Indian Institute of Technology Kanpur
National Programme on Technology Enhanced Learning (NPTEL)
Course Title
Error Control Coding: An Introduction to Convolutional Codes

Lecture-10B
Problem solving sessions-III

by
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Welcome to the codes on error control coding, an introduction to convolutional code. So in this lecture we will try to solve some problems

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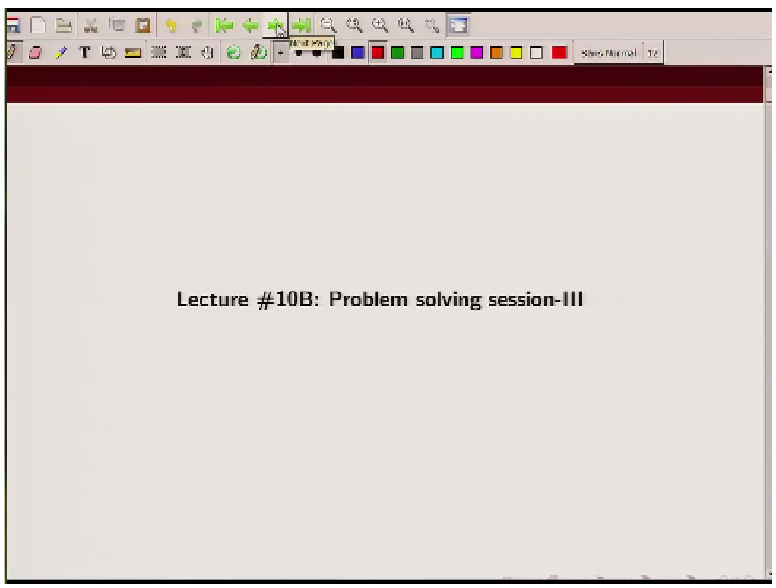

Related to convolutional code and in general.

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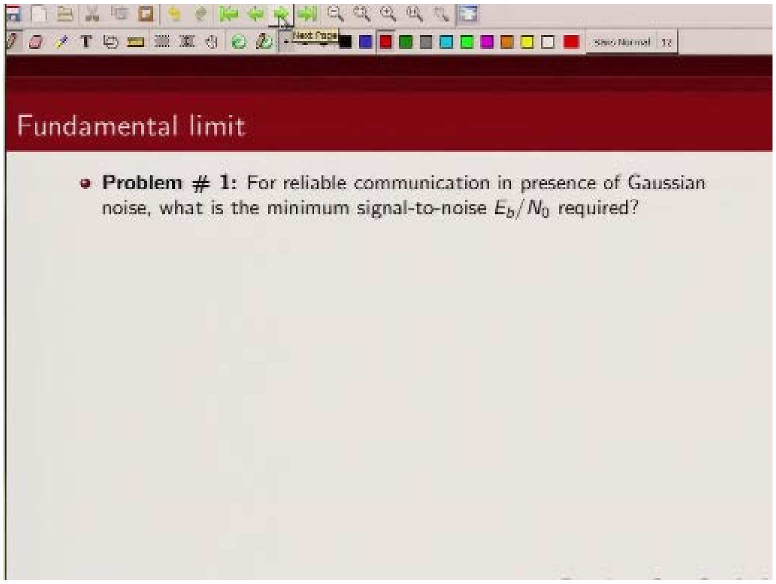

So the first question that we will try to answer is, if you want to do reliable communication in presence of an additive white Gaussian noise channel and of course we have infinite bandwidth, what is the minimum signal to noise ratio required?

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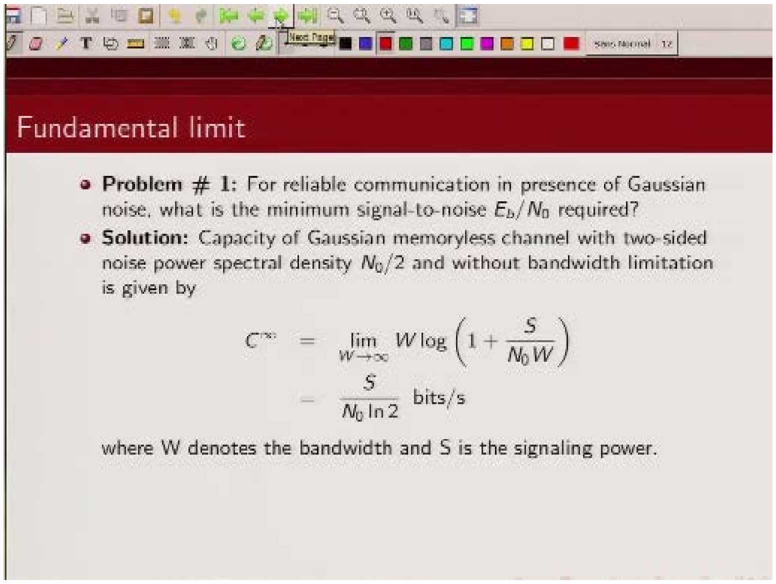

So for that we first need the expression for capacity of additive white Gaussian noise channel.

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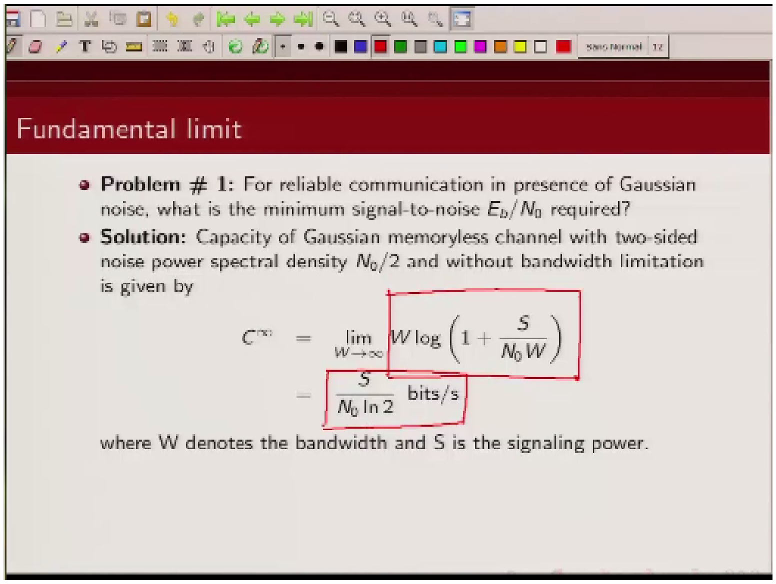

