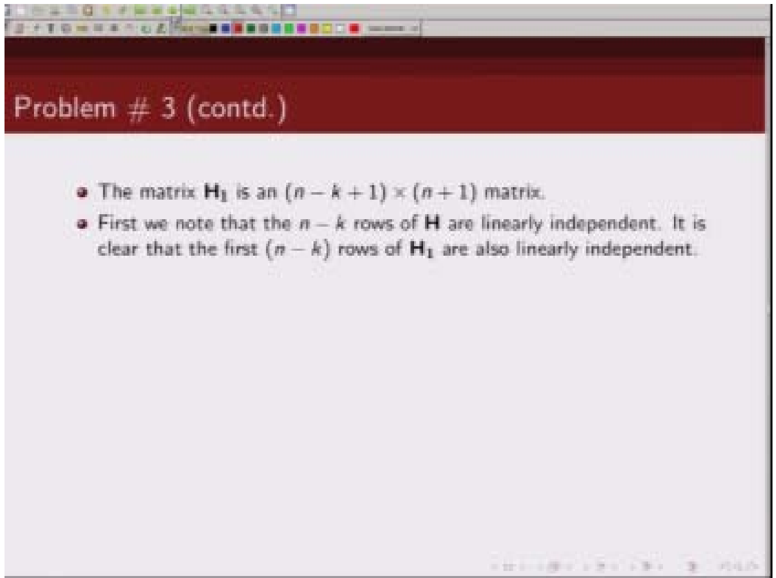


So as we know that this H matrix has these dimensions because we are adding a new column and we are adding a new row. Next now what is the rank of original matrix H, the rank of the original matrix H is n –k that means the n – k rows of the original parity check matrix H are linearly independent okay.

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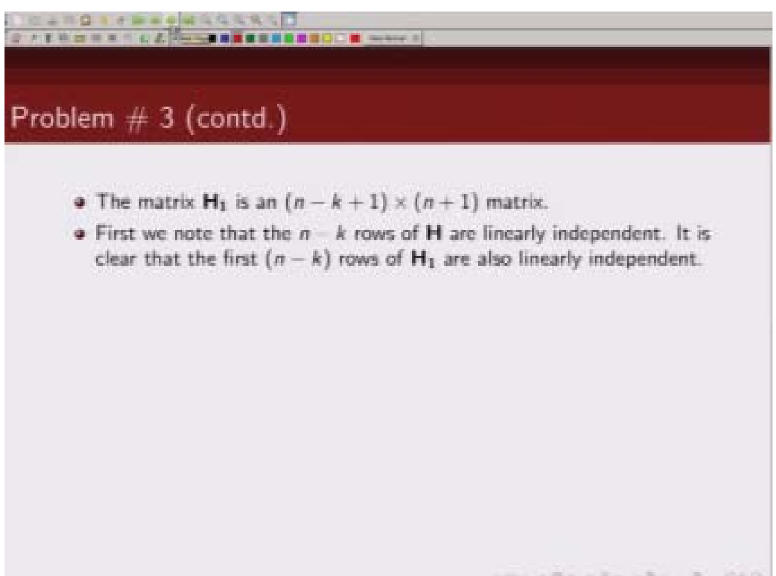

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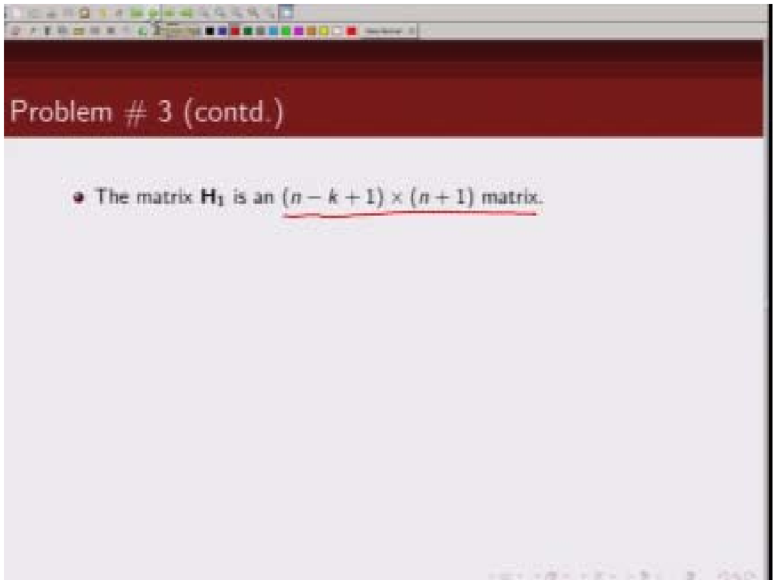


Now go back and look at the new construction.

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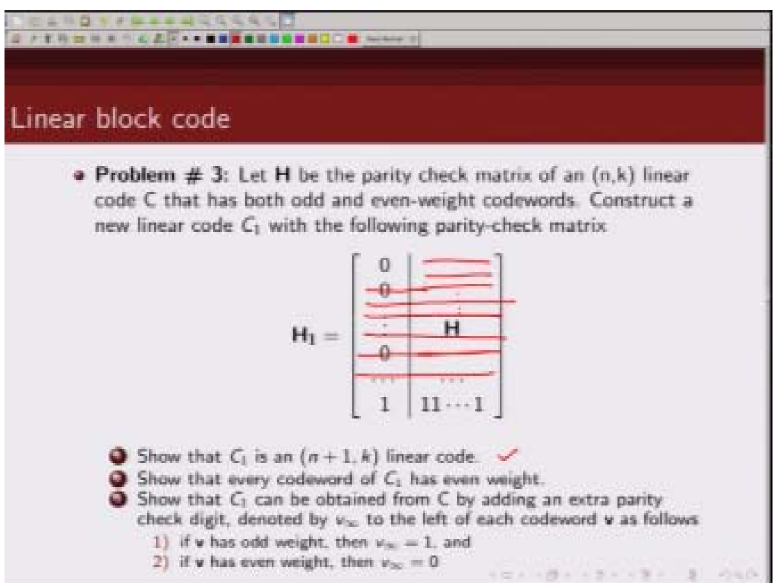


So these n-k rows are linearly independent and what have we added here, we have added 0 here. So these new rows will, these new n-k rows will also be linearly independent.

