

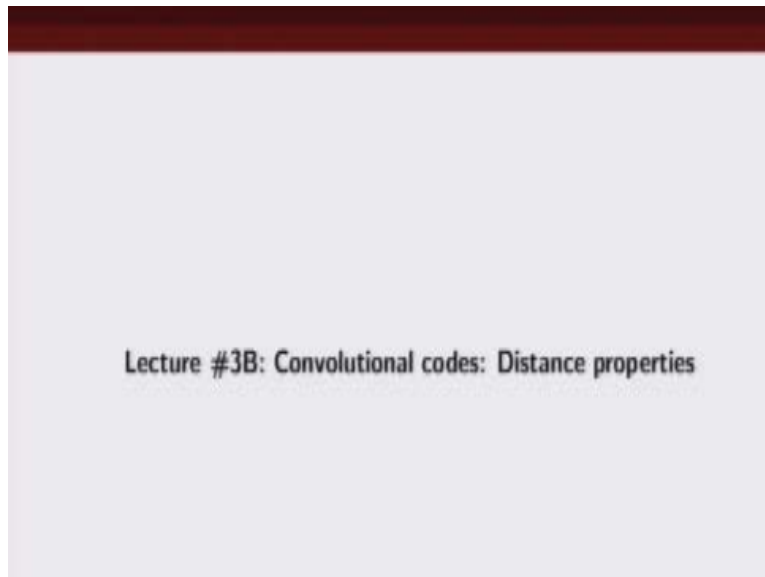
**Indian Institute of Technology Kanpur**  
**National Programme on Technology Enhanced Learning (NPTEL)**  
**Course Title**  
**Error Control Coding: An Introduction to Convolutional Codes**

**Lecture – 3 B**  
**Convolutional Codes: Distance Properties**

by  
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**Dept. Electrical Engineering, IIT Kanpur**

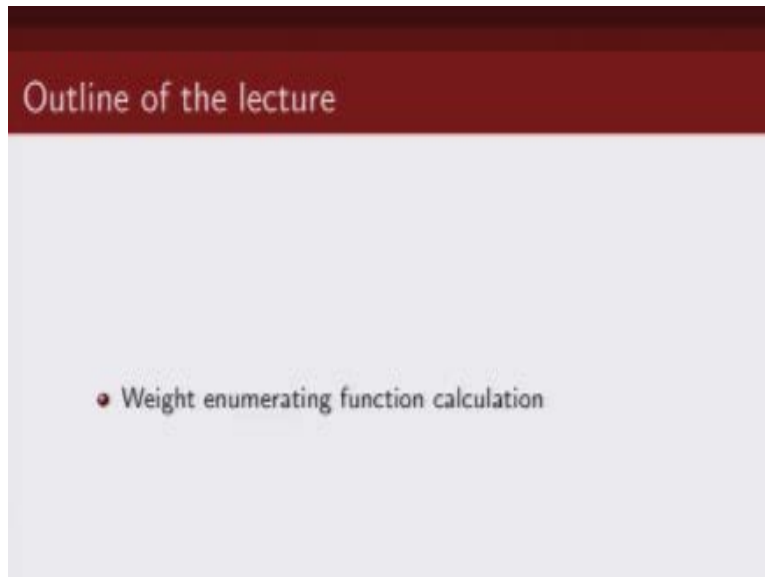
Welcome to the course on Error Control Coding, an introduction to Convolutional Codes.

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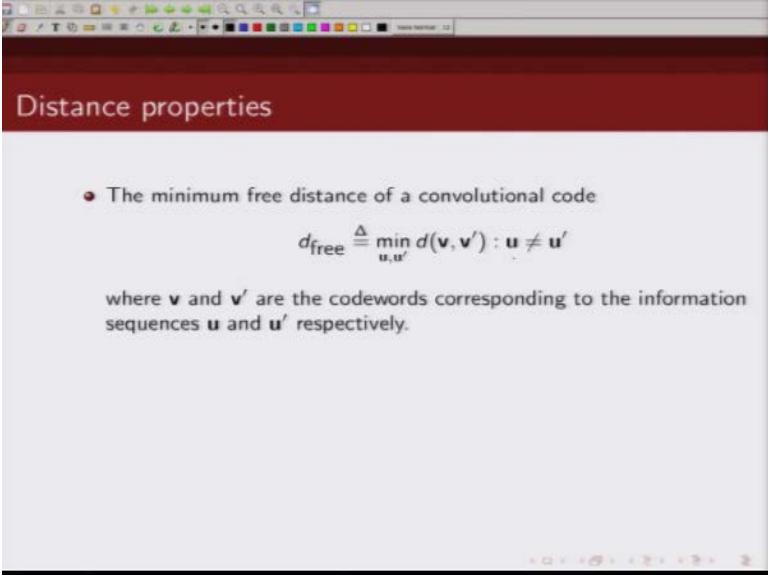
So in this lecture we are going to talk about how we can find out the way distribution of a convolutional codes and we are going to discuss about the distance properties of convolutional code.

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So this lecture deals with how we can enumerate the distance profile or the way distribution of a convolutional code.

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Distance properties

- The minimum free distance of a convolutional code

$$d_{\text{free}} \triangleq \min_{\mathbf{u}, \mathbf{u}'} d(\mathbf{v}, \mathbf{v}') : \mathbf{u} \neq \mathbf{u}'$$

where  $\mathbf{v}$  and  $\mathbf{v}'$  are the codewords corresponding to the information sequences  $\mathbf{u}$  and  $\mathbf{u}'$  respectively.

So we can define before we discuss a technique to find out the way distribution let us define what do we mean by a minimum free distance of a convolutional code, so minimum free distance of a convolutional code is defined as, minimum hamming distance between two codes  $\mathbf{v}$  and  $\mathbf{v}'$ , where  $\mathbf{v}$  and  $\mathbf{v}'$  are two code words corresponding to information sequence  $\mathbf{u}$  and  $\mathbf{u}'$  where  $\mathbf{u}$  and  $\mathbf{u}'$  are two different information sequences.

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Distance properties

- The minimum free distance of a convolutional code

$$d_{\text{free}} \triangleq \min_{\mathbf{u}, \mathbf{u}'} d(\mathbf{v}, \mathbf{v}') : \mathbf{u} \neq \mathbf{u}'$$

where  $\mathbf{v}$  and  $\mathbf{v}'$  are the codewords corresponding to the information sequences  $\mathbf{u}$  and  $\mathbf{u}'$  respectively.

- $d_{\text{free}}$  is the minimum Hamming distance between any two code sequences in the code.

So it is basically free distance is the minimum hamming distance between any two code words in a convolutional code.

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Distance properties

- The minimum free distance of a convolutional code

$$d_{\text{free}} \triangleq \min_{\mathbf{u}, \mathbf{u}'} d(\mathbf{v}, \mathbf{v}') : \mathbf{u} \neq \mathbf{u}'$$

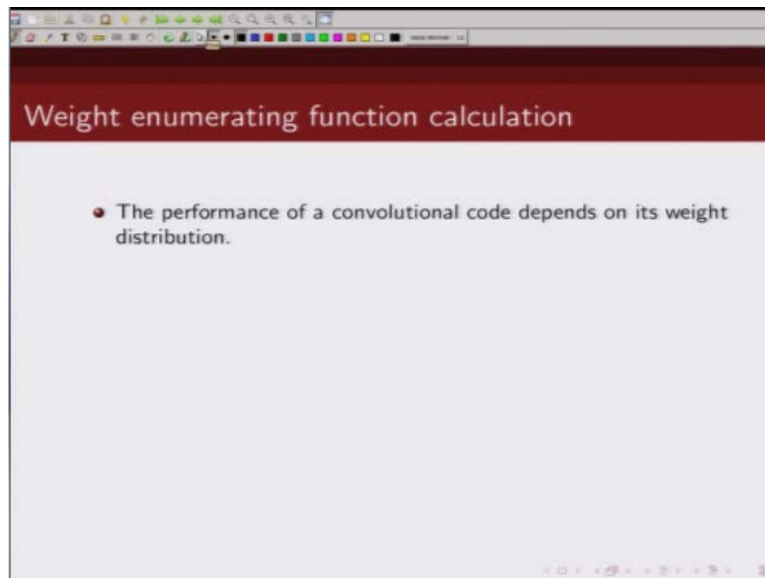
where  $\mathbf{v}$  and  $\mathbf{v}'$  are the codewords corresponding to the information sequences  $\mathbf{u}$  and  $\mathbf{u}'$  respectively.

- $d_{\text{free}}$  is the minimum Hamming distance between any two code sequences in the code.
- $d_{\text{free}}$  is also the minimum weight non-zero sequence, i.e.

$$d_{\text{free}} = \min\{w(\mathbf{v}) : \mathbf{u} \neq 0\}$$

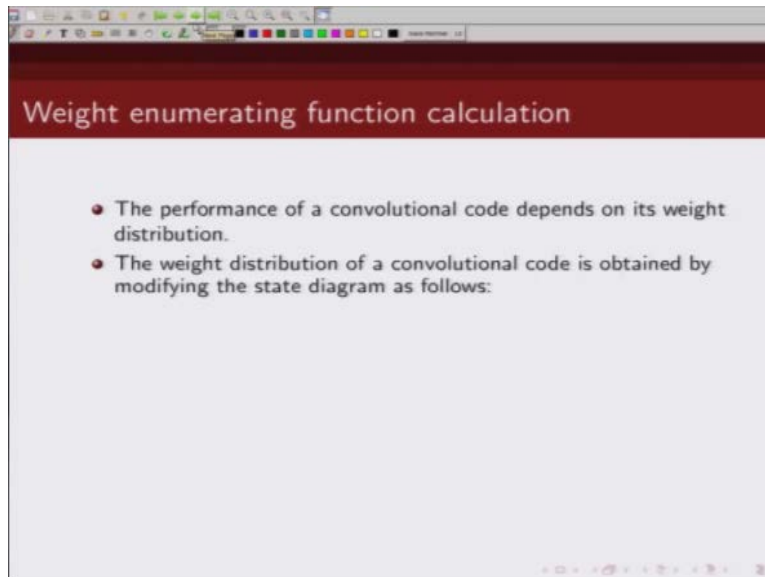
And this is same as minimum weight non-zero, so it is a minimum weight of a non-zero code word.

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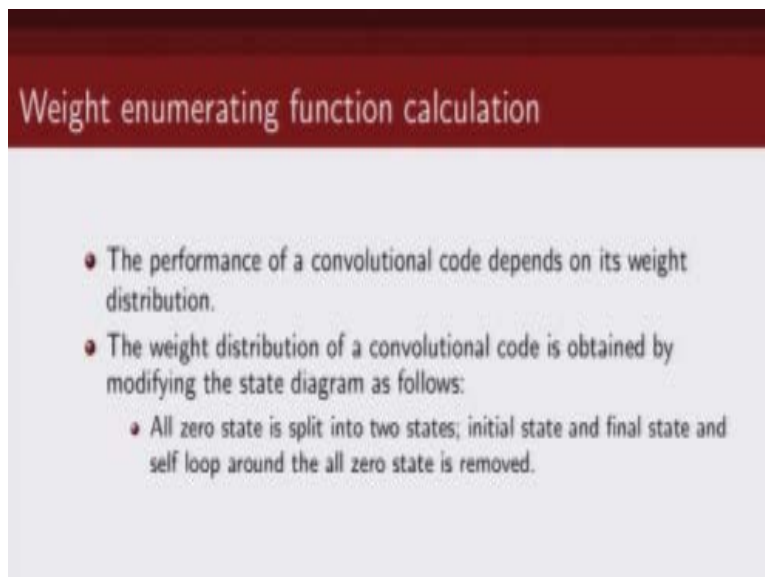
We know that performance of any convolutional code depends on its weight distribution and that is why we are interested to find out what is a weight distribution of a convolutional code. In this lecture we are going to talk about a method based on Mason's gain formula to compute a weight distribution for convolutional code.

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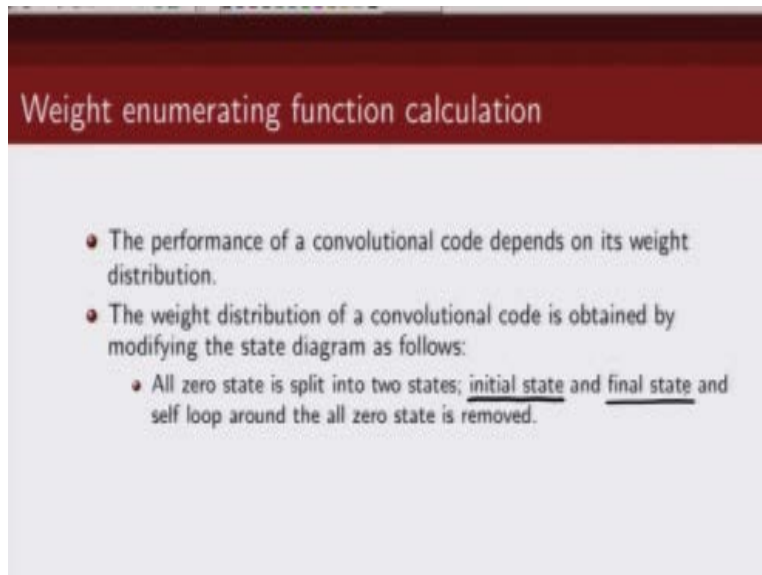
So in this we are first going to modify this state diagram of a convolutional code.

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And how are we going to modify this state diagram? We are going to split the all zero state into two state.

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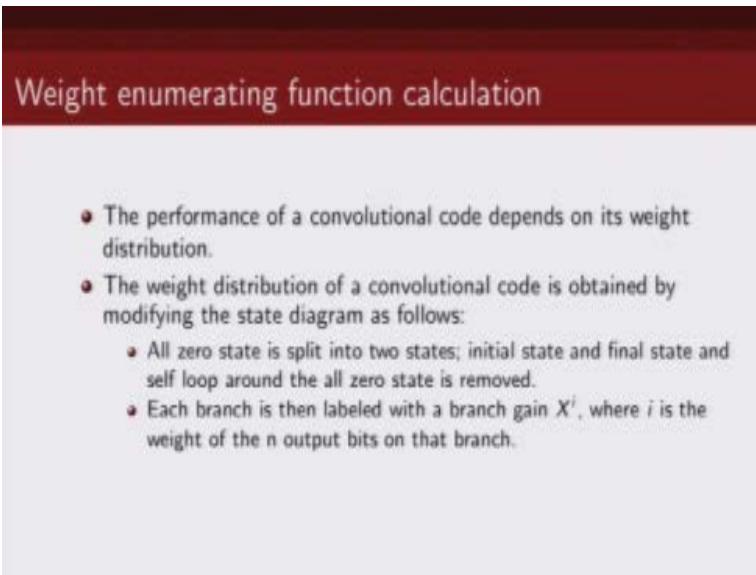


Weight enumerating function calculation

- The performance of a convolutional code depends on its weight distribution.
- The weight distribution of a convolutional code is obtained by modifying the state diagram as follows:
  - All zero state is split into two states; initial state and final state and self loop around the all zero state is removed.