And the capacity of additive white Gaussian noise channel is given by this expression, where w is a bandwidth, s is my signaling power; $N_0/2$ is two sided power spectral density. Now – so we are considering when bandwidth is infinite so this can be written, so this will be log(1+ when w is infinite then this will go to zero, so log (1) will be zero and 1/w will also go to zero, so it is 0/0 form.

So we will differentiate and we can find out that the capacity when bandwidth is infinite is given by this expression, S/N_0 natural log of two bits per second.

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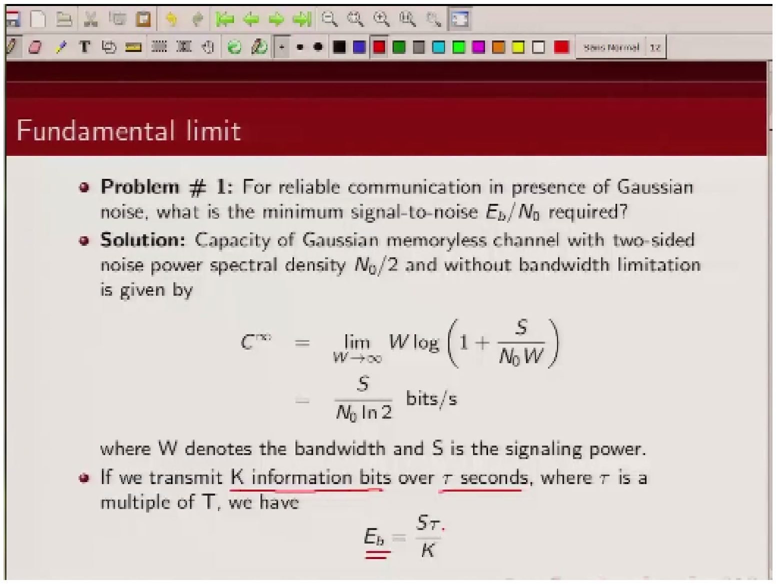

So we are interested in transmitting k bits over $\mathcal{T}$ seconds. So if we do that where $\mathcal{T}$ is a multiple of time period T, so if we do that our energy per bit is given by S $\mathcal{T}$/K this S was my signaling power we are sending over time T, $\mathcal{T}$ and total number of information bits was K, so energy per bit is S $\mathcal{T}$/K.

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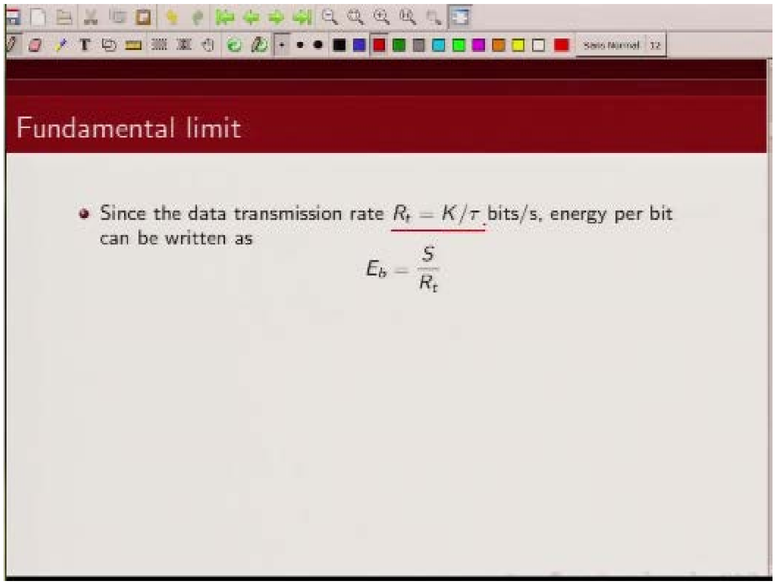

Now what transmission because we are transmitting K bits what time τ so our transmission rate is K/ τ bits per second.

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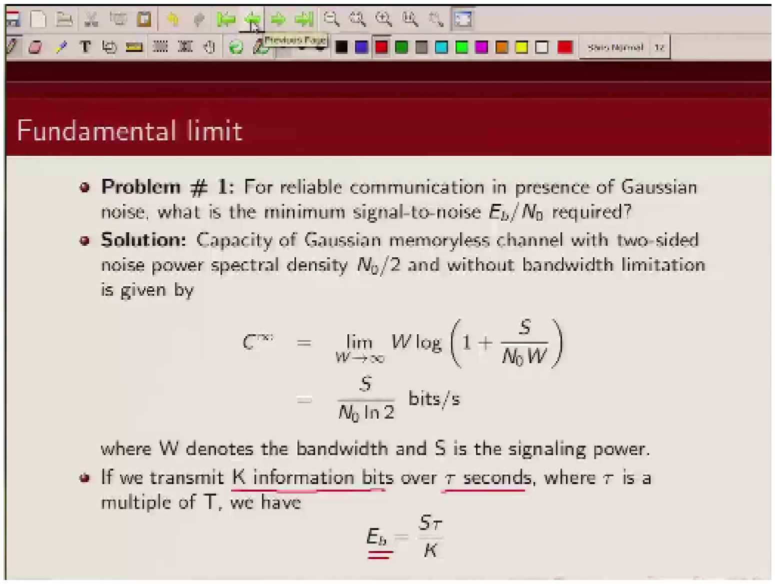

And our energy per bit that we wrote here is basically S τ/K.