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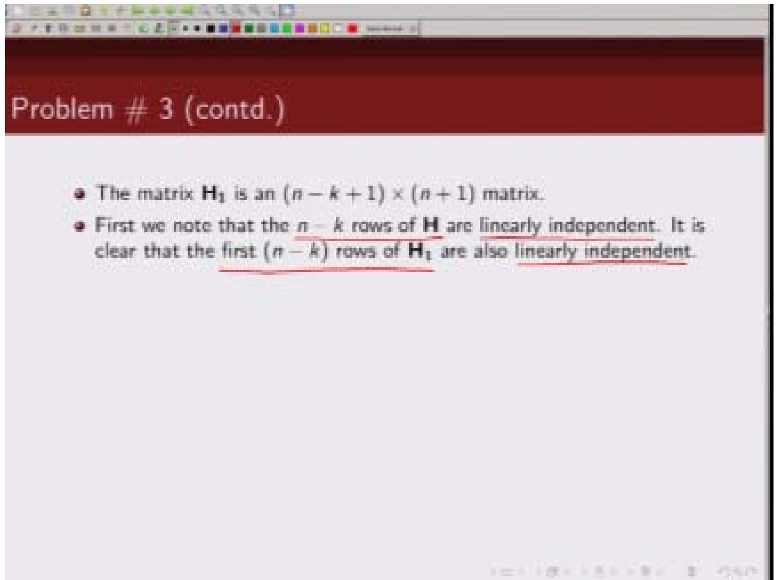


So that is what we are saying that since n-k rows of the original parity check matrix H are linearly independent. So the first n-k rows of the original parity check matrix $H_1$ will also be linearly independent.

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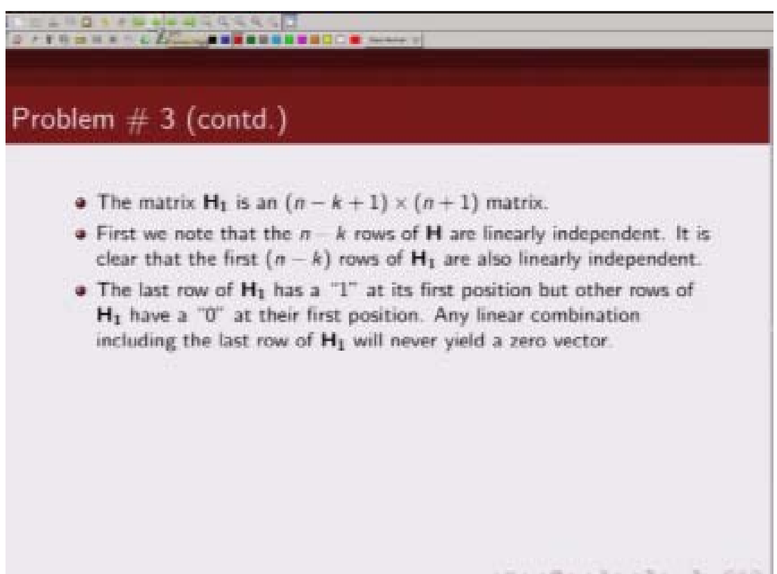

Now let us look at the last row of

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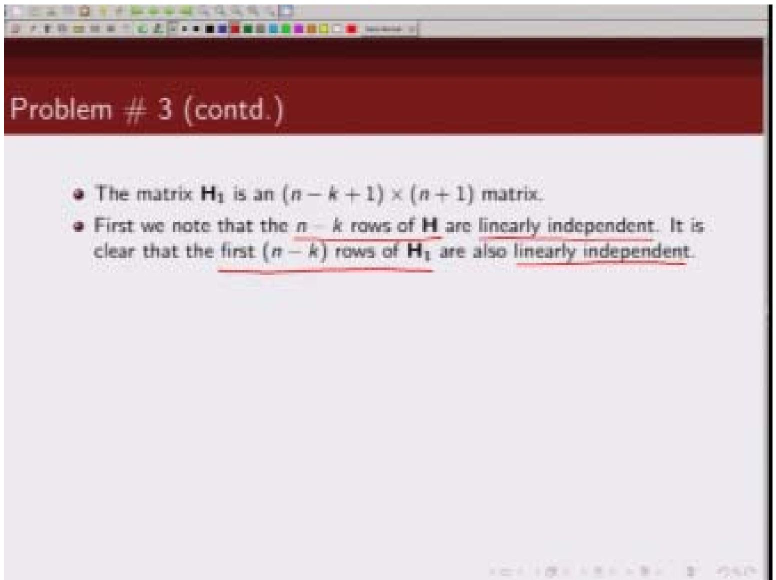


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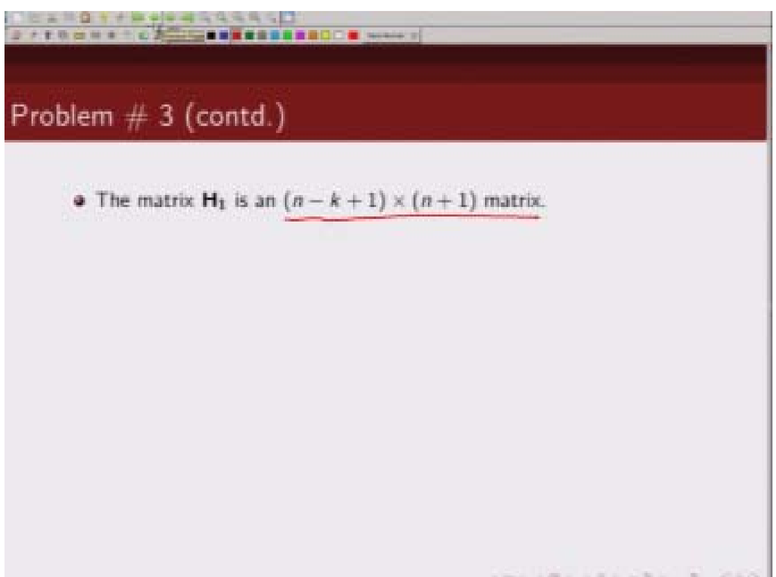


This new parity check matrix $H_1$.