One initial state and second final state and we are going to remove the self loop around the all zero state.

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### Weight enumerating function calculation

- The performance of a convolutional code depends on its weight distribution.
- The weight distribution of a convolutional code is obtained by modifying the state diagram as follows:
  - All zero state is split into two states; initial state and final state and self loop around the all zero state is removed.
  - Each branch is then labeled with a branch gain  $X^i$ , where  $i$  is the weight of the  $n$  output bits on that branch.

Next each branch which is linking from one state to another state this branch will be labeled by the output weight of the code words so we are going to denote by  $x^i$ , where  $i$  will be the weight of the coded bits.



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### Weight enumerating function calculation

- The performance of a convolutional code depends on its weight distribution.
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  - All zero state is split into two states; initial state and final state and self loop around the all zero state is removed.
  - Each branch is then labeled with a branch gain  $X^i$ , where  $i$  is the weight of the  $n$  output bits on that branch.

So for example if we make a transition from state 0 to state 01 when an input 1 comes and output is 11 so in that case.

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### Weight enumerating function calculation

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  - All zero state is split into two states; initial state and final state and self loop around the all zero state is removed.
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Since the output is 11 we will label that branch by  $x^2$ .

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### Weight enumerating function calculation

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- The weight distribution of a convolutional code is obtained by modifying the state diagram as follows:
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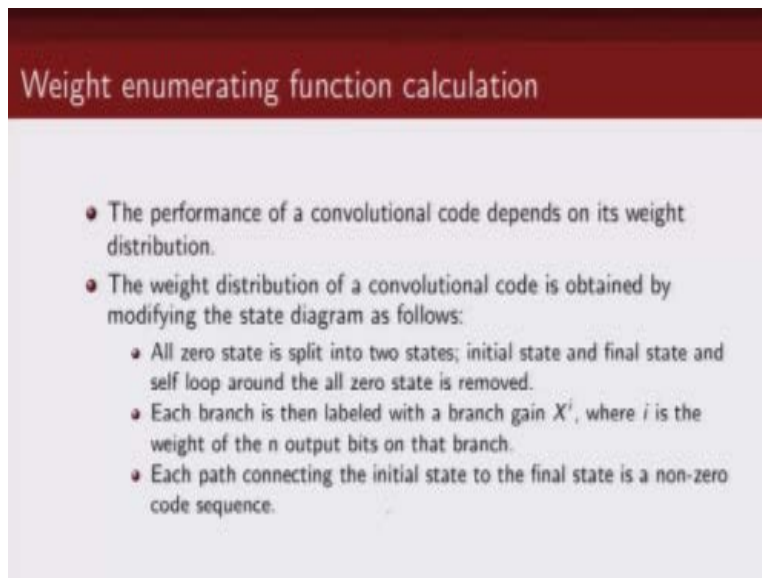
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### Weight enumerating function calculation

- The performance of a convolutional code depends on its weight distribution.
- The weight distribution of a convolutional code is obtained by modifying the state diagram as follows:
  - All zero state is split into two states; initial state and final state and self loop around the all zero state is removed.
  - Each branch is then labeled with a branch gain  $X^i$ , where  $i$  is the weight of the  $n$  output bits on that branch.
  - Each path connecting the initial state to the final state is a non-zero code sequence.

So now after we split this all zero sequence, all zero state into two states initial state and final state what we will have is each path starting from this initial state to the final state that will be our valid code word.

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Weight enumerating function calculation

- The performance of a convolutional code depends on its weight distribution.
- The weight distribution of a convolutional code is obtained by modifying the state diagram as follows:
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  - Each branch is then labeled with a branch gain  $X^i$ , where  $i$  is the weight of the  $n$  output bits on that branch.
  - Each path connecting the initial state to the final state is a non-zero code sequence.

So each path connecting the initial state to the final state is a valid non-zero code word because we have removed the self loop around the all zero state.

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### Weight enumerating function calculation

- The performance of a convolutional code depends on its weight distribution.
- The weight distribution of a convolutional code is obtained by modifying the state diagram as follows:
  - All zero state is split into two states; initial state and final state and self loop around the all zero state is removed.
  - Each branch is then labeled with a branch gain  $X^i$ , where  $i$  is the weight of the  $n$  output bits on that branch.
  - Each path connecting the initial state to the final state is a non-zero code sequence.

So this modified state diagram is now going to show us all possible non-zero code words.

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## Weight enumerating function calculation

- The performance of a convolutional code depends on its weight distribution.
- The weight distribution of a convolutional code is obtained by modifying the state diagram as follows:
  - All zero state is split into two states; initial state and final state and self loop around the all zero state is removed.
  - Each branch is then labeled with a branch gain  $X^i$ , where  $i$  is the weight of the  $n$  output bits on that branch.
  - Each path connecting the initial state to the final state is a non-zero code sequence.
  - The path gain is the product of the branch gains along a path.

We define a path gain as product of branch gains along a path.

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## Weight enumerating function calculation

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  - All zero state is split into two states; initial state and final state and self loop around the all zero state is removed.
  - Each branch is then labeled with a branch gain  $X^i$ , where  $i$  is the weight of the  $n$  output bits on that branch.
  - Each path connecting the initial state to the final state is a non-zero code sequence.
  - The path gain is the product of the branch gains along a path.
  - The weight of a code sequence is the power of  $X$  in the path gain of the corresponding path.

And the weight of a code sequence will be nothing but the power of  $x$ , in the path gain of the corresponding path because what we are doing is, so each branch is labeled by its corresponding output weight. So if we look at each path going from the initial state to the final state and we look at the power of  $x$ .

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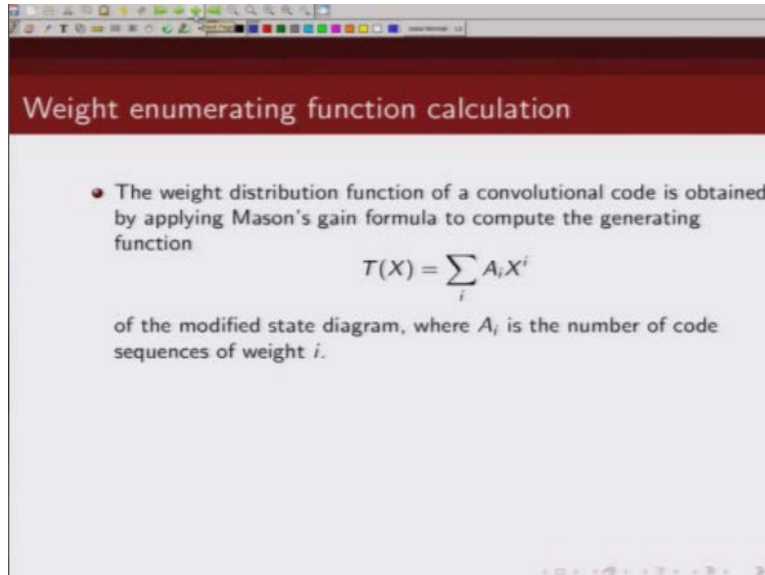
### Weight enumerating function calculation

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  - Each branch is then labeled with a branch gain  $X^i$ , where  $i$  is the weight of the  $n$  output bits on that branch.
  - Each path connecting the initial state to the final state is a non-zero code sequence.
  - The path gain is the product of the branch gains along a path.
  - The weight of a code sequence is the power of  $X$  in the path gain of the corresponding path.

That will give us the overall weight of that particular non-zero code sequence.

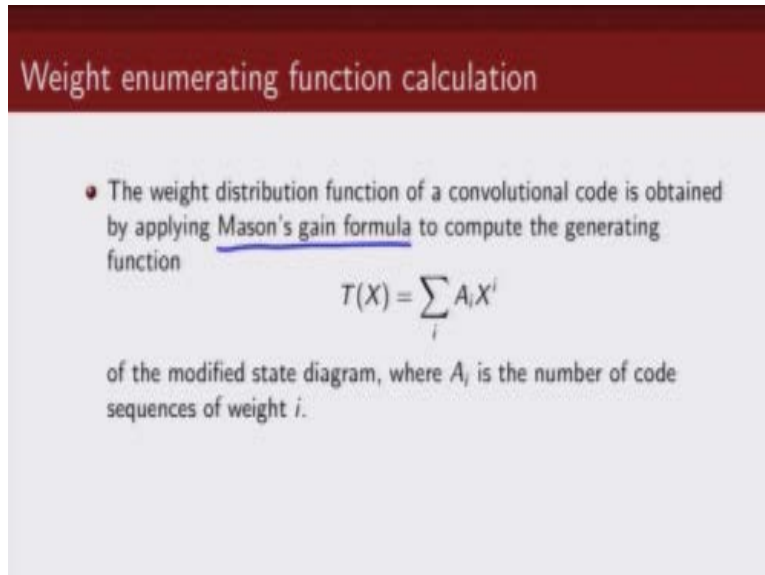


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The slide features a dark red header with the title "Weight enumerating function calculation" in white. Below the header, a bullet point states: "The weight distribution function of a convolutional code is obtained by applying Mason's gain formula to compute the generating function". This is followed by the equation 
$$T(X) = \sum_i A_i X^i$$
 and the text "of the modified state diagram, where  $A_i$  is the number of code sequences of weight  $i$ ." The slide also includes a standard Windows taskbar at the top and navigation icons at the bottom right.

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This slide is identical to the one above, but with the phrase "Mason's gain formula" underlined in the bullet point text. The rest of the content, including the title, equation, and explanatory text, remains the same.

As we said we are going to use Mason's gain formula to compute the weight enumerating function for the convolutional code. So we are going to describe how we are going to use the Mason gain formula. So we are representing by  $T(X)$  the generating function which will basically enumerate all code words of weight  $i$ .

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### Weight enumerating function calculation

- The weight distribution function of a convolutional code is obtained by applying Mason's gain formula to compute the generating function

$$T(X) = \sum_i A_i X^i$$

of the modified state diagram, where  $A_i$  is the number of code sequences of weight  $i$ .

- *Forward path*: a path connecting the initial state to the final state that does not go through any state twice.

Now let us define few terms that we are going to use in Mason's gain formula, the first term that we are going to define is what is known as forward path.

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### Weight enumerating function calculation

- The weight distribution function of a convolutional code is obtained by applying Mason's gain formula to compute the generating function
$$T(X) = \sum_i A_i X^i$$
of the modified state diagram, where  $A_i$  is the number of code sequences of weight  $i$ .
- Forward path: a path connecting the initial state to the final state that does not go through any state twice.

So a forward path is a path from the initial all zero state to the final state and the condition is this path should not go over any state twice that is our forward path.

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## Weight enumerating function calculation

- The weight distribution function of a convolutional code is obtained by applying Mason's gain formula to compute the generating function

$$T(X) = \sum_i A_i X^i$$

of the modified state diagram, where  $A_i$  is the number of code sequences of weight  $i$ .

- *Forward path*: a path connecting the initial state to the final state that does not go through any state twice.
- *Loop*: a closed path that starts and ends in the same state without going over any other state twice.

Next term that we define is basically a loop what is a loop? A loop is a close path that starts and ends in the same state without going over any state twice, that is a loop.

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## Weight enumerating function calculation

- The weight distribution function of a convolutional code is obtained by applying Mason's gain formula to compute the generating function

$$T(X) = \sum_i A_i X^i$$

of the modified state diagram, where  $A_i$  is the number of code sequences of weight  $i$ .

- *Forward path*: a path connecting the initial state to the final state that does not go through any state twice.
- Loop: a closed path that starts and ends in the same state without going over any other state twice.

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## Weight enumerating function calculation

- The weight distribution function of a convolutional code is obtained by applying Mason's gain formula to compute the generating function

$$T(X) = \sum_i A_i X^i$$

of the modified state diagram, where  $A_i$  is the number of code sequences of weight  $i$ .

- *Forward path*: a path connecting the initial state to the final state that does not go through any state twice.
- *Loop*: a closed path that starts and ends in the same state without going over any other state twice.
- Two or more loops are non-touching if they don't have any states in common.

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## Weight enumerating function calculation

- The weight distribution function of a convolutional code is obtained by applying Mason's gain formula to compute the generating function

$$T(X) = \sum_i A_i X^i$$

of the modified state diagram, where  $A_i$  is the number of code sequences of weight  $i$ .

- *Forward path*: a path connecting the initial state to the final state that does not go through any state twice.
- *Loop*: a closed path that starts and ends in the same state without going over any other state twice.
- Two or more loops are non-touching if they don't have any states in common.

When do we say two loops are non touching, we say two loops are non touching if they do not have any state in common so again I repeat these three definitions.

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## Weight enumerating function calculation

- The weight distribution function of a convolutional code is obtained by applying Mason's gain formula to compute the generating function

$$T(X) = \sum_i A_i X^i$$

of the modified state diagram, where  $A_i$  is the number of code sequences of weight  $i$ .

- *Forward path*: a path connecting the initial state to the final state that does not go through any state twice.
- *Loop*: a closed path that starts and ends in the same state without going over any other state twice.
- Two or more loops are non-touching if they don't have any states in common.

Forward path is a path from initial state to the final state without visiting any state twice, a loop is the close path starting and ending at the same state without going over the same state twice and two or more loops are non-touching if they do not have any state in common.