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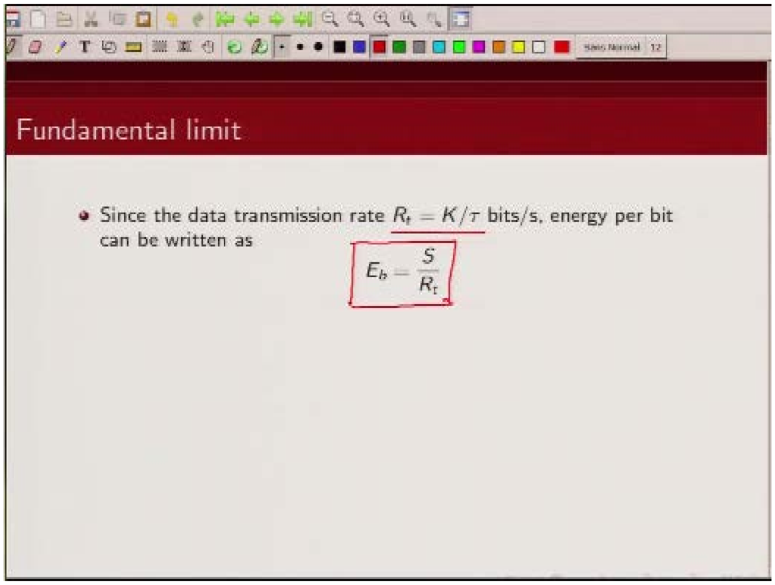

And K/ τ is R_t so we can write energy per bit in terms of signaling power and transmission rate RF_t.

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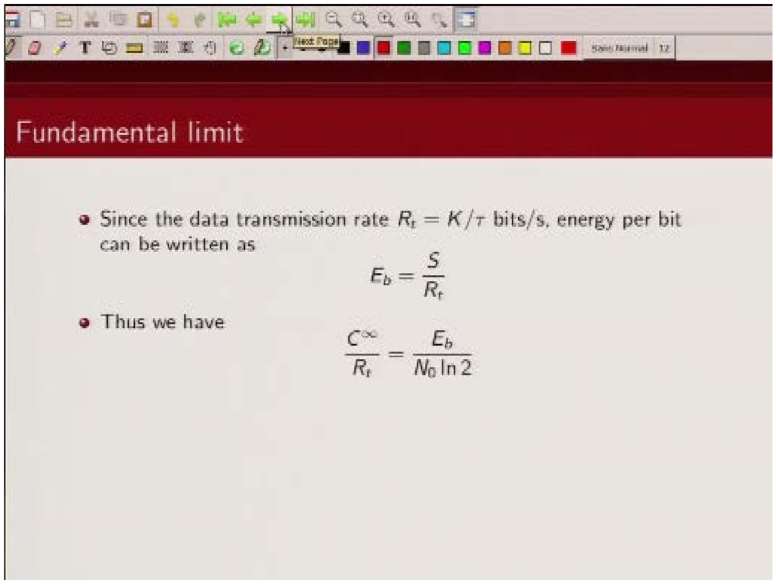

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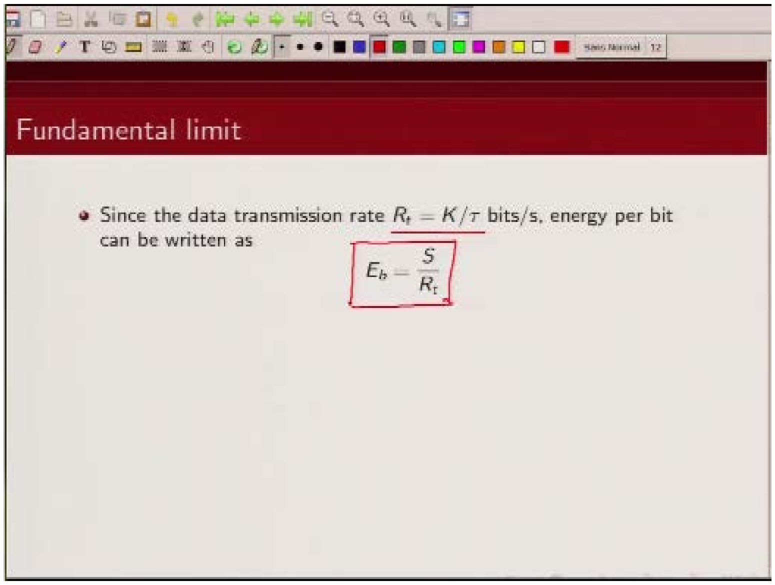

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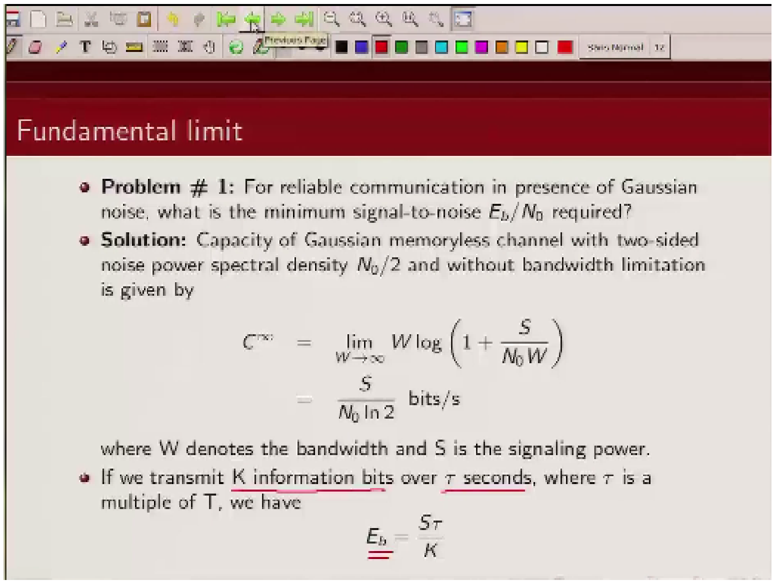