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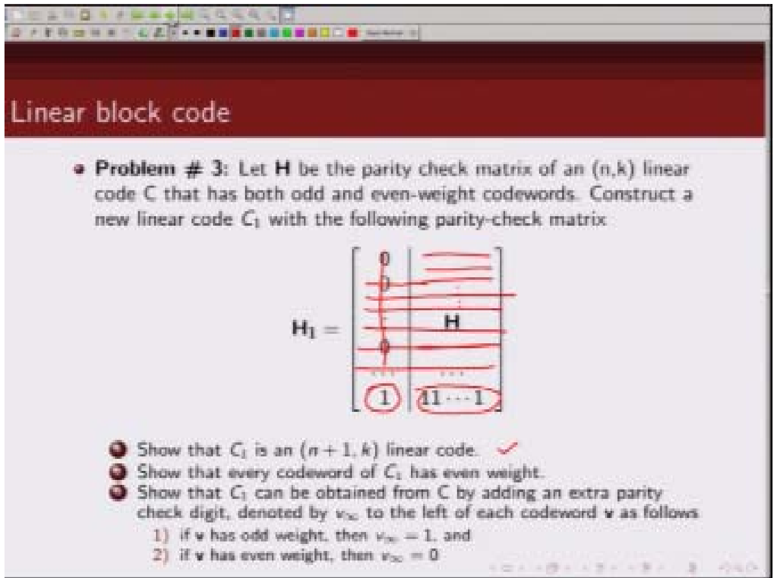


Note that we have a 1 here and these are all ones here. Whereas here all of these are zeros so this new row will also be linearly independent from any of the other rows of this parity check matrix $H_1$.

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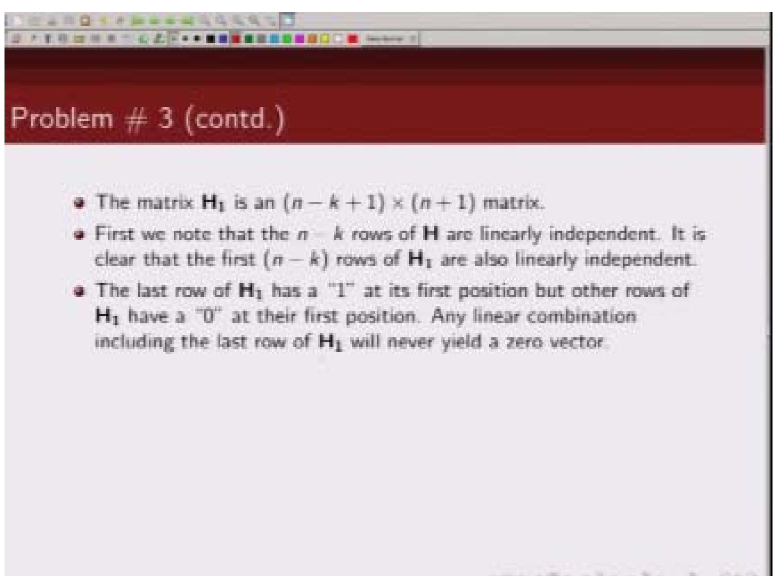

So any linear combination including the last row of $H_1$ will never result in a all zero vector. So what does it mean? It means that n-k+1 rows of a new parity check matrix $H_1$ are linearly independent.

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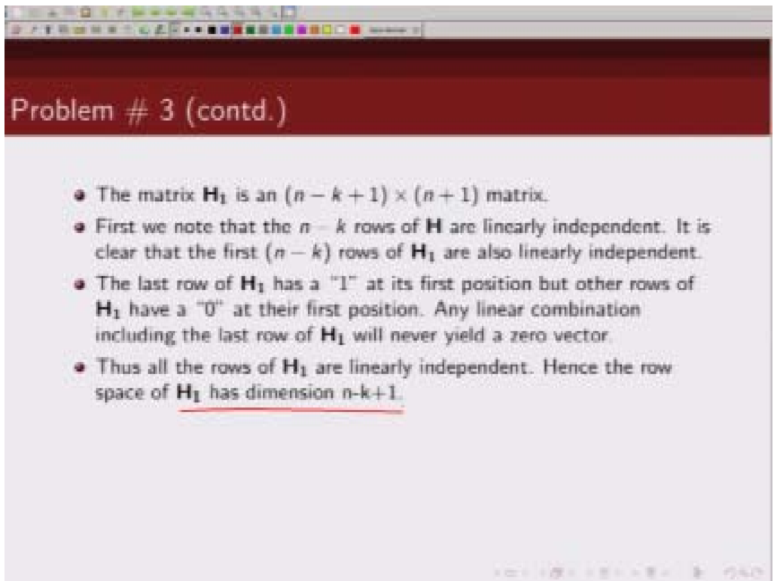


Hence, the dimension of $H_1$ is n-k+1.

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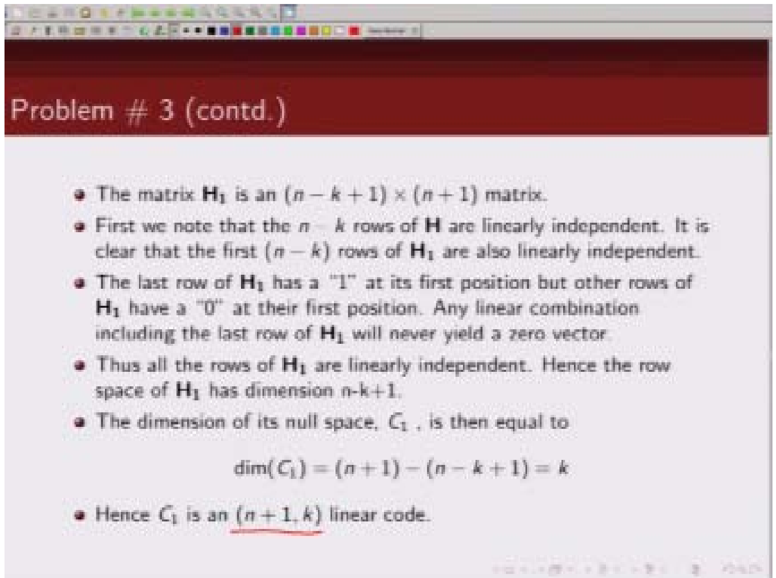


Now how do we find the dimension of basically the null space of this parity check matrix $H_1$, this is given by so number of columns is n+1, the dimension of $H_1$ is given by this, so this is the dimension of the null space of this parity check matrix. So then basically the number of information bits is then k and number of coded bit is n+1. So this proves that $C_1$ is an (n+1, k) linear code.