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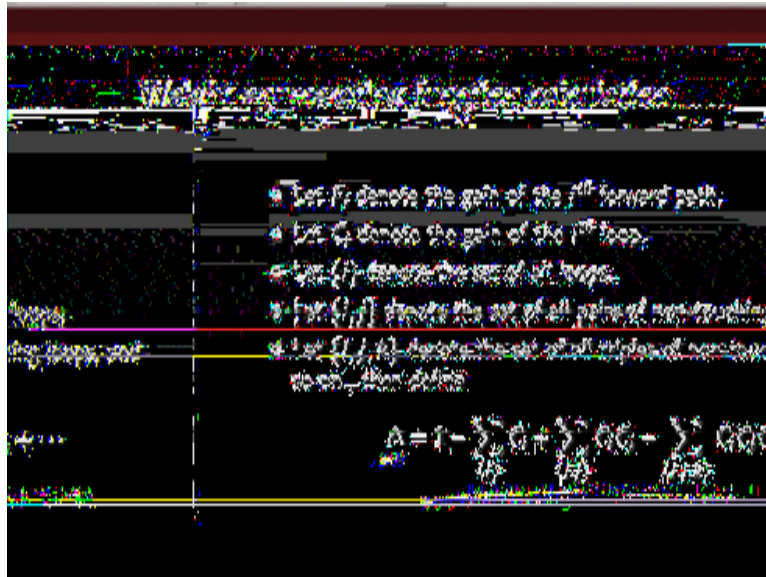
Weight enumerating function calculation

- Let  $F_i$  denote the gain of the  $i^{\text{th}}$  forward path.
- Let  $C_i$  denote the gain of the  $i^{\text{th}}$  loop.
- Let  $\{i\}$  denote the set of all loops.
- Let  $\{i, j\}$  denote the set of all pairs of non-touching loops.
- Let  $\{i, j, k\}$  denote the set of all triples of non-touching loops, and so on., then define

$$\Delta = 1 - \sum_{\{i\}} C_i + \sum_{\{i, j\}} C_i C_j - \sum_{\{i, j, k\}} C_i C_j C_k + \dots$$

Now let us denote by  $F_i$  the gain of the  $i^{\text{th}}$  forward path and let  $C_i$  denote the gain for the  $i^{\text{th}}$  loop, we denote by this the set of all loops similarly this set of  $i$  and  $j$  will denote set of all pairs of non – touching loops. This triplet will define set of all triplets of non-touching loops. So if we use this we define a term  $\Delta$  which is defined as follows.

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This  $1 - \Sigma$  of all the gains of the loops plus product of gains of all those non-touching loops minus this is product of set of all triplets of non-touching loop and it goes on like this, so that is our  $\Delta$ .

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### Weight enumerating function calculation

- Let  $F_i$  denote the gain of the  $i^{\text{th}}$  forward path.
- Let  $C_i$  denote the gain of the  $i^{\text{th}}$  loop.
- Let  $\{i\}$  denote the set of all loops.
- Let  $\{i, j\}$  denote the set of all pairs of non-touching loops.
- Let  $\{i, j, k\}$  denote the set of all triples of non-touching loops, and so on., then define

$$\Delta = 1 - \sum_{\{i\}} C_i + \sum_{\{i, j\}} C_i C_j - \sum_{\{i, j, k\}} C_i C_j C_k + \dots$$

- Let  $G_i$  be the graph obtained by removing all the states on the  $i^{\text{th}}$  forward path and all the branches connected to these states, and let  $\Delta_i$  be defined similarly as  $\Delta$  for the graph  $G_i$ .

We define our graph that is obtained after we remove all states belonging to an  $i^{\text{th}}$  forward path by  $G_i$ .

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### Weight enumerating function calculation

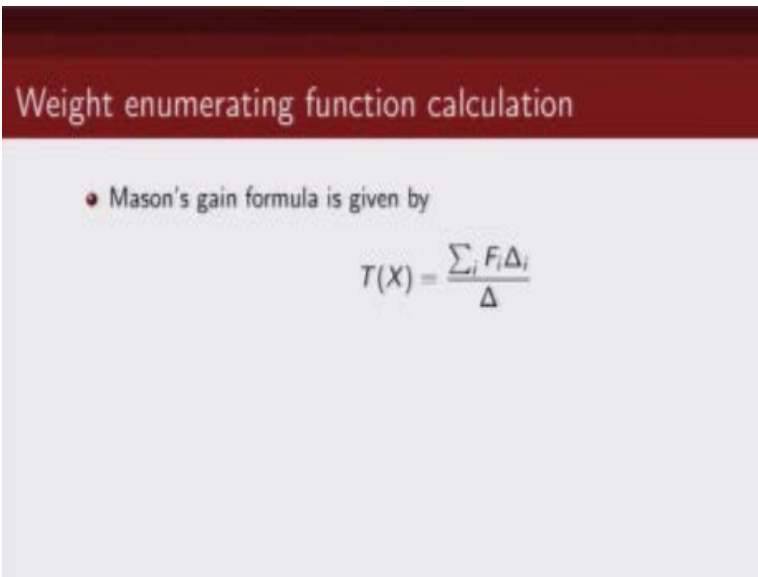
- Let  $F_i$  denote the gain of the  $i^{\text{th}}$  forward path.
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$$\Delta = 1 - \sum_{\{i\}} C_i + \sum_{\{i, j\}} C_i C_j - \sum_{\{i, j, k\}} C_i C_j C_k + \dots$$

- Let  $G_i$  be the graph obtained by removing all the states on the  $i^{\text{th}}$  forward path and all the branches connected to these states, and let  $\Delta_i$  be defined similarly as  $\Delta$  for the graph  $G_i$ .

So  $G_i$  is basically the graph remaining after we remove the  $i^{\text{th}}$  forward path. And the  $\Delta$  corresponding to this modified graph will be denoted by delta  $\Delta_i$ .

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Weight enumerating function calculation

- Mason's gain formula is given by

$$T(X) = \frac{\sum_i F_i \Delta_i}{\Delta}$$

So the mason gains formula then says that the generator function for this convolutional encoder can then be given by this expression.

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## Weight enumerating function calculation

- Mason's gain formula is given by

$$T(X) = \frac{\sum_i F_i \Delta_i}{\Delta}$$

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### Weight enumerating function calculation

- Mason's gain formula is given by

$$T(X) = \frac{\sum_i F_i \Delta_i}{\Delta}$$

- The modified state diagram can be augmented by labeling each branch corresponding to nonzero message bit with Y and labeling every branch with Z.

So this modified state diagram can be augmented to include more information, now what we had done so far was we labeled the branches of this state diagram by the overall total weight.

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### Weight enumerating function calculation

- Mason's gain formula is given by

$$T(X) = \frac{\sum_i F_i \Delta_i}{\Delta}$$

- The modified state diagram can be augmented by labeling each branch corresponding to nonzero message bit with Y and labeling every branch with Z.

Now we can also augment this by mentioning what is the input that results in that output weight so we can label the weight of the input by Y.



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## Weight enumerating function calculation

- Mason's gain formula is given by

$$T(X) = \frac{\sum_i F_i \Delta_i}{\Delta}$$

- The modified state diagram can be augmented by labeling each branch corresponding to nonzero message bit with Y and labeling every branch with Z.

So the power of Y will denote what is the input weight and similarly we can label each branch by Z. So in a path gain formula, the degree of Z will tell us, like what is the length of nonzero path. So degree of Z will tell us like once it diverge from all zero state after how much time it comes back into all zero state.

So we can augment our state diagram by what I call a modified state diagram by adding two additional information, one is the weight of the message bit which will be denoted by power of  $Y^i$  and other is branch which should be denoted by Z.

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### Weight enumerating function calculation

- Mason's gain formula is given by

$$T(X) = \frac{\sum_i F_i \Delta_i}{\Delta}$$

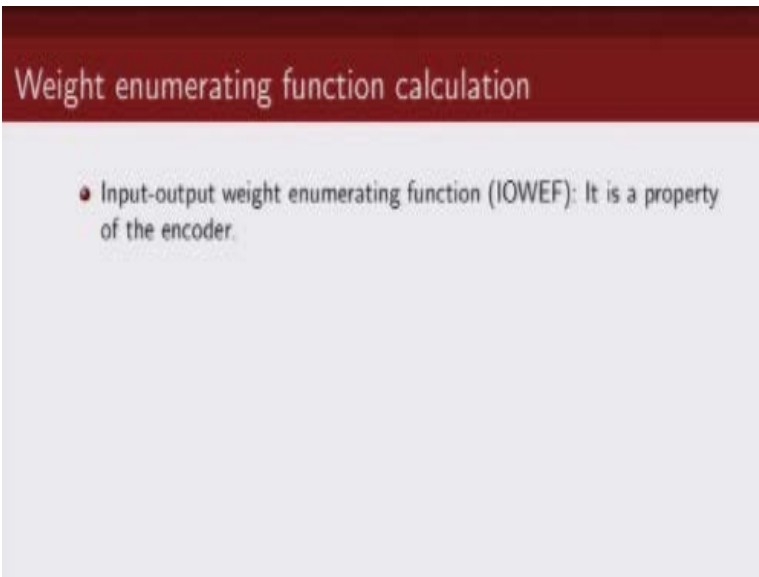
- The modified state diagram can be augmented by labeling each branch corresponding to nonzero message bit with Y and labeling every branch with Z.
- The augmented transfer function is given by

$$T(X, Y, Z) = \sum_{i,j,l} A_{i,j,l} X^i Y^j Z^l$$

where  $A_{i,j,l}$  is the number of code sequences of weight  $i$ , whose corresponding information sequence has weight  $j$ , and which has length  $l$  branches.

So we can then similarly define an augmented transfer function which will not only tell us the code word weight but it also tell us what is the input weight that results in that particular output weight and it also tell us length of that particular code word, by length I mean the time it diverges from all zero state until it cause back to all zero state.

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Weight enumerating function calculation

- Input-output weight enumerating function (IOWEF): It is a property of the encoder.

So this is basically what we call input-output weight enumerating function.

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### Weight enumerating function calculation

- Mason's gain formula is given by

$$T(X) = \frac{\sum_i F_i \Delta_i}{\Delta}$$

- The modified state diagram can be augmented by labeling each branch corresponding to nonzero message bit with Y and labeling every branch with Z.
- The augmented transfer function is given by

$$T(X, Y, Z) = \sum_{i,j,l} A_{i,j,l} X^i Y^j Z^l$$

where  $A_{i,j,l}$  is the number of code sequences of weight  $i$ , whose corresponding information sequence has weight  $j$ , and which has length  $l$  branches.

Because this function is enumerating for what input you get what output, okay so this will give us weight input-output weight enumerating function.

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## Weight enumerating function calculation

- Input-output weight enumerating function (IOWEF): It is a property of the encoder.
- An alternative version of IOWEF that contains only information about input and output weights but not the length of each codeword is obtained as

$$T(X, Y) = T(X, Y, Z)|_{Z=1}$$

- Weight enumerating function (WEF): It is a code property.
- WEF  $T(X)$  is related to IOWEF as follows

$$T(X) = T(X, Y)|_{Y=1} = T(X, Y, Z)|_{Y=Z=1}$$

And it is the property of the encoder an alternative version of this input-output weight enumerating function is one that contains only information about the input and output weight and not the length of each code word, so if we put  $z = 1$  basically this is going to give us [indiscernible][00:12:05] in diversion of input-output weight enumerating function.

And what is weight enumerating function? The weight enumerating function will only tell us what is the overall code word weight and this is a property of the convolutional code.

(Refer Slide Time: 12:24)

### Weight enumerating function calculation

- Input-output weight enumerating function (IOWEF): It is a property of the encoder.
- An alternative version of IOWEF that contains only information about input and output weights but not the length of each codeword is obtained as
$$T(X, Y) = T(X, Y, Z)|_{Z=1}$$
- Weight enumerating function (WEF): It is a code property.
- WEF  $T(X)$  is related to IOWEF as follows
$$T(X) = T(X, Y)|_{Y=1} = T(X, Y, Z)|_{Y=Z=1}$$

So weight enumerating function is related to input output weight enumerating function in this particular way so if you put z and y as 1 in the input output weight enumerating function we will get back our weight enumerating function.

(Refer Slide Time: 12:44)

### Weight enumerating function calculation

- Input-output weight enumerating function (IOWEF): It is a property of the encoder.
- An alternative version of IOWEF that contains only information about input and output weights but not the length of each codeword is obtained as
$$T(X, Y) = T(X, Y, Z)|_{Z=1}$$
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- WEF  $T(X)$  is related to IOWEF as follows
$$T(X) = T(X, Y)|_{Y=1} = T(X, Y, Z)|_{Y=Z=1}$$
- Conditional weight enumerating function (CWEF): Enumerates weights of all codewords associated with particular information weights.

Similarly we can define what is known as conditional weight enumerating function so what is conditional weight enumerating function? The conditional weight enumerating function it enumerates weights of all code words associated with a particular information weight, so if you are interested in knowing what is the output weight correspond to weight four input sequence, so from the input output weight enumerating function by collecting all terms which will have  $w_4$ .

(Refer Slide Time: 13:21)

**Weight enumerating function calculation**

- Input-output weight enumerating function (IOWEF): It is a property of the encoder.
- An alternative version of IOWEF that contains only information about input and output weights but not the length of each codeword is obtained as
 
$$T(X, Y) = T(X, Y, Z)|_{Z=1}$$
- Weight enumerating function (WEF): It is a code property.
- WEF  $T(X)$  is related to IOWEF as follows
 
$$T(X) = T(X, Y)|_{Y=1} = T(X, Y, Z)|_{Y=Z=1}$$
- Conditional weight enumerating function (CWEF): Enumerates weights of all codewords associated with particular information weights.

We can find out what is the weight of all code words corresponding to a output weight input weight of four and again we are using  $Y$  to denote the input weights so if you are interested in input weight four we should look for terms containing  $Y^4$ .

(Refer Slide Time: 13:51)

### Weight enumerating function calculation

- Input-output weight enumerating function (IOWEF): It is a property of the encoder.
- An alternative version of IOWEF that contains only information about input and output weights but not the length of each codeword is obtained as

$$T(X, Y) = T(X, Y, Z)|_{Z=1}$$

- Weight enumerating function (WEF): It is a code property.
- WEF  $T(X)$  is related to IOWEF as follows:

$$T(X) = T(X, Y)|_{Y=1} = T(X, Y, Z)|_{Y=Z=1}$$

- Conditional weight enumerating function (CWEF): Enumerates weights of all codewords associated with particular information weights.

So this denotes the input weight this denotes the output coded weight and this denotes the length.



(Refer Slide Time: 14:02)

**Weight enumerating function calculation**

- For information weight of  $j$ , the CWEF is obtained as
$$T_j(X) = \sum_i A_{i,j} X^i$$
where  $A_{i,j}$  represents number of code sequences with weight  $i$ , and information weight  $j$ .
- IOWEF can be expressed in terms of CWEF as follows
$$T(X, Y) = \sum_j Y^j T_j(X)$$
- Input-redundancy weight enumerating function (IRWEF): It is defined for systematic encoders as follows
$$T(W, Y, Z) = \sum_{w,j,l} A_{w,j,l} W^w Y^j Z^l$$
where  $A_{w,j,l}$  is the number of code sequences of length  $l$  with information weight  $j$ , and parity weight  $w$ .