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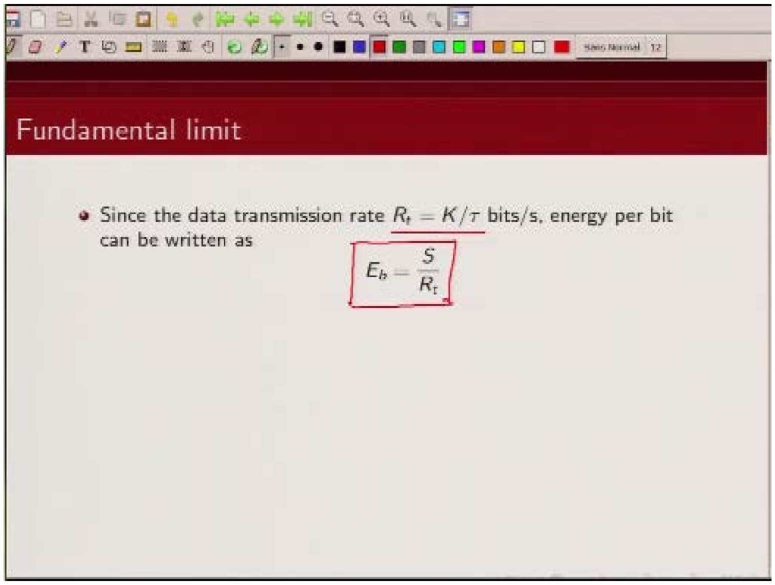

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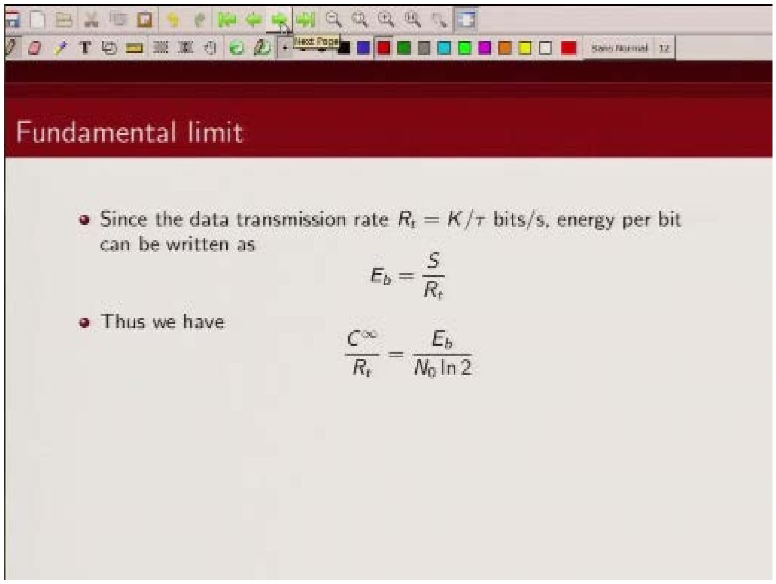

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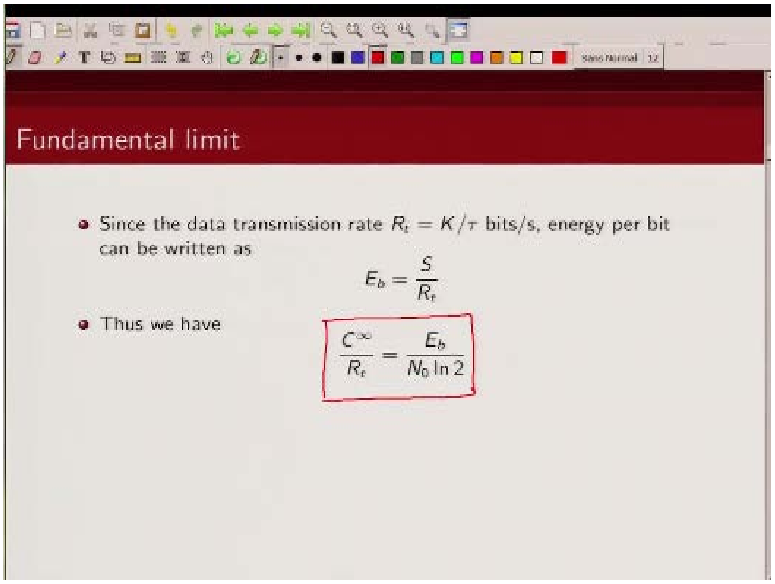

So if we divide our expression for general capacity by R_t what we get is this expression. Now we know from Shannon noisy channel coding theorem that as long as transmission rate is less than channel capacity we can reliably communicate over the communication channel. So we want R_t to be less than this channel capacity.

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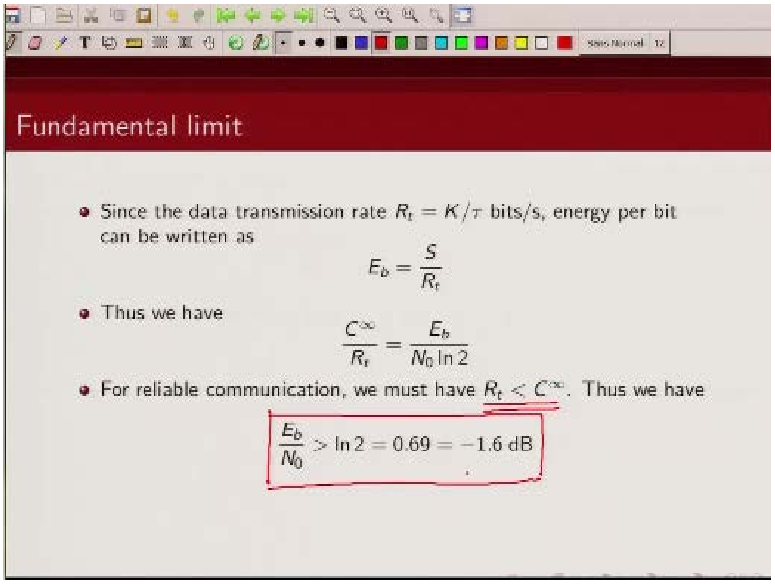

So if we want this to hold then we get a condition on signal to noise ratio, so we get this following condition on energy per bit, so snr per information bit we get this condition that snr per information bit should be greater than -1.6 dB. This is for the case when we have infinite bandwidth.