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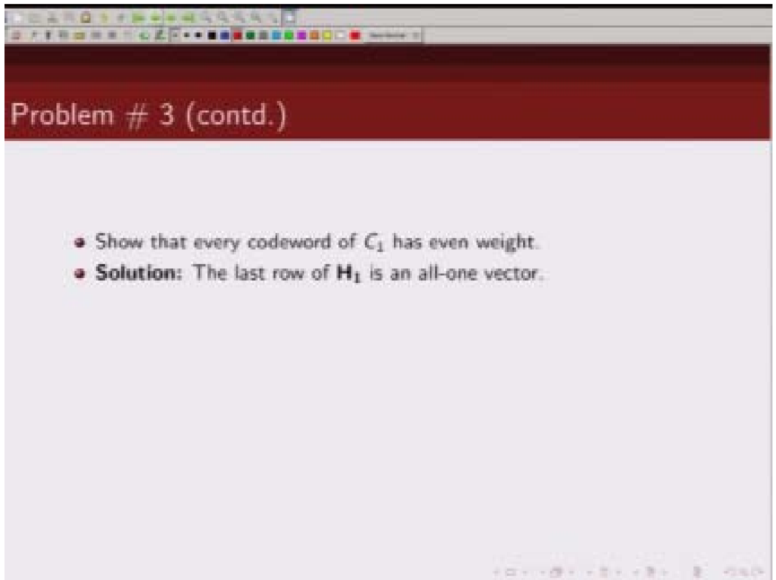


Next we are going to show is, every code word of $C_1$ has even weight, so how do we prove this? Please note that the last row of this parity check matrix $H_1$ contain all-one vector, if you go back.

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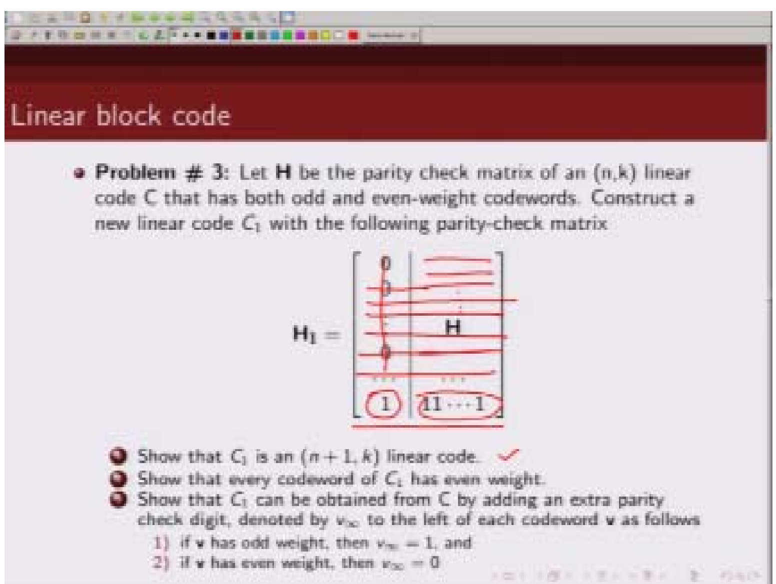


Recall the last row of this parity check matrix has all ones.

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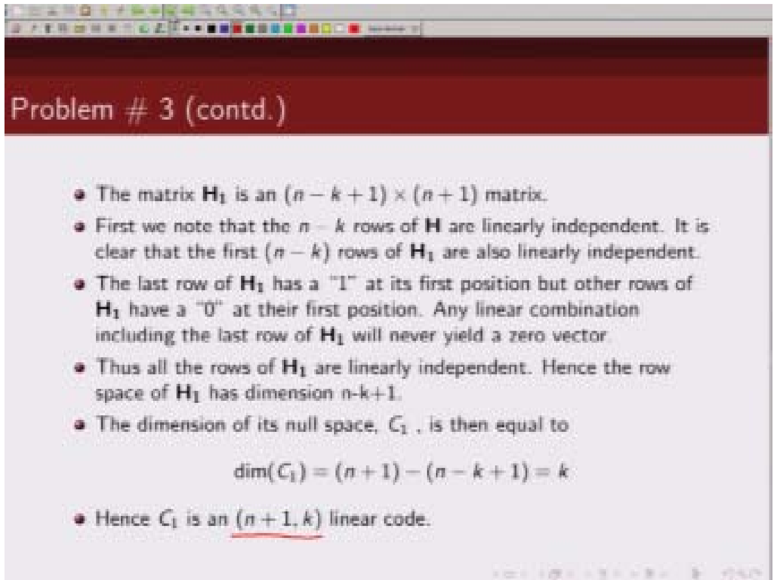


And if v is a valid code word what property does it satisfy?