So as I said for a input weight of  $j$  conditional enumerating function will give us what is the output code weight that you can achieve for a input weight of  $j$ . And we write our input output weight enumerating function in terms of conditional weight enumerating function so this is basically input of weight  $j$  will result in conditional weight enumerating function and we show it for all  $j$  that will be our input output weight enumerating function.

There is another property which is define for systematic encoders which is called input redundancy weight enumerating function. So here because the output weight of a systematic encoder consist of weight of the information bits and weight of the parity bits.

(Refer Slide Time: 15:10)

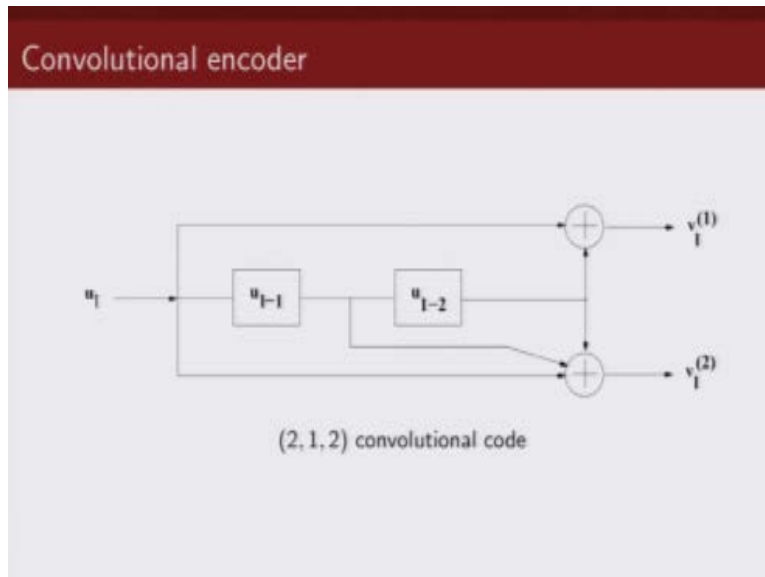
**Weight enumerating function calculation**

- For information weight of  $j$ , the CWEF is obtained as
$$T_j(X) = \sum_i A_{i,j} X^i$$
where  $A_{i,j}$  represents number of code sequences with weight  $i$ , and information weight  $j$ .
- IOWEF can be expressed in terms of CWEF as follows
$$T(X, Y) = \sum_j Y^j T_j(X)$$
- Input-redundancy weight enumerating function (IRWEF): It is defined for systematic encoders as follows
$$T(W, Y, Z) = \sum_{w,j,l} A_{w,j,l} W^w Y^j Z^l$$
where  $A_{w,j,l}$  is the number of code sequences of length  $l$  with information weight  $j$ , and parity weight  $w$ .

Now since the power of  $Y$  already is denoting the weight of the information bits so when you are asked to show the output weight you can just instead of saying the output weight you can just specify the weight of the parity bits, because it is systematic encoder the overall weight will be weight of the parity bits and weight of the information bits.

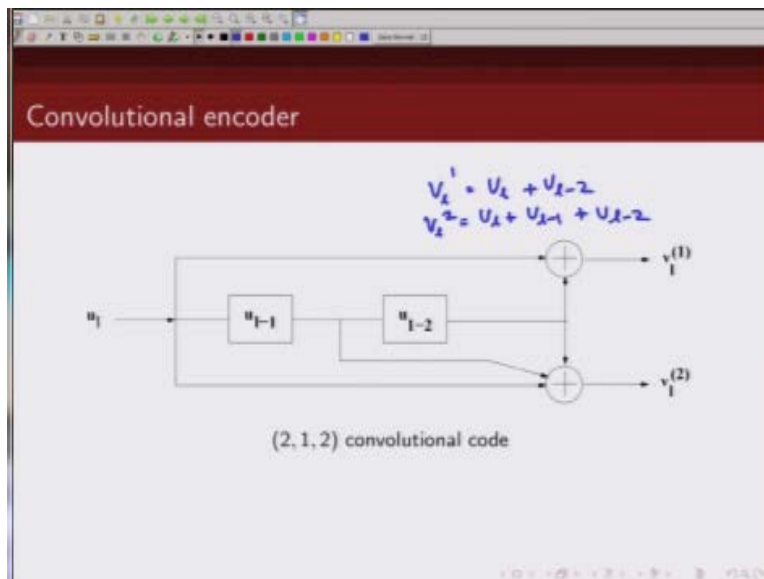
So overall weight would be  $w+j$  so input redundancy weight enumerating function is defined for systematic encoders. So where instead of specifying the overall coded weight here you only specify the weight of the parity bits.

(Refer Slide Time: 15:59)



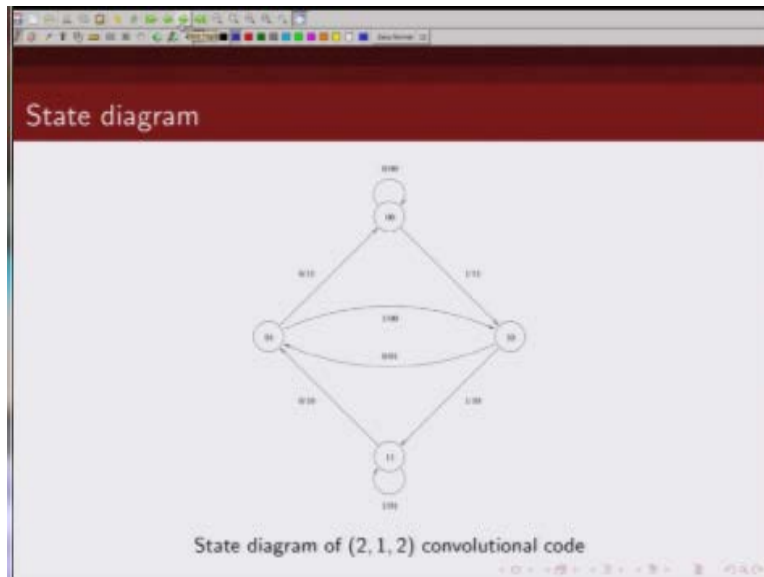
Now let us take an example to illustrate how we can find the weight enumerating function of a convolutional code.

(Refer Slide Time: 16:10)



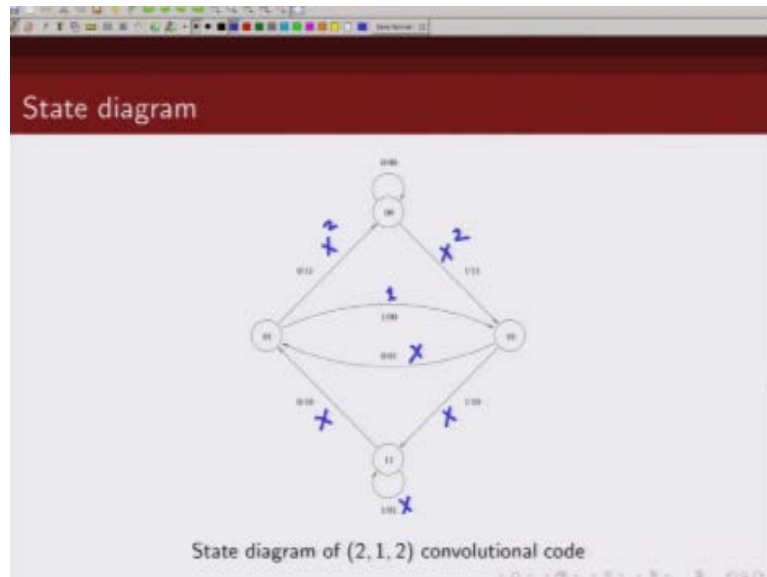
So we are going to consider a rate 1/2 convolutional code whose memory is 2 to this is the convolutional code we can see basically  $v_1^{(1)}$  is  $u_1 + u_{1-2}$  and  $v_1^{(2)}$  is nothing but  $u_1 + u_{1-1} + u_{1-2}$  the convolutional code.

(Refer Slide Time: 16:45)



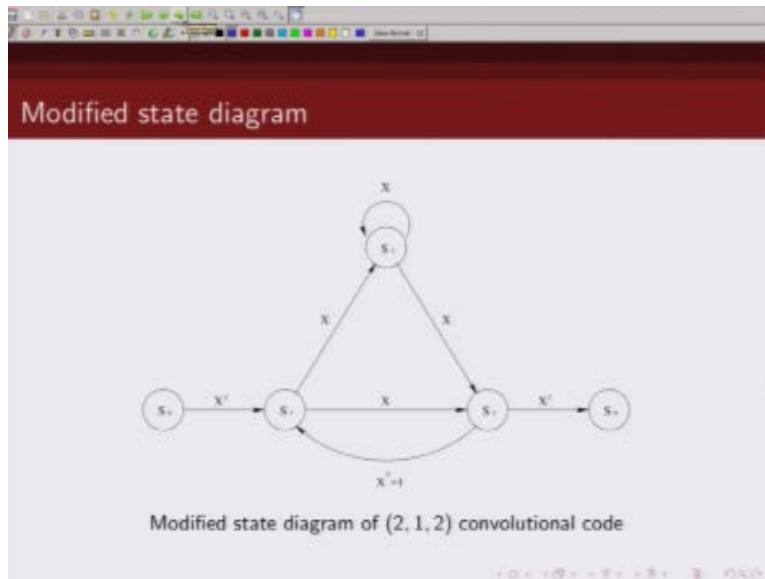
Now the state diagram for this convolutional encoder is given by this. Now we are going to modify this state diagram for the purpose of calculating the weight enumerating function. So recall what are the modifications we have to do.

(Refer Slide Time: 17:04)



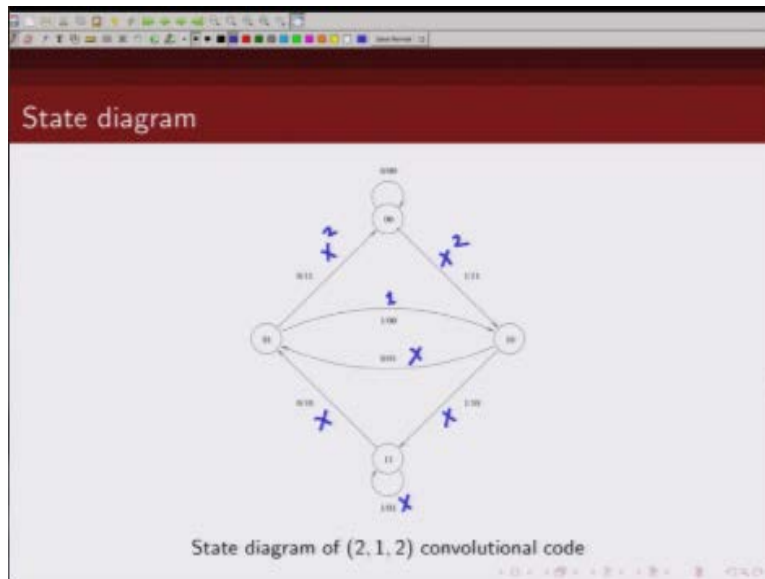
We have to remove this self loop around all 0 state and we have to split this all 0 state into 2 state, initial state and final state. Next we have to label all these branches by weight of the output bit. So this will be  $x^2$ . This will be  $x^0$  which is 1, this will be  $x$ , this will be  $x^2$ , this will be  $x$ , this will be  $x$ , this will be  $x$ . if you go back.

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This is how my augmented will modified state diagram will look like. So what I did was.

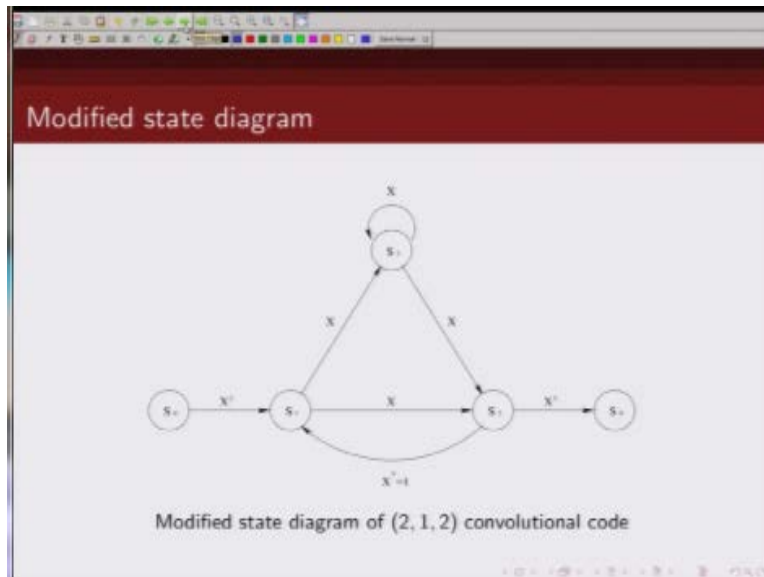
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I have these states here, I split this all 0 state into 2 state.