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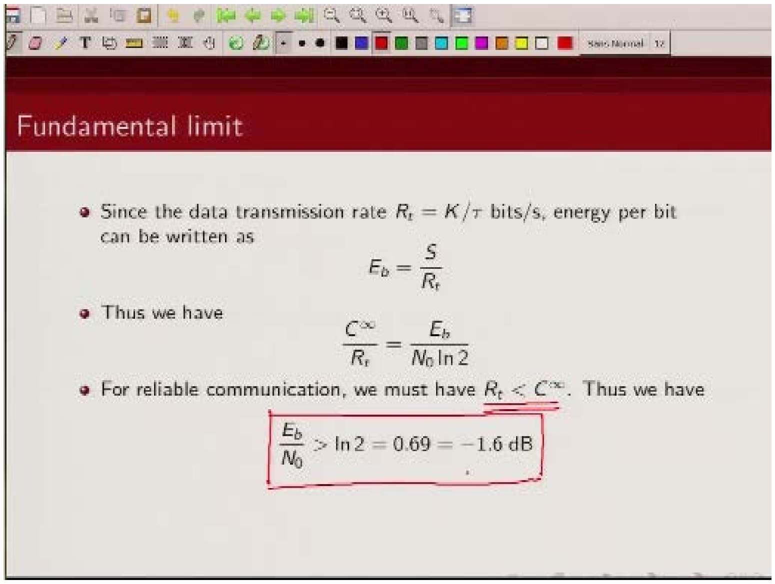

And our code rate can actually go to zero.

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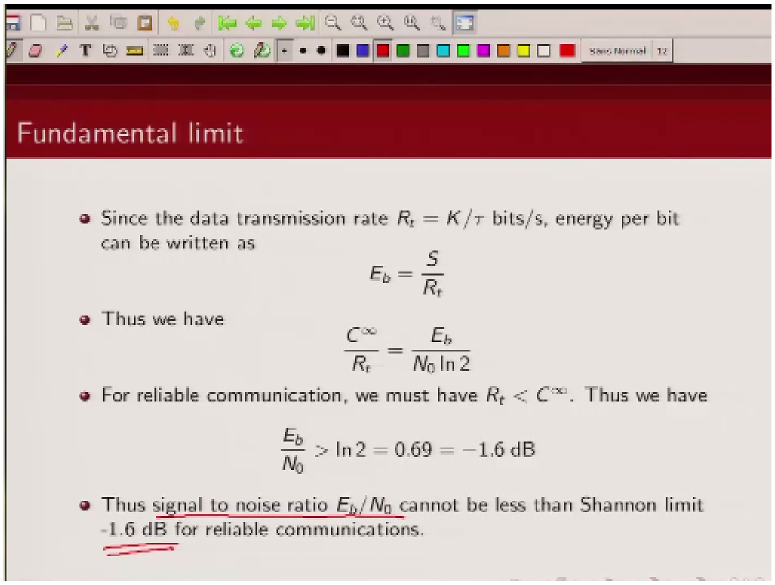

So signal to noise ratio then cannot be less than this limit which is -1.6dB. Okay let us look at the next problem.

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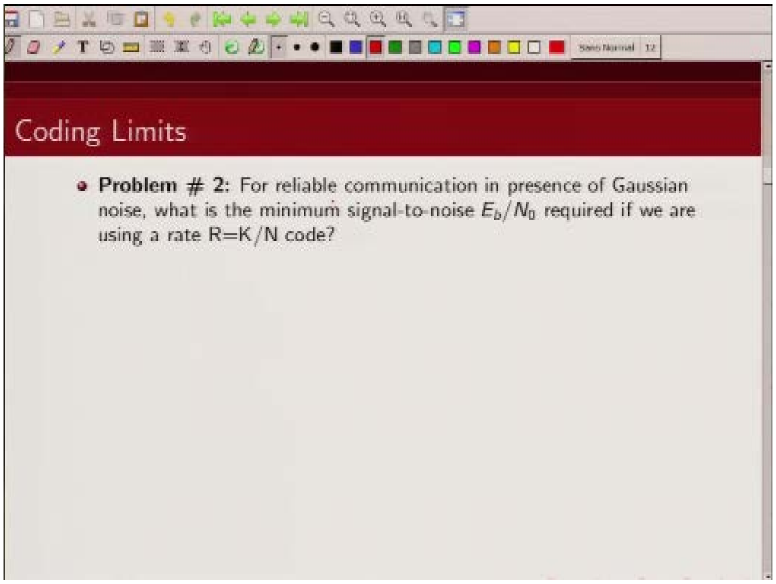

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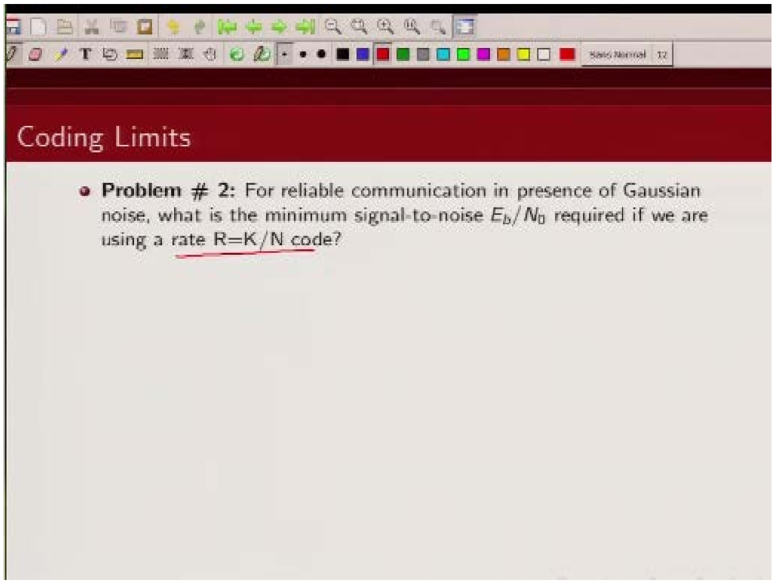

So we are interested in reliable communication over additive white Gaussian noise channel and we are transmitting using a rate K/N code.