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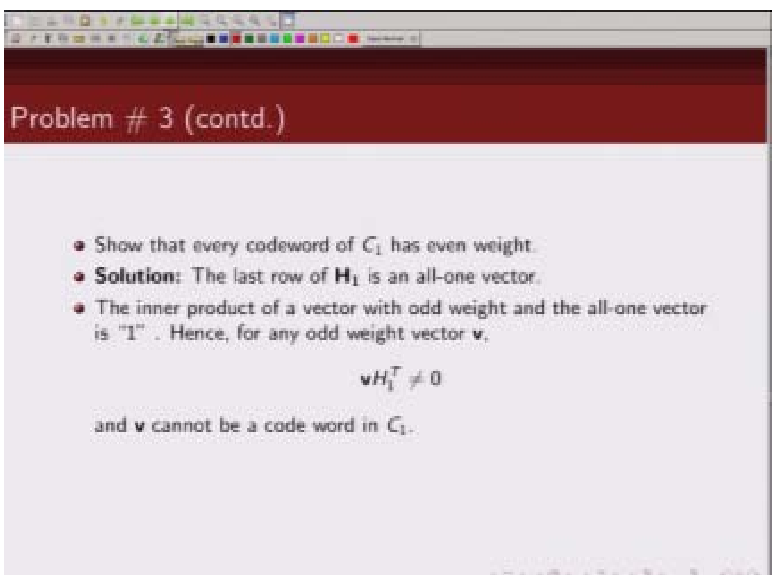


If v is a valid code word then $vH^T$ should be 0.

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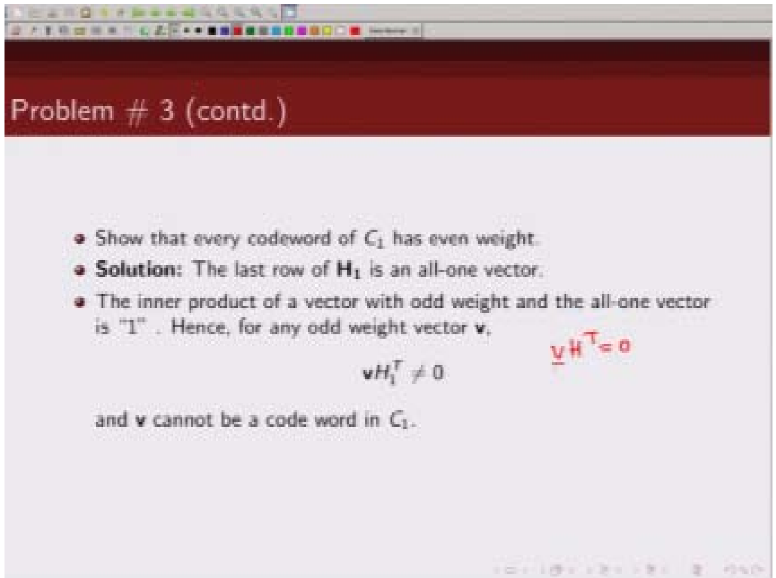


Now let us take a code word, let us say there exists a code word with odd weight which is generated by, described by this parity check matrix $H_1$. Now if we do $vH^T$ so when you are going to take the other product of this code vector v with the last row of this parity check matrix what will you get? You will essentially get sum of

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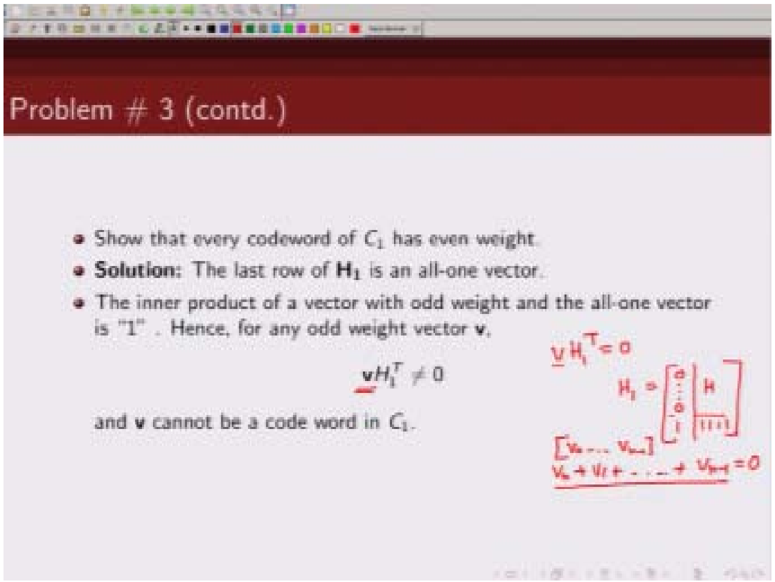


So basically if you do $vH^T$ so your H is of the, so $vH^T$ where H is $H_1^T$ H1 is given like this, this is all zeros, here we have parity check matrix H and you have here all one, sorry you have your this is one, this is one and this all-one vector. Now when you do $vH^T$ so let us say v is your $v_0$ to $v_{n-1}$. When you do $vH_1^T$ what you will get is $v_0+v_1+v_2$ up to $v_{n-1}$ is going to be zero. Now if this v has odd number of ones, this sum cannot be zero right. Hence we proved that v has to have even number of ones.

Because we know if v is a valid code word then $vH_1^T$ should be zero. So if we do $vH^T$ because the last row of this para digit matrix $H_1$ is all one, the condition that we will get is individual components of this parity code vector v, $v_0+v_1+v_2+v_3$ up to $v_{n-1}$ basically $v_{n-1}$ they should all add up to zero. Hence we cannot have an odd weight vector which will give $vH^T$, $vH_1^T$ to be zero.

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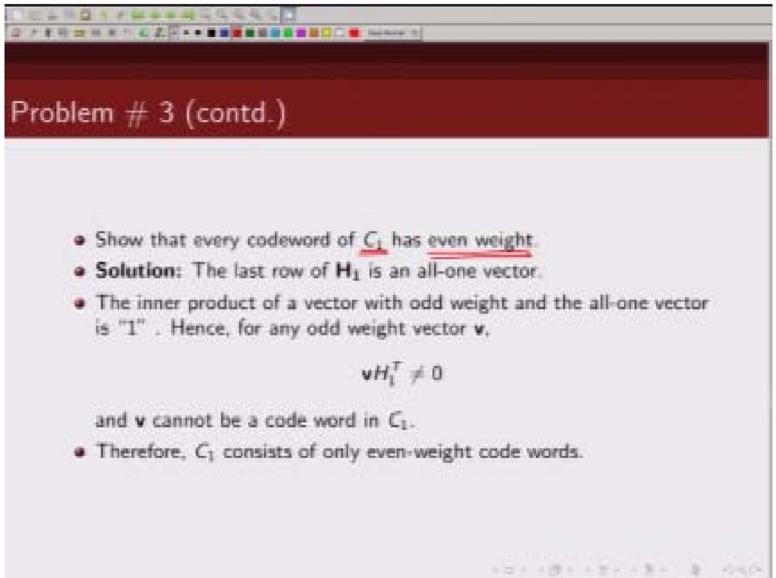


Hence every code word in $C_1$ has even weight.