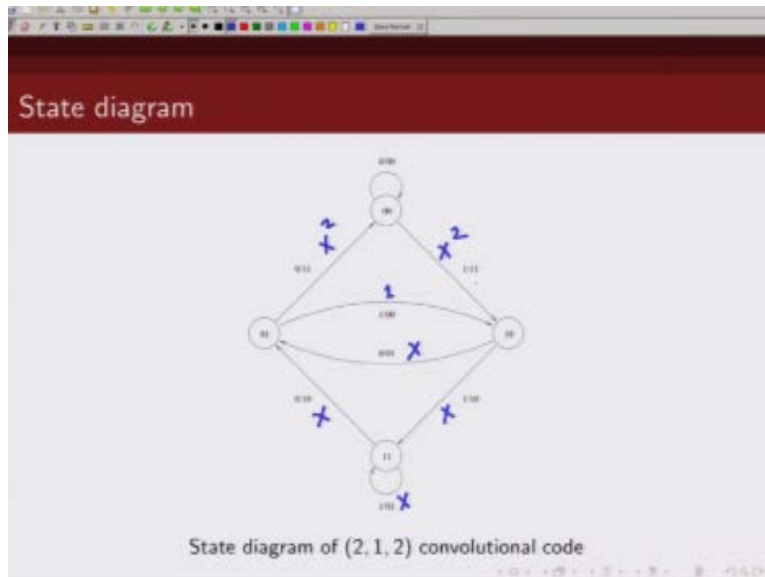


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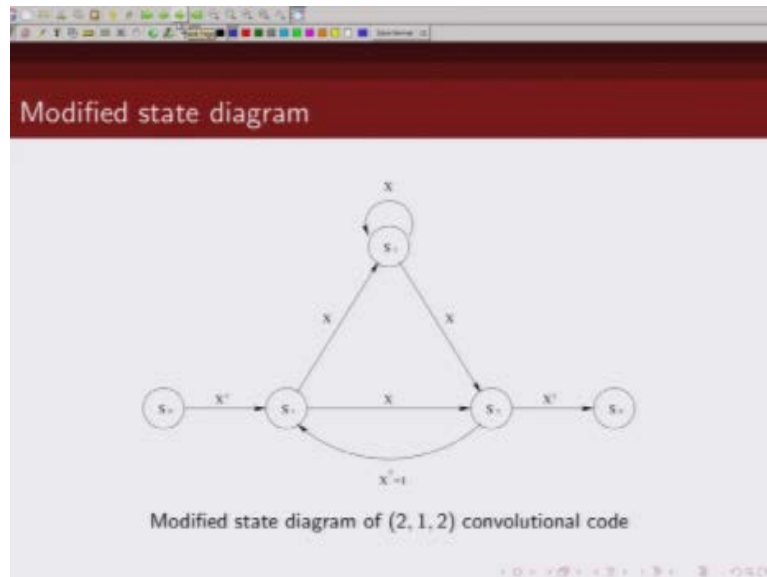
So this state was split into initial state and final state okay. Next what did I do?

(Refer Slide Time: 18:14)



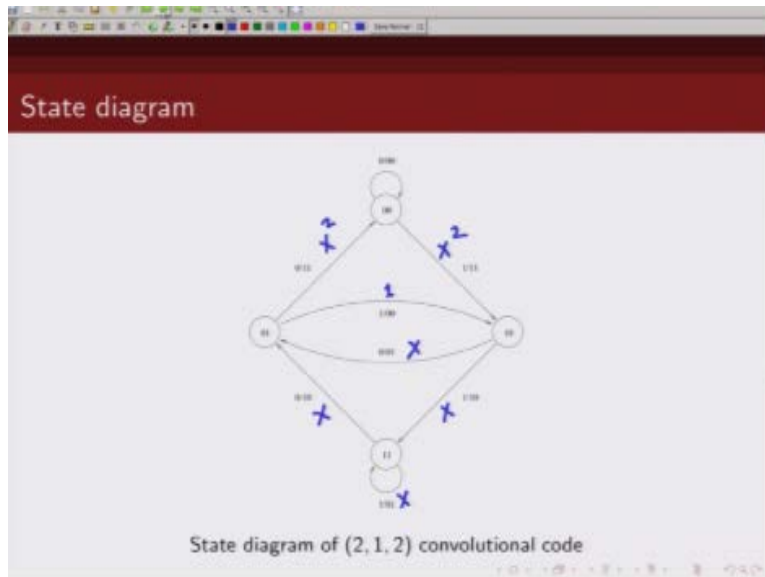
I redrew the same diagram but I labeled each transition by the weight of the outputs so you can see from 00 I am going to 10 and its output weight is 11, which is  $x^2$ . So let us go back here.

(Refer Slide Time: 18:32)



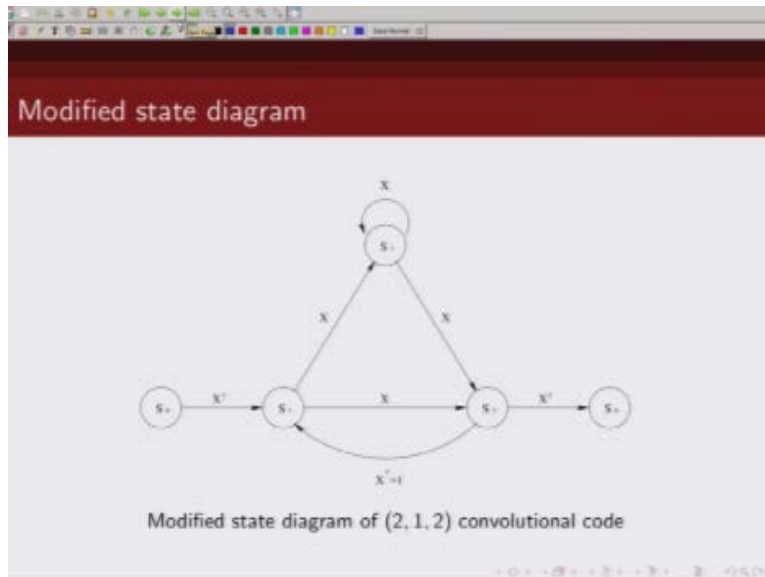
From 00 I am going to this state and output is  $x^2$  this branch is labeled by  $x^2$ .

(Refer Slide Time: 18:42)



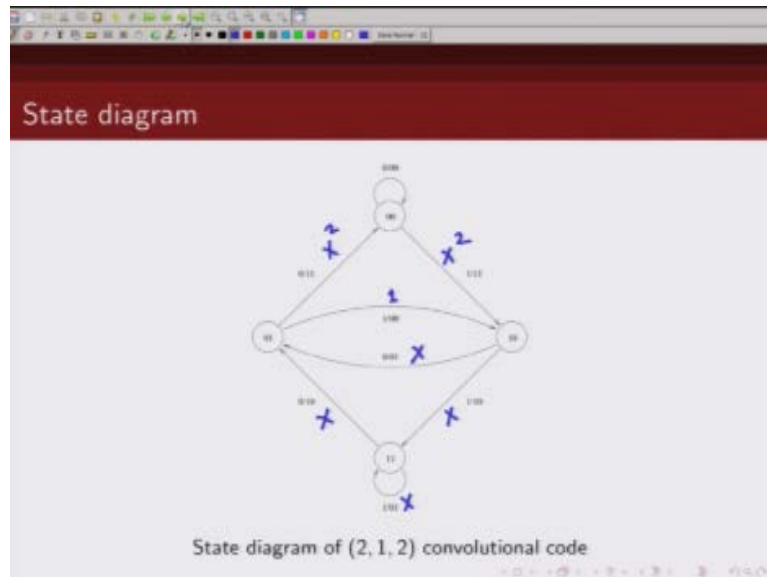
Similarly, you can see from let us say from this state you are going to this state and the weight is x, you can see.

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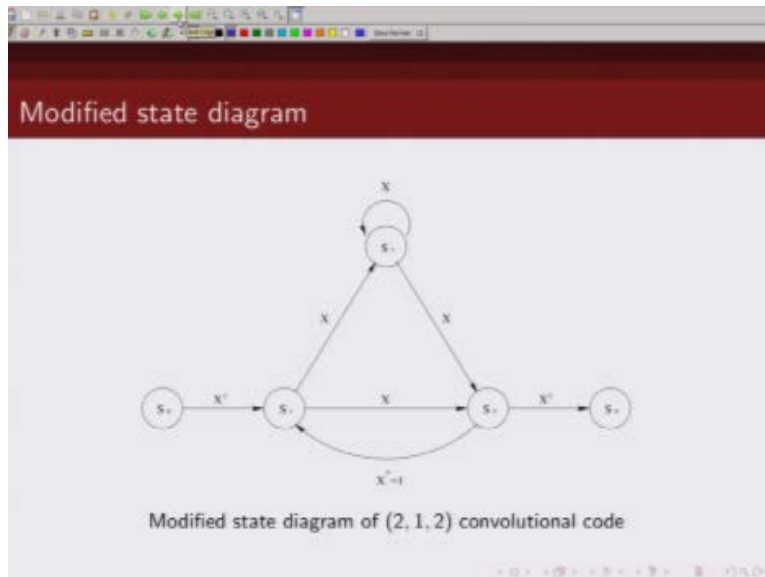
From this I am going to this state and the branch is labeled by x okay.

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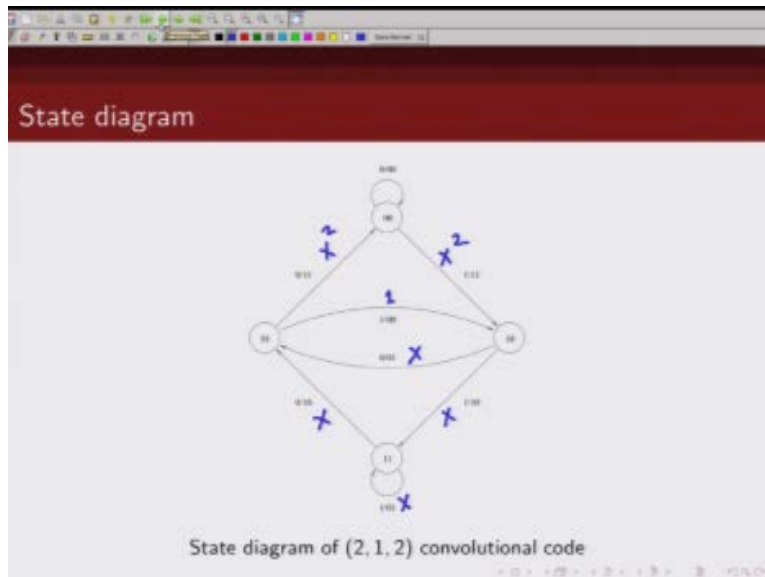
From this state you are going to this 11 state, and the branch is labeled by x.

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From this you are going to this 11 state and it is labeled by x.

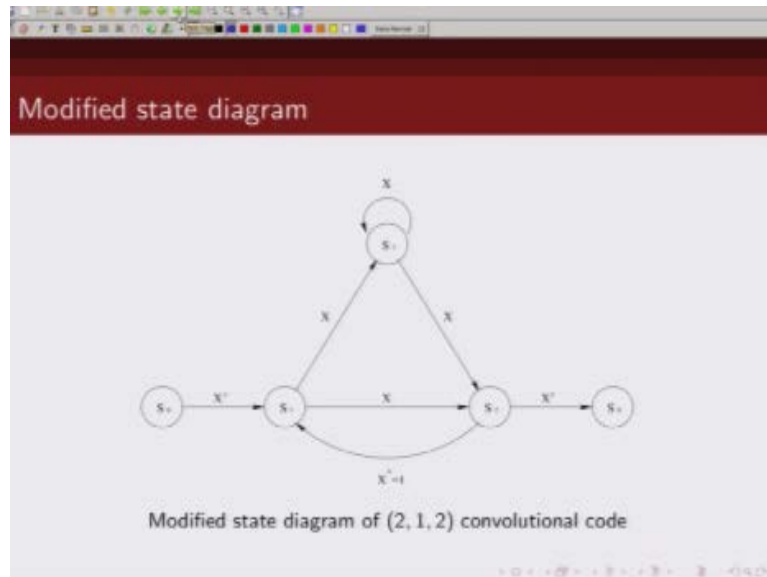
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Around the state 11, there is a loop which has weight 1 x.

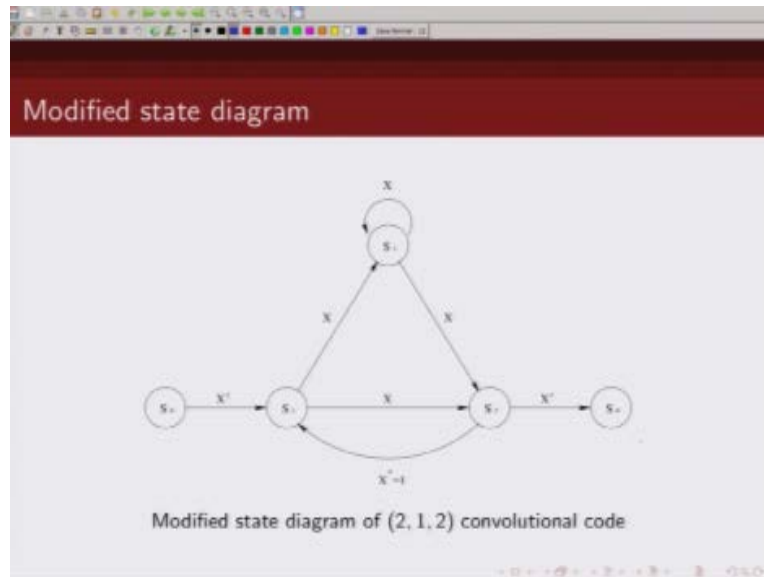


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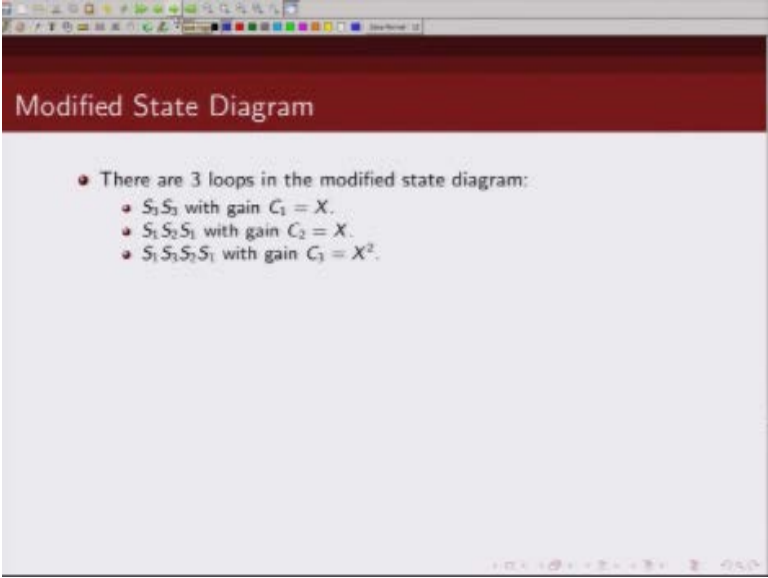
So this is my loop around 11, which has weight  $x$ , so like that basically we modify the state diagram and this is how our modified state diagram of a rate  $\frac{1}{2}$  convolutional code that we just showed looks like okay.

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Now the next step is, we need to find out what are the forward parts, what are loops, what are the non-touching loops, and then we need to find out the path gains along those forward paths. We need to find out delta is corresponding to this forward paths, and then we need to apply Mason's gain formula to get the weight enumerating function.