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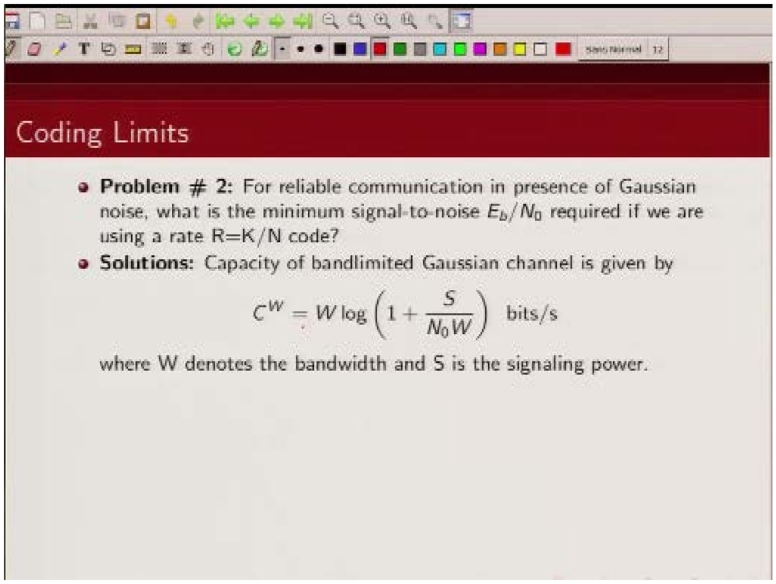

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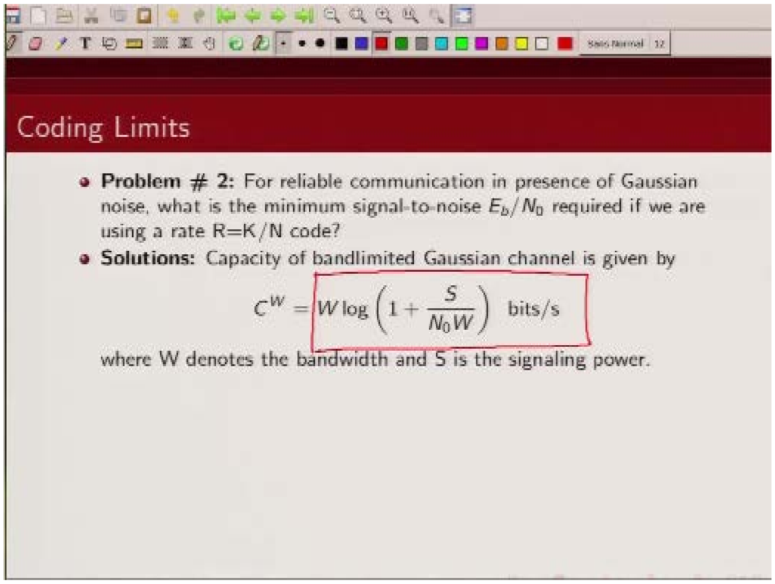

So what is the effect of the bandwidth w, now we know for a band limited channel the capacity is given by this expression okay.

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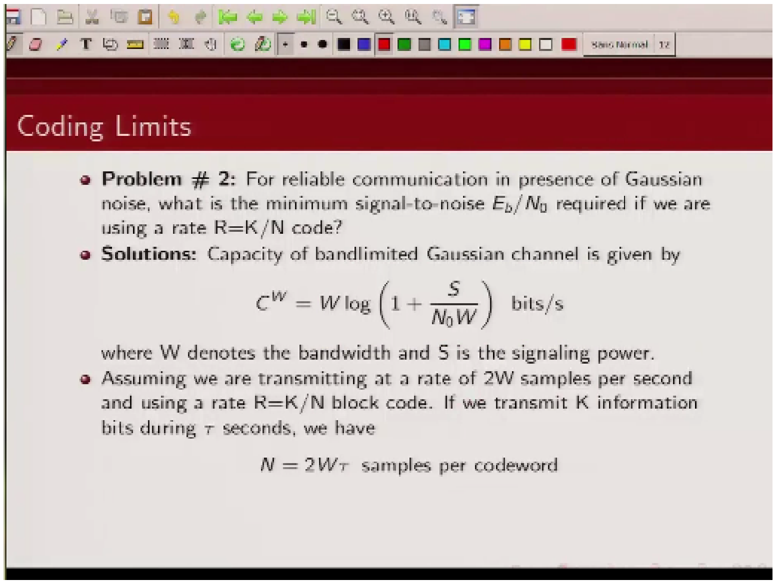

So let us say we are transmitting at Nyquist rate.