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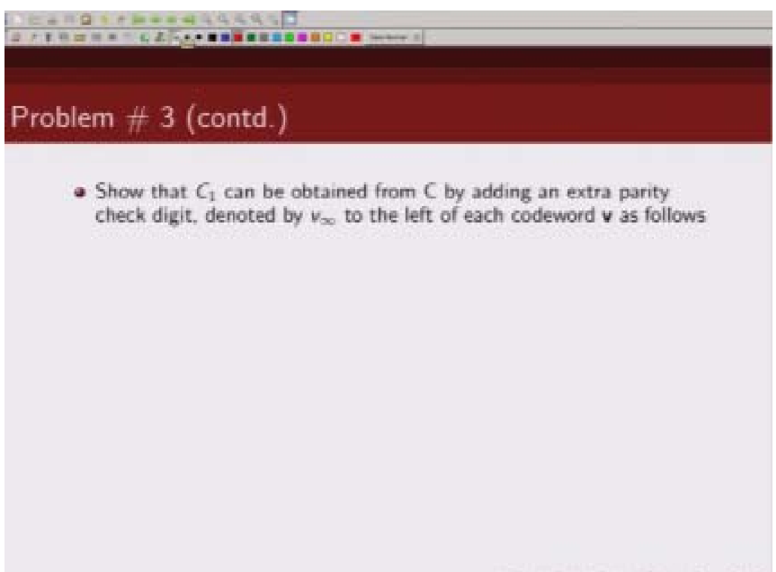


Next we are going to show, prove how we can generate this new code $C_1$ from the original code C. and what did we

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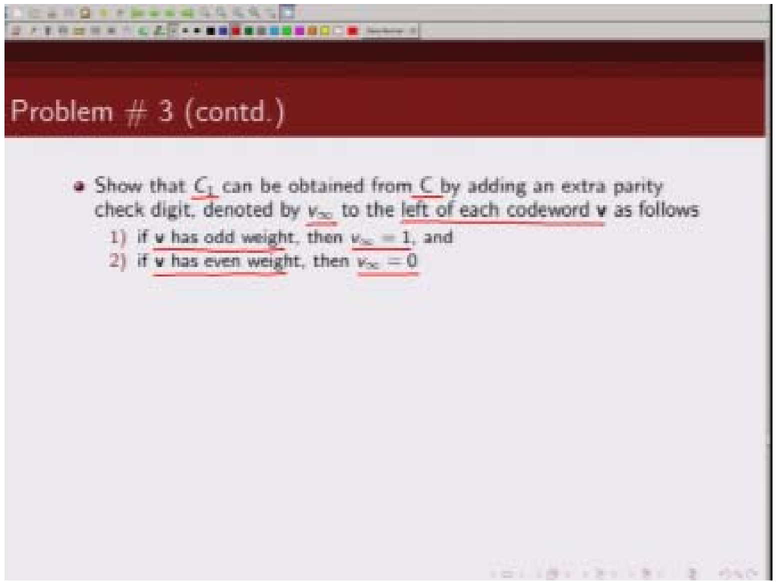


Mention, we mentioned that this new code $C_1$ can be obtained from the original code C by adding an extra parity bit which we are denoting by $v_\infty$ to the left of the original code word v in this fashion. If v has odd weight then $v_\infty$ is odd parity and if v has even weight then $v_\infty$ is 0 parity, is zero is even parity. So let us prove this

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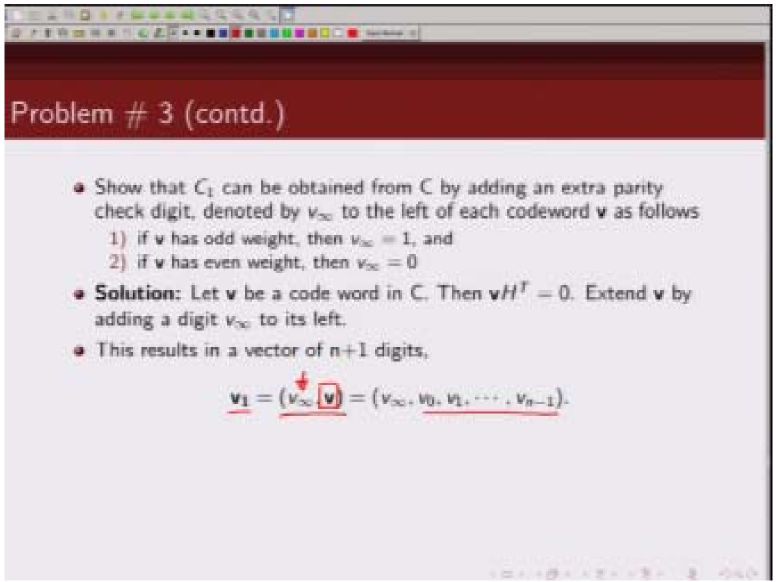


So let us see, let v be a code word in C then $vH^T$ will be zero. Now we are extending this original code v by adding a bit $v_\infty$ to its left. So we are defining a new code word of length n+1 which is defined as follows. So this is your original code word v which is basically $v_0$ to $v_{n-1}$ and then this is the additional parity bit that you added to the left.