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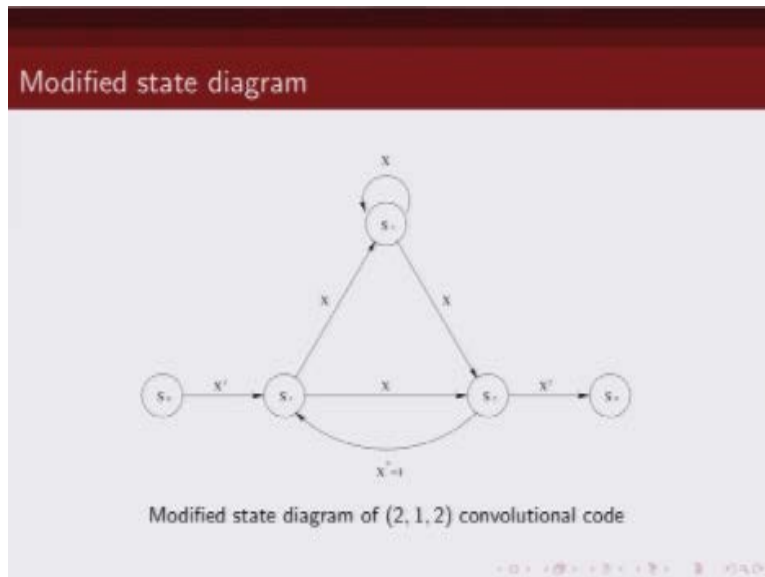


Modified State Diagram

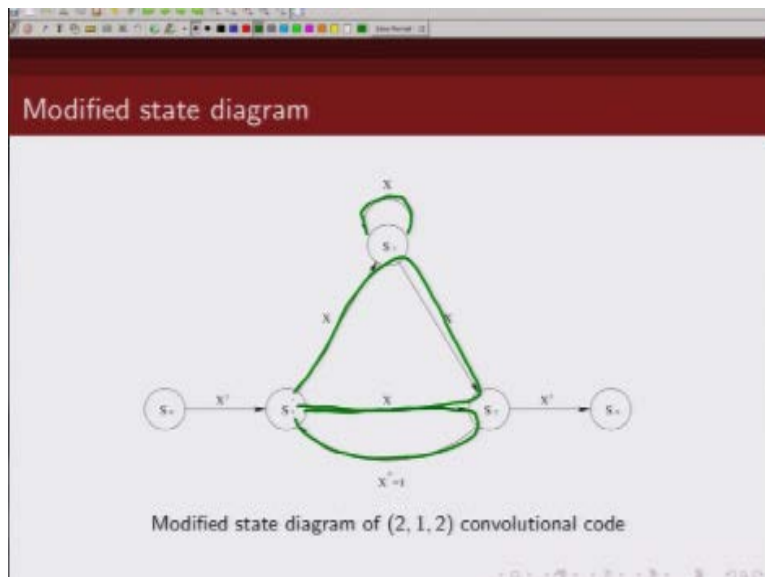
- There are 3 loops in the modified state diagram:
  - $S_3 S_3$  with gain  $C_1 = X$ .
  - $S_1 S_2 S_1$  with gain  $C_2 = X$ .
  - $S_1 S_3 S_2 S_1$  with gain  $C_3 = X^2$ .

So first we find out what are the loops, so there are three loops in this, 1 is a self loop around the state  $S_3$  you can see.

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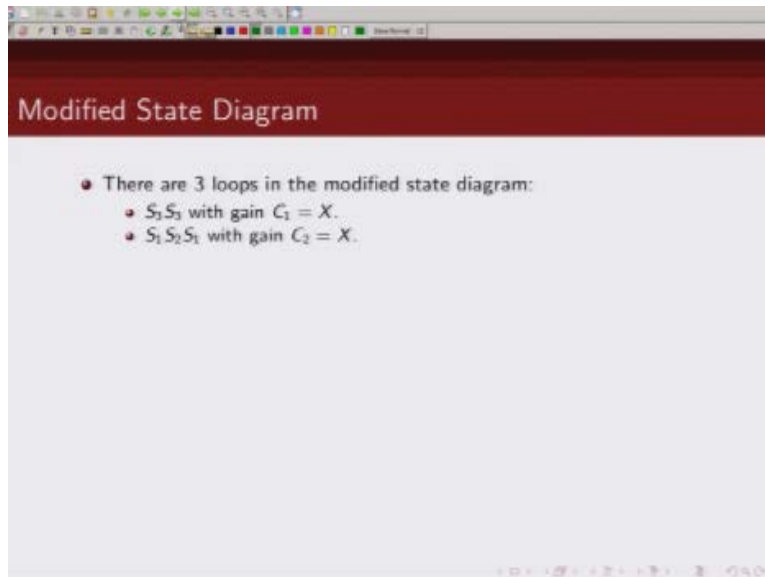


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This 1 loop right, there is another loop right, and then there is another loop. So there are three loops here and that is what I am denoting.

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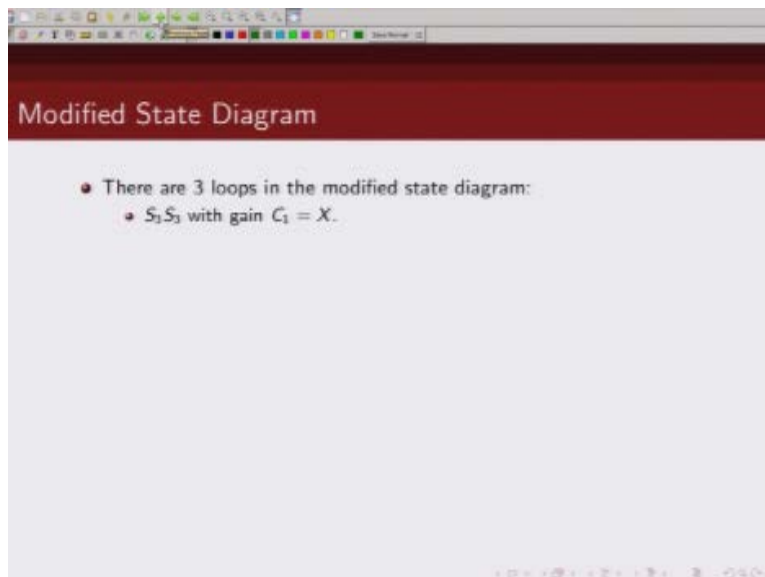


Modified State Diagram

- There are 3 loops in the modified state diagram:
  - $S_3S_3$  with gain  $C_1 = X$ .
  - $S_1S_2S_1$  with gain  $C_2 = X$ .

It by  $S_3S_3$  is gain X, next one is  $S_1S_2S_1$ .

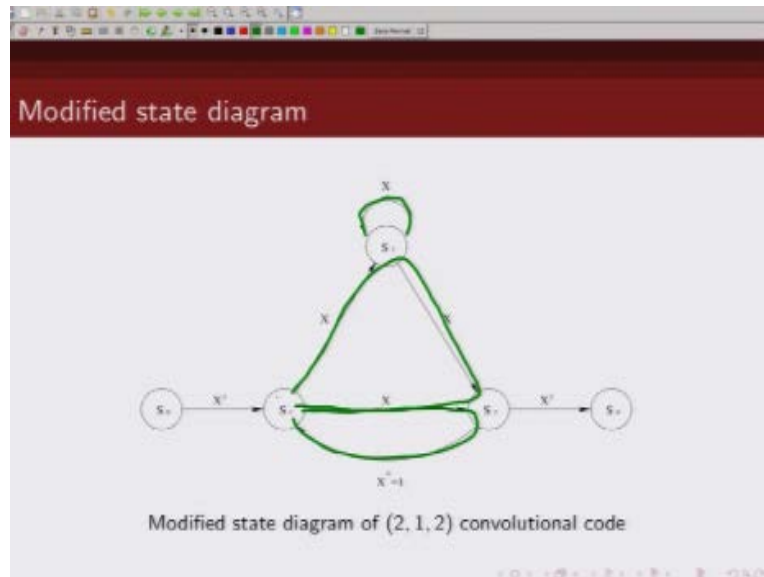
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Modified State Diagram

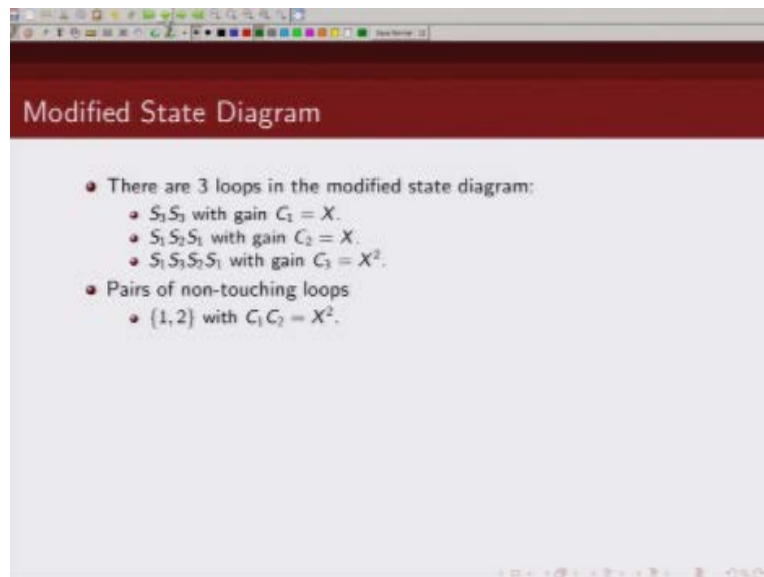
- There are 3 loops in the modified state diagram:
  - $S_3S_3$  with gain  $C_1 = X$ .

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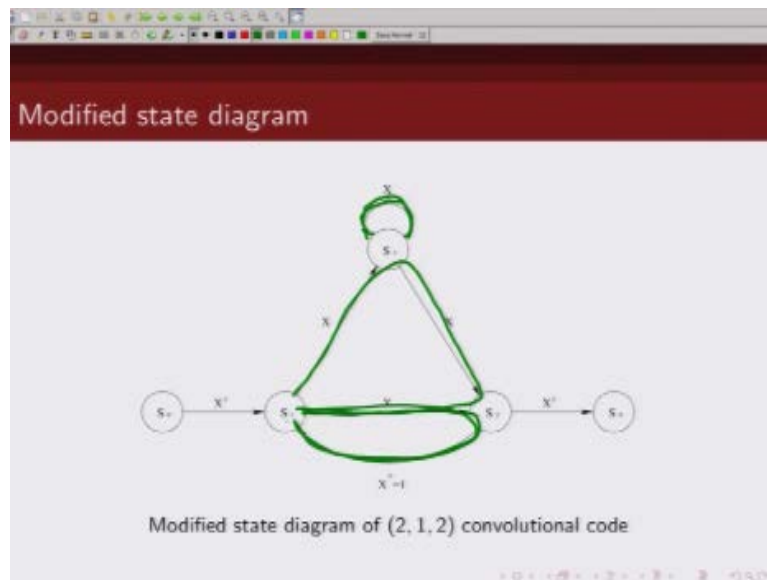
$S_1S_2S_1$  is this one okay. And the third loop is given by this.

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So these are the three loops and corresponding to these three loops these are the gains. Next, what are the pair of non-touching loops? Now, only these two  $C_1$  and  $C_2$  are non touching loops you can see and go back to this example.

(Refer Slide Time: 21:37)



This loop and this loop are non-touching why this loop contains  $S_3$  and this loop contains  $S_1$  and  $S_2$ . So they do not have any state common between these two loops.

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**Modified State Diagram**

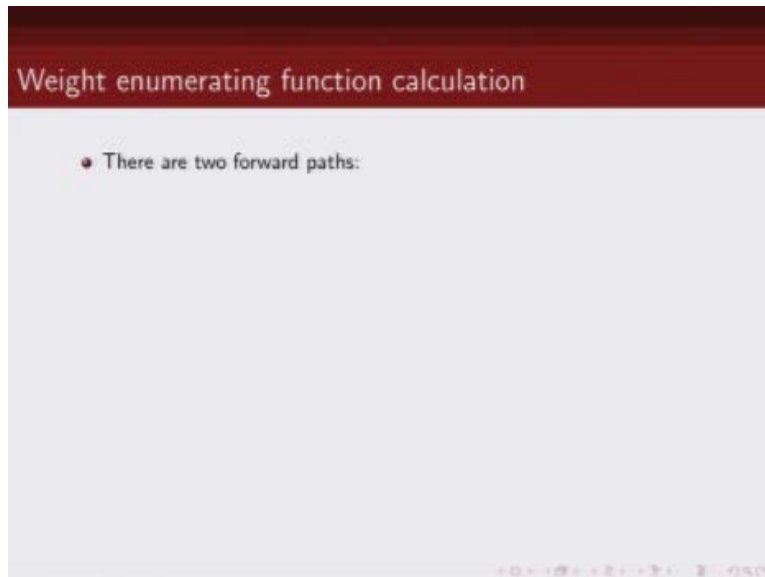
- There are 3 loops in the modified state diagram:
  - $S_3 S_1$  with gain  $C_1 = X$ .
  - $S_1 S_2 S_1$  with gain  $C_2 = X$ .
  - $S_1 S_3 S_2 S_1$  with gain  $C_3 = X^2$ .
- Pairs of non-touching loops
  - $\{1, 2\}$  with  $C_1 C_2 = X^2$ .
- No triples of non-touching loops.
- Hence,

$$\begin{aligned}\Delta &= 1 - (C_1 + C_2 + C_3) + C_1 C_2 \\ &= 1 - 2X - X^2 + X^2 \\ &= 1 - 2X\end{aligned}$$

So the set of non-touching loops is basically the  $C_1$  and  $C_2$  and the gain corresponding to them is basically  $X^2$ . And there is no set of three loops which are non-touching. So now, we can then find out the value of delta which is  $1 - \Sigma$  of these loop gains and plus set of non-touching loops so this comes out to be  $1 - 2X$ .