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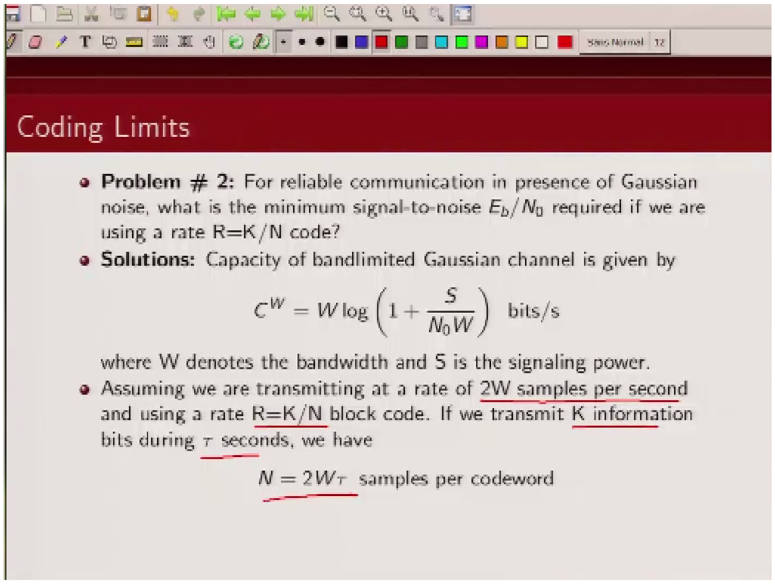

So we are transmitting at 2W samples per second and we are using a rate K/N code. So if we transmit K information bits during time $\mathcal{T}$ then number of samples per code word that we are sending is 2W $\mathcal{T}$.

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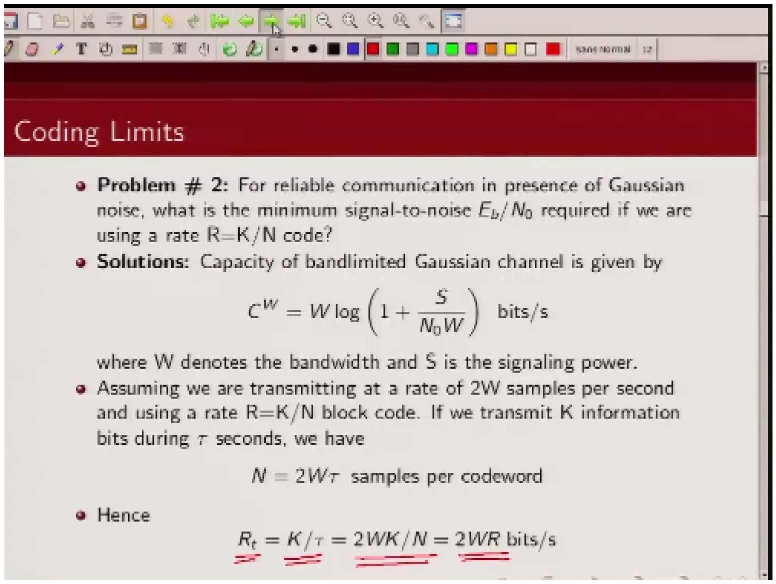

Now our transmission rate is K information bits over τ time, so a transmission rate is K/ τ and this we can write in terms of these samples per code word n which is given by this expression and K/N is nothing but our code rate, so this transmission rate is given by 2WR where R is my code rate, so this many bits per second is my transmission rate.

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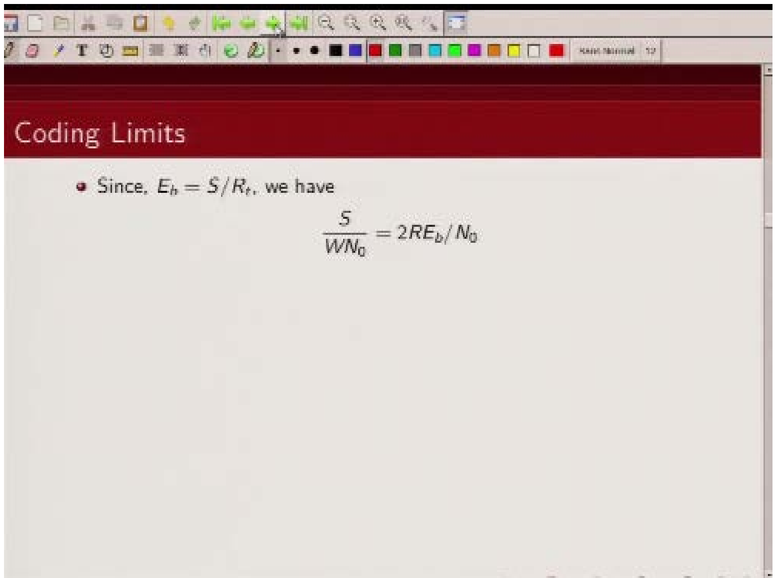

Now we know we that

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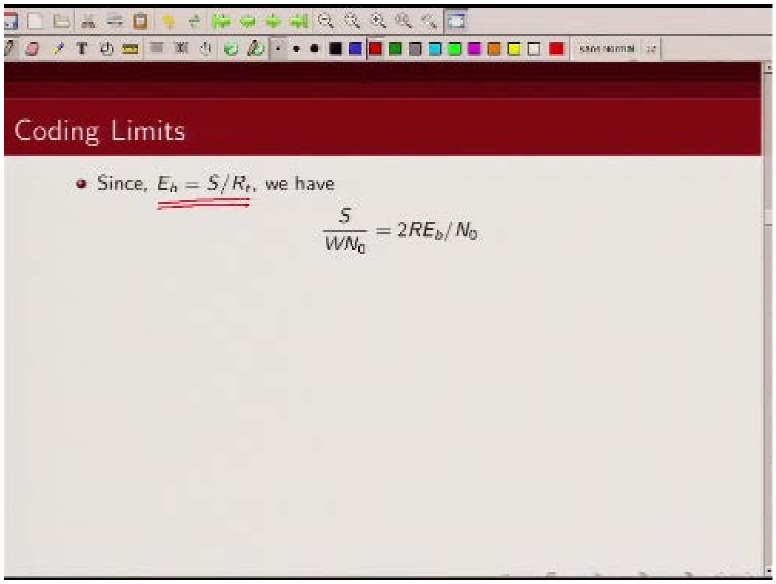

Energy per bit is given by signaling power by transmission rate, this we have done in the last example.