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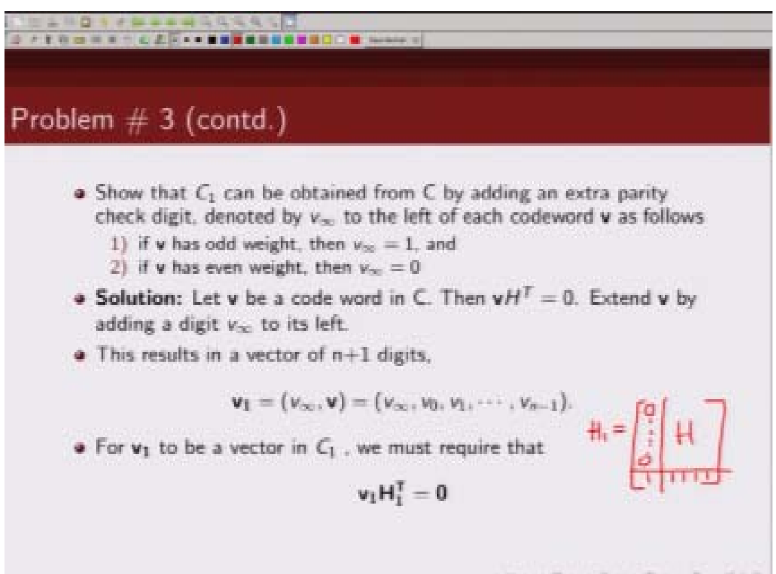

Now if $v^1$ is a valid code word then it should satisfy the property that $v_1H_1^T$ should be zero. And what is our $H_1$, again please recall our $H_1$ is a form like this so the first column here is zero, then you have here the original H matrix H and this is all one vector. So when we do $vH^T$ so when $v_1$ will be multiplied by this last row what we will get is

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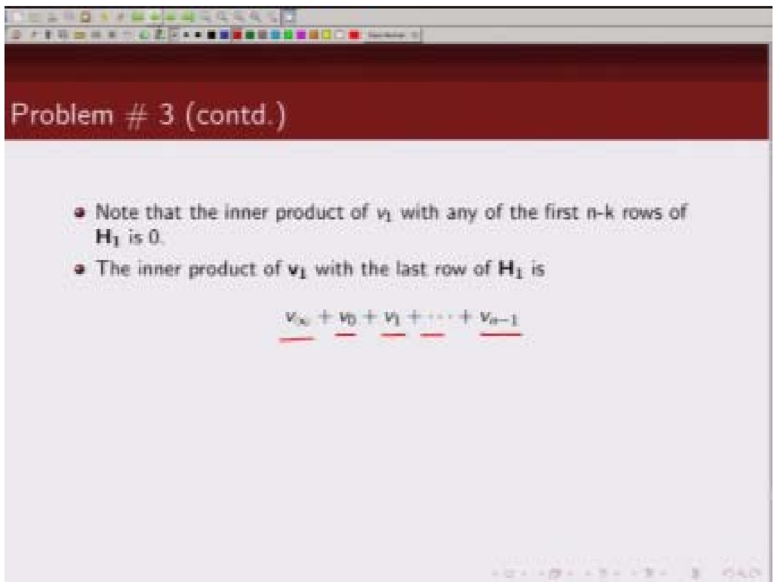


Condition of this form, $v_\infty + v_0 + v_1 + v_2 + v_{n-1}$ that is basically should be equal to zero okay. Now how are we getting this condition again?

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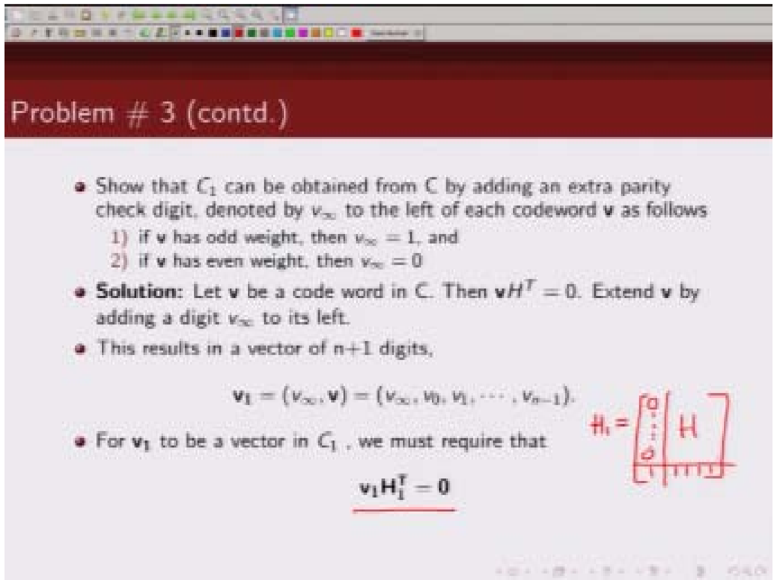


We are making use of the fact that $v_1H_1^T$ is zero and $H_1$ is a form like this. So when we do $v_1H_1^T$ the last row which will be in $H^T$ will be last column, if you multiple v with that $H_1^T$ column what we would get is

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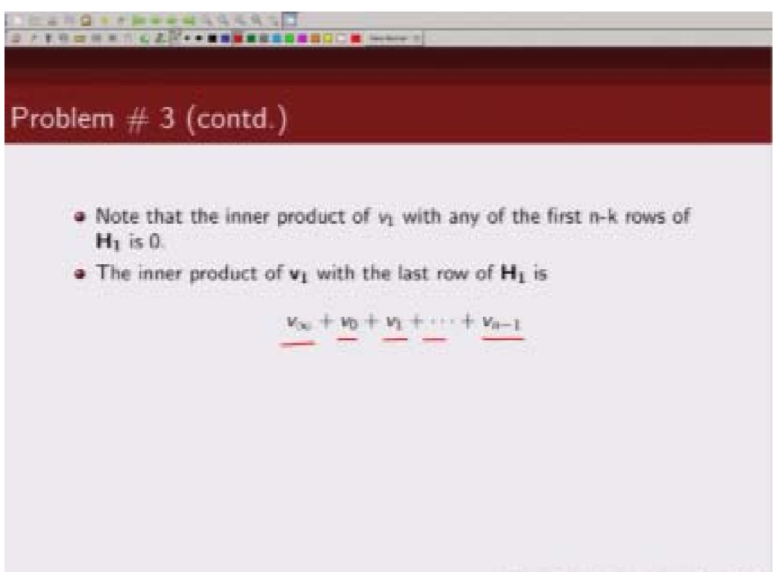

Something of this form. Now this should be equal to zero if v1 is a valid code word right.