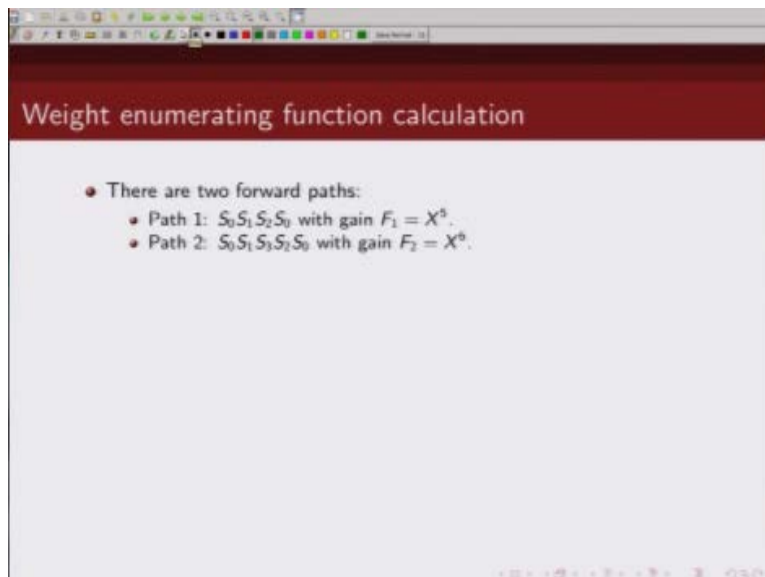


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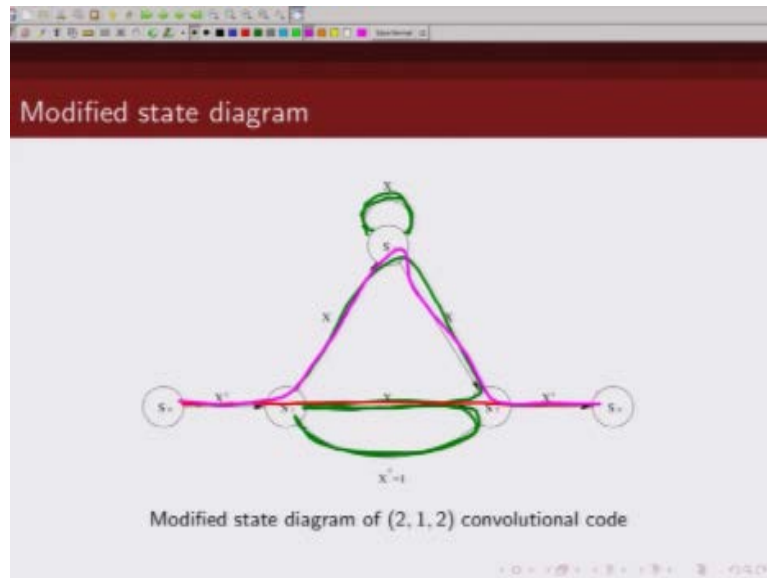
Next we are going to find out what are the forward paths.

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So there are two forward paths in this, and we are going to show you.

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So let us use a different color pen let us use a red color pen, remember what is the forward path, a path from the initial state to the final state without going over any state twice. So one forward path is this, find what about another forward path, the another forward path is this. Both the cases you can see I am not going over any state twice.

And there are only two forward path in this case. And what are the corresponding path gain for the one which I marked with red, this is  $x^2 x$  and  $x^2$ . So this will be  $X^5$  and this will be  $x^2 x x, x^2$ . So this will be  $X^6$ .

(Refer Slide Time: 23:53)

Weight enumerating function calculation

- There are two forward paths:
  - Path 1:  $S_0 S_1 S_2 S_0$  with gain  $F_1 = X^5$ .
  - Path 2:  $S_0 S_1 S_1 S_2 S_0$  with gain  $F_2 = X^6$ .
- The graph  $G_1$  obtained by removing the states on the forward path 1 is shown below.




Navigation icons: back, forward, search, etc.

So then we have to forward path the 1 with gain  $X^5$  another with gain  $X^6$ . Now what is the next step, we need to remove the forward path and see what is the graph remaining. And we need to compute the delta corresponding to that.

(Refer Slide Time: 24:16)

### Weight enumerating function calculation

- There are two forward paths:
  - Path 1:  $S_0 S_1 S_2 S_0$  with gain  $F_1 = X^3$ .
  - Path 2:  $S_0 S_1 S_1 S_2 S_0$  with gain  $F_2 = X^4$ .
- The graph  $G_1$  obtained by removing the states on the forward path 1 is shown below.



$X$

$S_0$

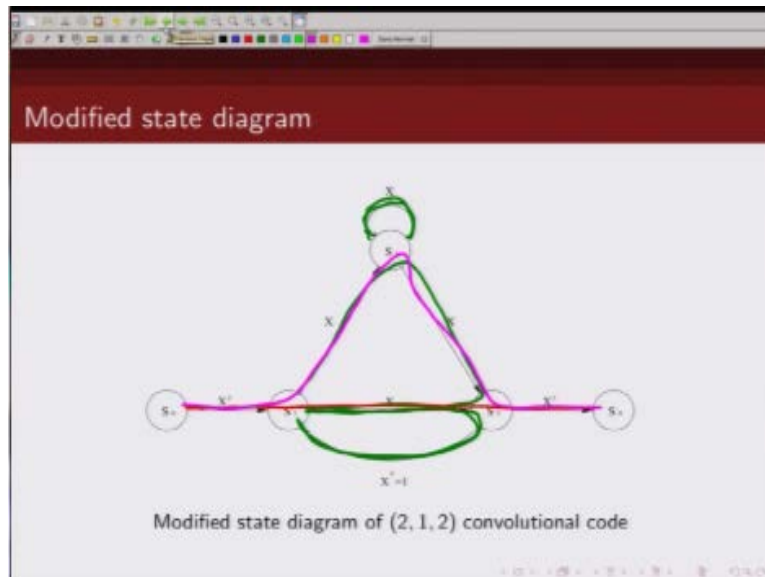
(Refer Slide Time: 24:17)

### Modified State Diagram

- There are 3 loops in the modified state diagram:
  - $S_1 S_1$  with gain  $C_1 = X$ .
  - $S_1 S_2 S_1$  with gain  $C_2 = X$ .
  - $S_1 S_1 S_2 S_1$  with gain  $C_3 = X^2$ .
- Pairs of non-touching loops
  - $\{1, 2\}$  with  $C_1 C_2 = X^2$ .
- No triples of non-touching loops.
- Hence,

$$\begin{aligned}\Delta &= 1 - (C_1 + C_2 + C_3) + C_1 C_2 \\ &= 1 - 2X - X^2 + X^2 \\ &= 1 - 2X\end{aligned}$$

(Refer Slide Time: 24:20)




Now again let us go back to the same diagram, if I remove this forward path what is left in the graph only this, this node only this is remaining. And what about if I remove this forward path, if I remove this forward path everything is gone there is nothing left in the graph.

(Refer Slide Time: 24:38)

Weight enumerating function calculation

- There are two forward paths:
  - Path 1:  $S_0 S_1 S_2 S_0$  with gain  $F_1 = X^3$ .
  - Path 2:  $S_0 S_1 S_1 S_2 S_0$  with gain  $F_2 = X^6$ .
- The graph  $G_1$  obtained by removing the states on the forward path 1 is shown below.



- $\Delta_1 = 1 - X$ .
- The graph  $G_2$  obtained by removing the states on the forward path 2 is empty.
- Hence  $\Delta_2 = 1$ .

So that is what I am seeing here, if I remove the forward path 1, the only graph remaining is this, and the delta corresponding to this is basically there is only one loop with gain X so this is  $1 - X$ . and for the second case, there is no graph left so  $\Delta_2$  will be 1 okay. So now I have  $F_1 \Delta_1$ ,  $F_2 \Delta_2$  and I also have the value of  $\Delta$ .

(Refer Slide Time: 25:17)

Weight enumerating function calculation

- The transfer function  $T(X)$  is given by

$$\begin{aligned} T(X) &= \frac{F_1 \Delta_1 + F_2 \Delta_2}{\Delta} \\ &= \frac{X^5(1-X) + X^6 \cdot 1}{1-2X} \\ &= \frac{X^5}{1-2X} \\ &= X^5 + 2X^6 + 4X^7 + \dots + 2^k X^{k+5} + \dots \end{aligned}$$

So I can then apply Mason's gain formula to get the weight enumerating function. So the weight enumerating function is given by this expression so I plug-in the value of  $F_1 \Delta_1$ ,  $F_2 \Delta_2$  and  $\Delta$  and what I get is this expression which I can write like this.

So you can see basically my output consist of one code word of weight 5, two code words of weight 6, four code words of weight 7. So you can see this transfer function is completely enumerating the weight distribution of my convolutional code. In the same thing I can do with augmented transfer function.

(Refer Slide Time: 26:04)

Weight enumerating function calculation

- The transfer function  $T(X)$  is given by

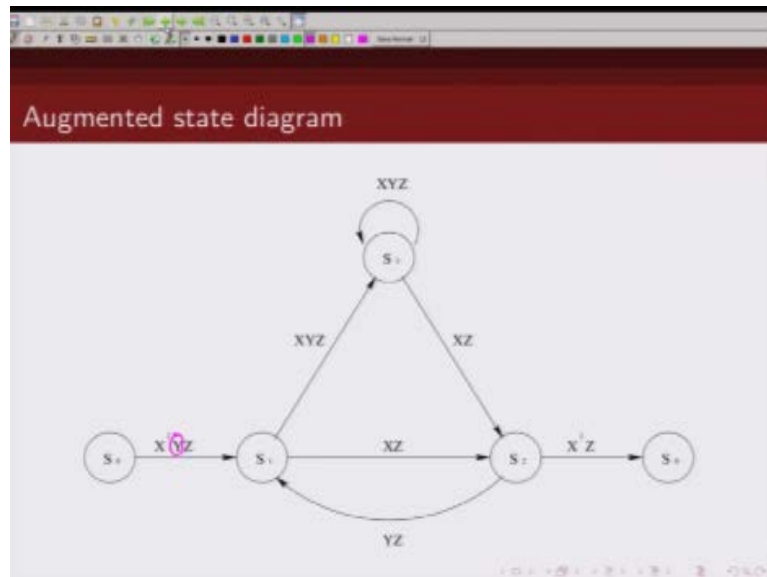
$$\begin{aligned} T(X) &= \frac{F_1 \Delta_1 + F_2 \Delta_2}{\Delta} \\ &= \frac{X^5(1-X) + X^6 \cdot 1}{1-2X} \\ &= \frac{X^5}{1-2X} \\ &= X^5 + 2X^6 + 4X^7 + \dots + 2^k X^{k+5} + \dots \end{aligned}$$

- $d_{\text{free}} = 5$

And again because the minimum weight is 5 so free distance of this convolutional code is 5.

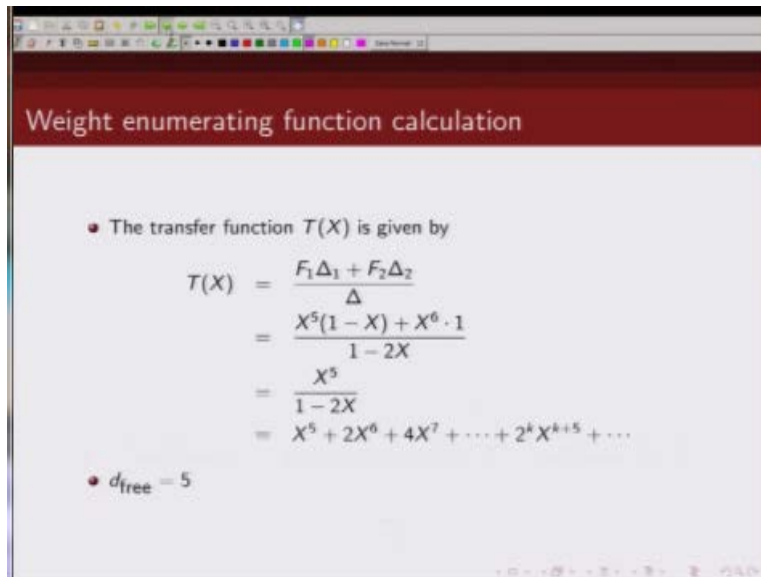


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Now we repeat the same exercise with augmented state diagram. Now what was augmented state diagram each valid branch we added a, z to denote this you can reach from one state to another in one step. And we also added in each of these branches the weight corresponding to the information bits. So the information bit weight was 0, so with y0 so that was 1. So you can see in some cases the information sequence weight is 0.

(Refer Slide Time: 26:56)



Weight enumerating function calculation

- The transfer function  $T(X)$  is given by

$$\begin{aligned} T(X) &= \frac{F_1 \Delta_1 + F_2 \Delta_2}{\Delta} \\ &= \frac{X^5(1-X) + X^6 \cdot 1}{1-2X} \\ &= \frac{X^5}{1-2X} \\ &= X^5 + 2X^6 + 4X^7 + \dots + 2^k X^{k+5} + \dots \end{aligned}$$

- $d_{\text{free}} = 5$

(Refer Slide Time: 26:57)

Weight enumerating function calculation

- The transfer function  $T(X)$  is given by


$$\begin{aligned} T(X) &= \frac{F_1 \Delta_1 + F_2 \Delta_2}{\Delta} \\ &= \frac{X^5(1-X) + X^6 \cdot 1}{1-2X} \\ &= \frac{X^5}{1-2X} \\ &= X^5 + 2X^6 + 4X^7 + \dots + 2^k X^{k+5} + \dots \end{aligned}$$

The final series expansion is underlined in pink, with the terms  $X^5$ ,  $2X^6$ , and  $4X^7$  individually circled in pink.

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Weight enumerating function calculation

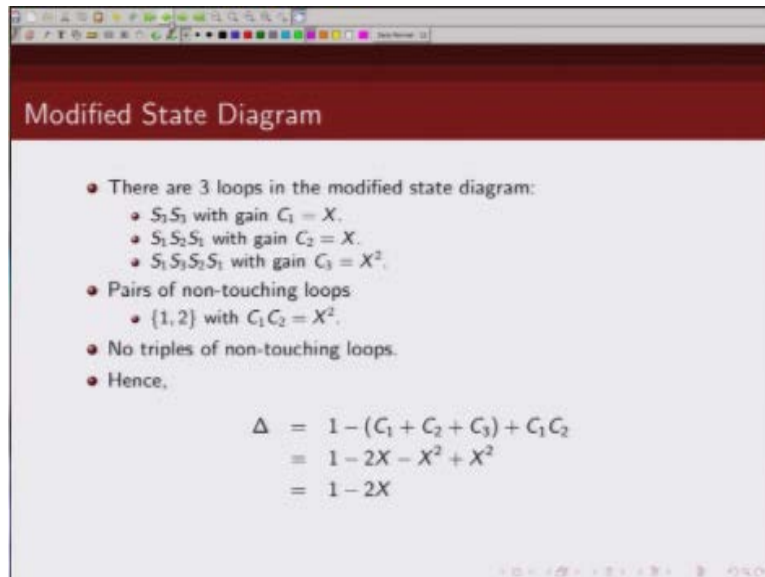
- There are two forward paths:
  - Path 1:  $S_0 S_1 S_2 S_0$  with gain  $F_1 = X^3$ .
  - Path 2:  $S_0 S_1 S_3 S_2 S_0$  with gain  $F_2 = X^5$ .
- The graph  $G_1$  obtained by removing the states on the forward path 1 is shown below.