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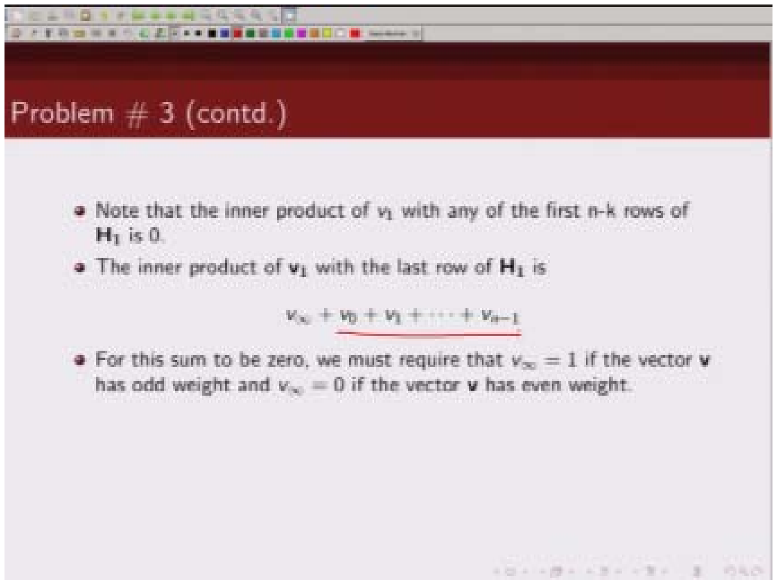


So if the sum has to be zero what do we need? So if the original code word is odd weight code word we need $v_\infty$ to be one. And if the original code word is even parity then this new parity bit should be zero. And that is basically the proof, how we can extend our original code to construct a new code.

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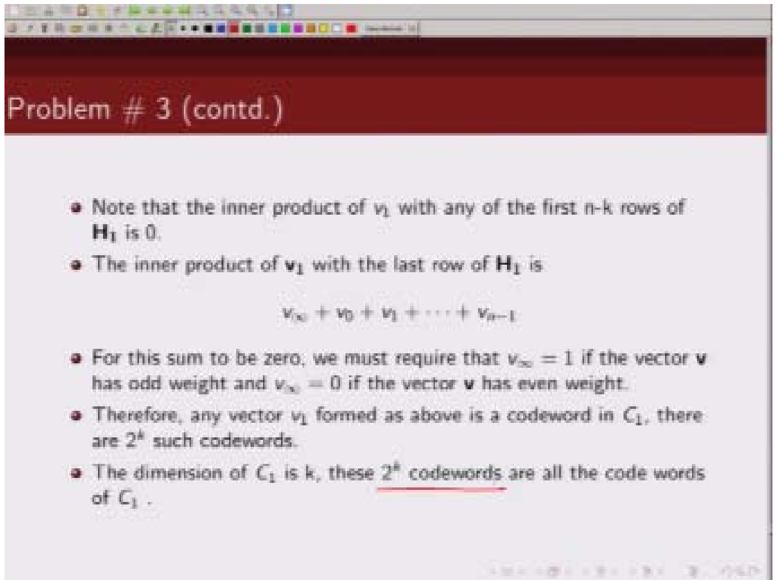


And this is basically if this is equal to 0 we know that $vH^T$, $v_1H_1^T$ is zero. So $v_1$ is a valid code word in $C_1$. And total there are $2^k$ code words. This we have already proved in the first part that there are total $2^k$ code words of length n+1okay.

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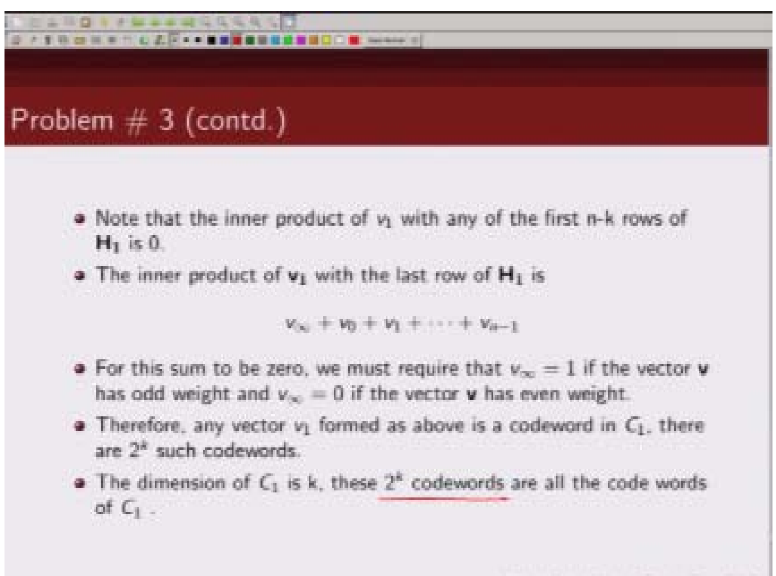

So with this I will conclude this lecture. Thank you.


**<u>Acknowledgement</u>**
**Ministry of Human Resource & Development**

**Prof. Satyaki Roy**
**Co-ordinator, NPTEL IIT Kanpur**

**NPTEL Team**
**Sanjay Pal**
**Ashish Singh**
**Badal Pradhan**
**Tapobrata Das**
**Ram Chandra**
**Dilip Tripathi**
**Manoj Shrivastava**
**Padam Shukla**
**Sanjay Mishra**
**Shubham Rawat**
**Shikha Gupta**
**K. K. Mishra**
**Aradhana Singh**
**Sweta**
**Ashutosh Gairola**
**Dilip Katiyar**
**Sharwan**
**Hari Ram**
**Bhadra Rao**
**Puneet Kumar Bajpai**
**Lalty Dutta**
**Ajay Kanaujia**
**Shivendra Kumar Tiwari**
**an IIT Kanpur Production**