- $\Delta_1 = 1 - X$ .
- The graph  $G_2$  obtained by removing the states on the forward path 2 is empty.

So let us just go back to the original state diagram.

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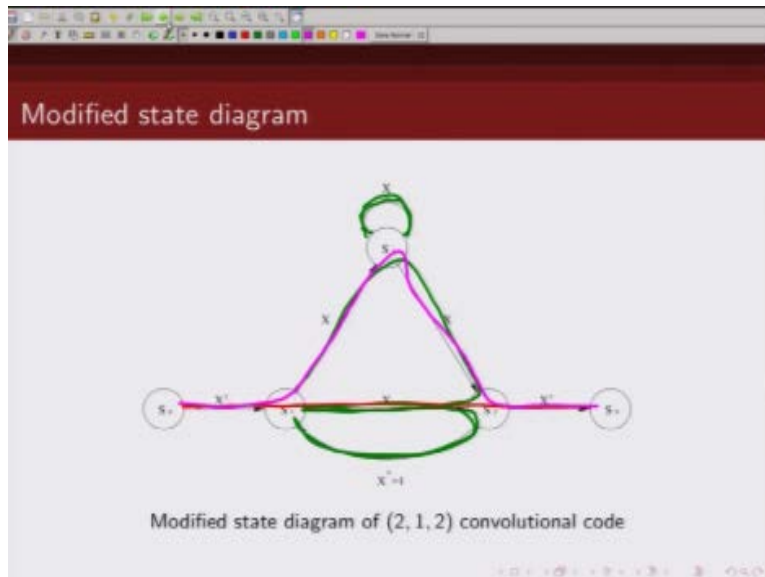


**Modified State Diagram**

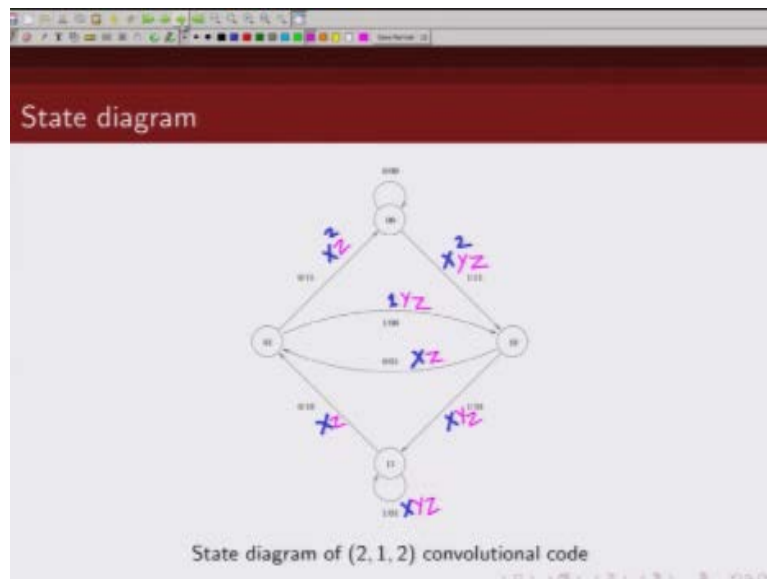
- There are 3 loops in the modified state diagram:
  - $S_3 S_3$  with gain  $C_1 = X$ ,
  - $S_1 S_2 S_1$  with gain  $C_2 = X$ ,
  - $S_1 S_3 S_2 S_1$  with gain  $C_3 = X^2$ .
- Pairs of non-touching loops
  - $\{1, 2\}$  with  $C_1 C_2 = X^2$ .
- No triples of non-touching loops.
- Hence,

$$\begin{aligned}\Delta &= 1 - (C_1 + C_2 + C_3) + C_1 C_2 \\ &= 1 - 2X - X^2 + X^2 \\ &= 1 - 2X\end{aligned}$$

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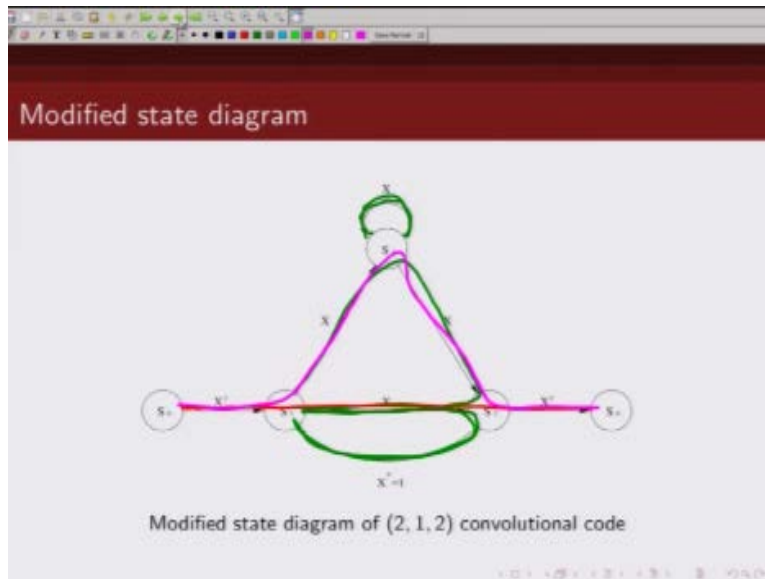
(Refer Slide Time: 27:06)



Yeah, let us go back to this, so you can see for this transition from 01 to 00, what is the weight of the information sequence that is 0. So  $y_0$  that is basically 1, what about this the weight of the information sequence here the input is 1, so this will be  $y$ . What is the weight of information sequence that is 1 so this will be  $y$ , this is weight information sequence is 0 so  $y_0$  is 1.

So wherever you had 1 here you are adding basically  $y$ . This is 1 and similarly at each of these transitions where there will be a  $z$  added to denote the length okay. So that is your augmented state diagram.

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### Modified State Diagram

- There are 3 loops in the modified state diagram:
  - $S_1S_3$  with gain  $C_1 = X$ .
  - $S_2S_2S_1$  with gain  $C_2 = X$ .
  - $S_1S_3S_2S_1$  with gain  $C_3 = X^2$ .
- Pairs of non-touching loops
  - $\{1, 2\}$  with  $C_1C_2 = X^2$ .
- No triples of non-touching loops.
- Hence,


$$\begin{aligned}\Delta &= 1 - (C_1 + C_2 + C_3) + C_1C_2 \\ &= 1 - 2X - X^2 + X^2 \\ &= 1 - 2X\end{aligned}$$

And that is what I mean, the completed augmented state diagram is what.

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Weight enumerating function calculation

- There are two forward paths:
  - Path 1:  $S_0 S_1 S_2 S_0$  with gain  $F_1 = X^3$ .
  - Path 2:  $S_0 S_1 S_1 S_2 S_0$  with gain  $F_2 = X^6$ .
- The graph  $G_1$  obtained by removing the states on the forward path 1 is shown below.



- $\Delta_1 = 1 - X$ .
- The graph  $G_2$  obtained by removing the states on the forward path 2 is empty.
- Hence  $\Delta_2 = 1$ .

I am showing you here.

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Weight enumerating function calculation

- The transfer function  $T(X)$  is given by

$$\begin{aligned} T(X) &= \frac{F_1 \Delta_1 + F_2 \Delta_2}{\Delta} \\ &= \frac{X^5(1-X) + X^6 \cdot 1}{1-2X} \\ &= \frac{X^5}{1-2X} \\ &= X^5 + 2X^6 + 4X^7 + \dots + 2^k X^{k+5} + \dots \end{aligned}$$

(Refer Slide Time: 28:17)

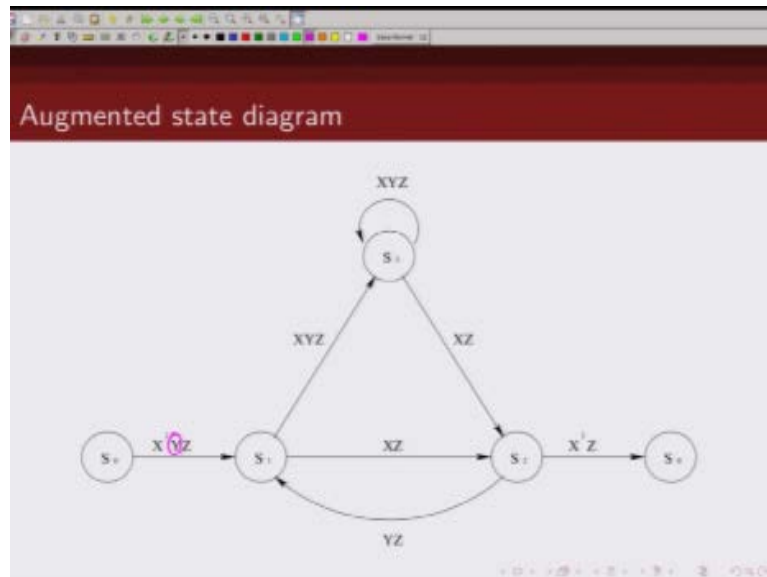
Weight enumerating function calculation

- The transfer function  $T(X)$  is given by

$$\begin{aligned} T(X) &= \frac{F_1 \Delta_1 + F_2 \Delta_2}{\Delta} \\ &= \frac{X^5(1-X) + X^6 \cdot 1}{1-2X} \\ &= \frac{X^5}{1-2X} \\ &= X^5 + 2X^6 + 4X^7 + \dots + 2^k X^{k+5} + \dots \end{aligned}$$

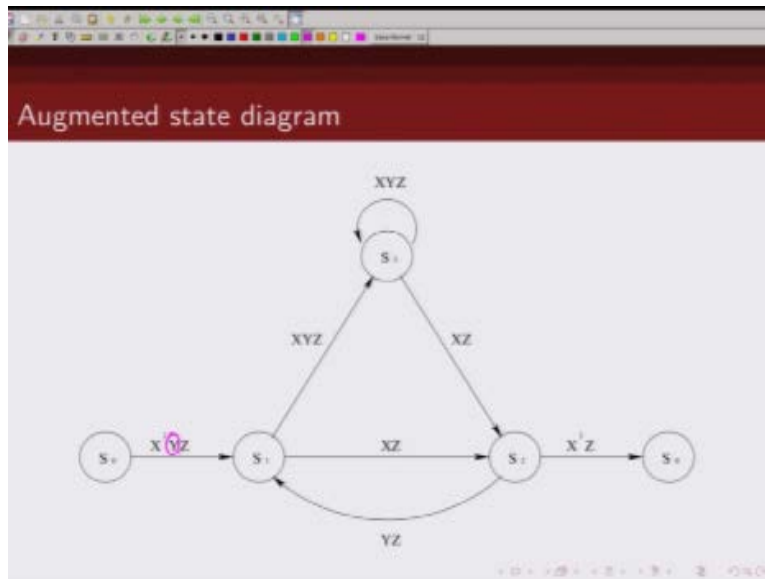
- $d_{\text{free}} = 5$

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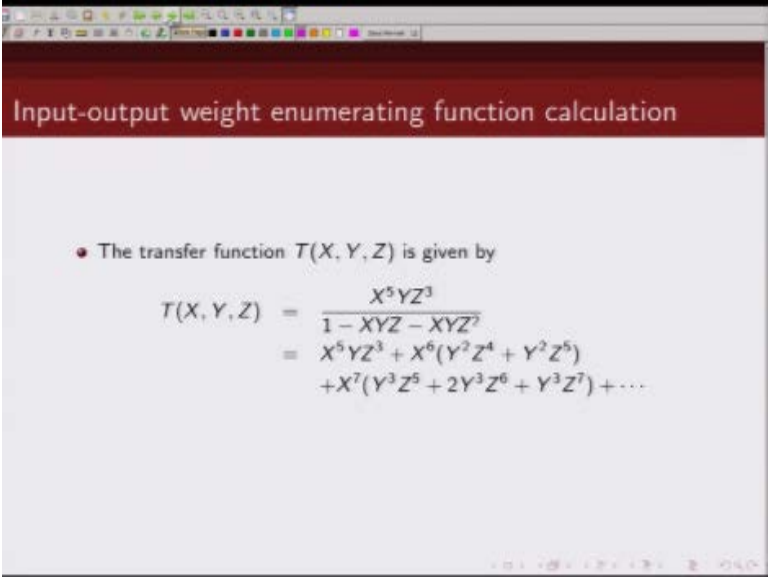
This is basically my augmented state diagram where I am not only specifying the coded weight but I am also specifying what input causes that output bit and z to denote the length. And I follow the same procedure using Mason's gain formula to compute.

(Refer Slide Time: 28:41)



The weight enumerating function.

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Input-output weight enumerating function calculation

- The transfer function  $T(X, Y, Z)$  is given by

$$\begin{aligned} T(X, Y, Z) &= \frac{X^5 Y Z^3}{1 - XYZ - XYZ^2} \\ &= X^5 Y Z^3 + X^6 (Y^2 Z^4 + Y^2 Z^5) \\ &\quad + X^7 (Y^3 Z^5 + 2Y^3 Z^6 + Y^3 Z^7) + \dots \end{aligned}$$

So I get this information I am skipping the steps is exactly the same procedure I just laid out for computing the weight enumerating function, and you can see it gives us lot more information.

(Refer Slide Time: 28:58)

Input-output weight enumerating function calculation

- The transfer function  $T(X, Y, Z)$  is given by

$$T(X, Y, Z) = \frac{X^5YZ^3}{1 - XYZ - XYZ^2}$$
$$= X^5YZ^3 + X^6(Y^2Z^4 + Y^2Z^5) + X^7(Y^3Z^5 + 2Y^3Z^6 + Y^3Z^7) + \dots$$

The weight enumerating function said we had one code word of weight 5. Now it says that code word of weight 5 basically was caused by message information bit 1 and the length of the [indiscernible][00:29:18] from all 0 state before it merge with all 0 state was 3. Similarly there we have shown that there were two code words of weight 6; this completely specifies what those two code words was.

One which was generated by message bit v2 of length 4 this was message bit to length 5. So you can see at the augmented state diagram, if we use it to generate the transfer function it gives us lot more information. So with this I will conclude this lecture. Thank you.

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