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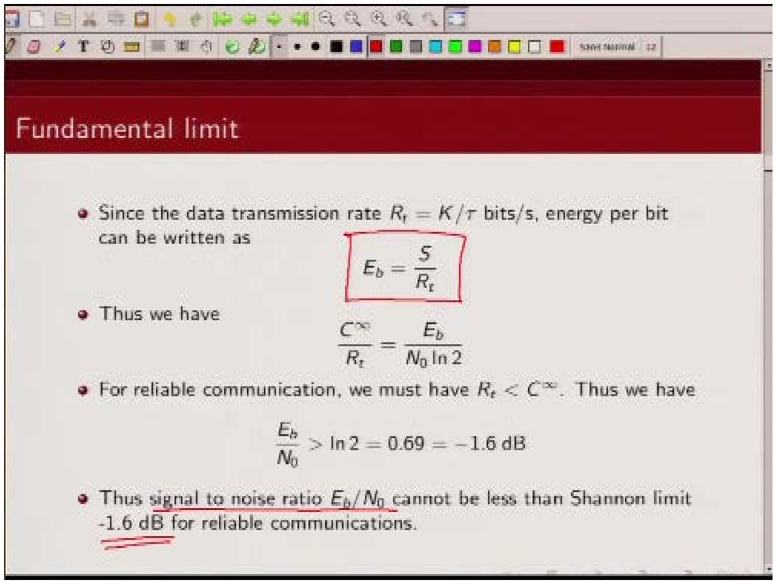

We had this right so using this, then so if we do that

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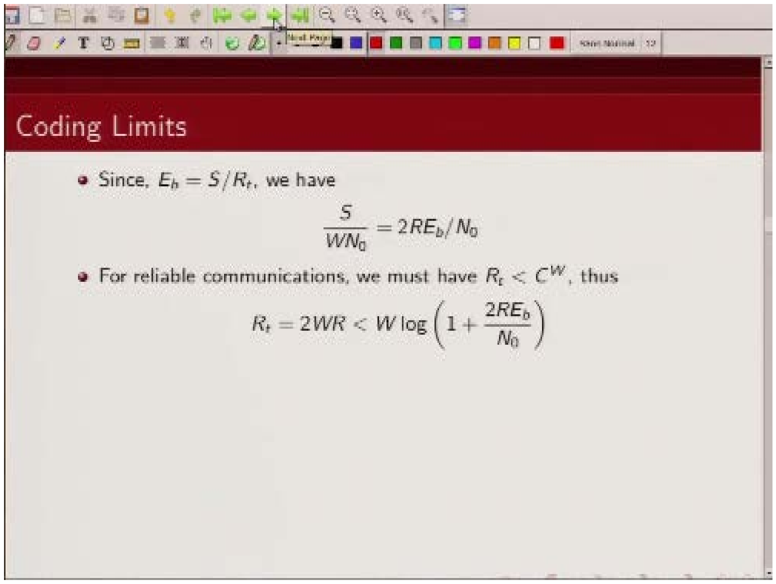

We can write

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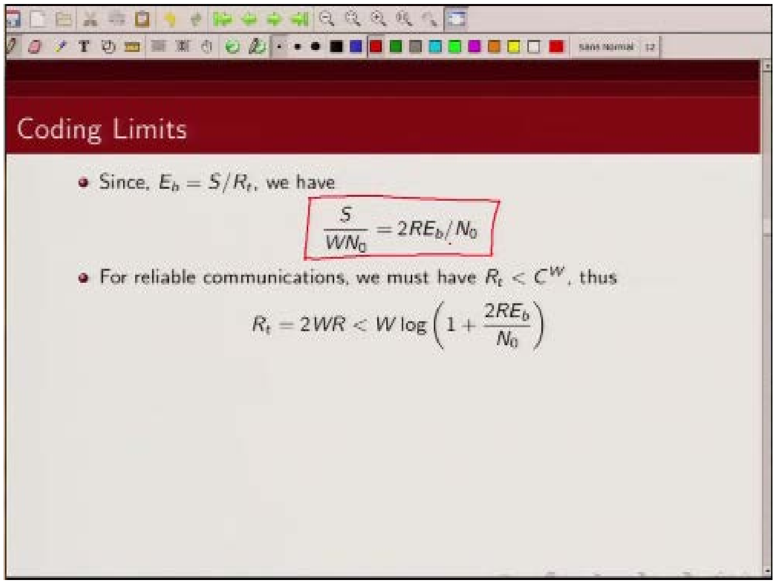

S/WN_0 in terms of E_b/N_0, this is equal to 2 times code rate by into SNR per information bit so in this expression of

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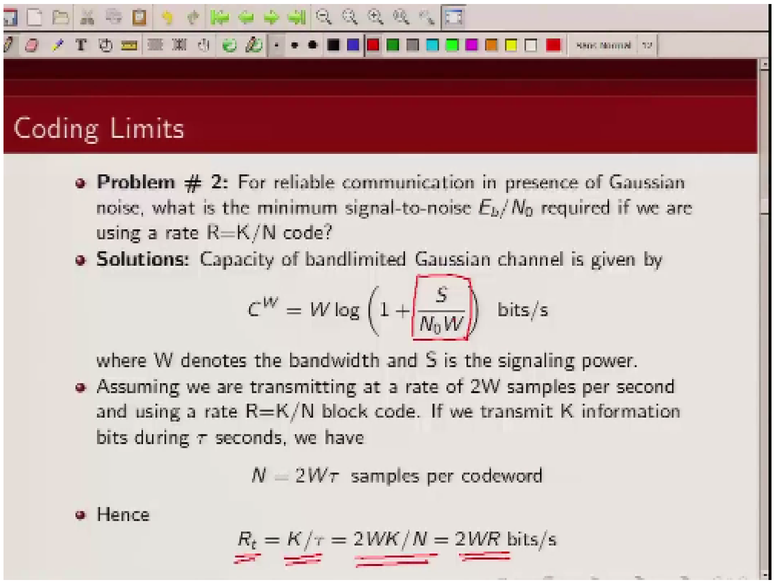

Capacity of band limited channel we can replace this expression by

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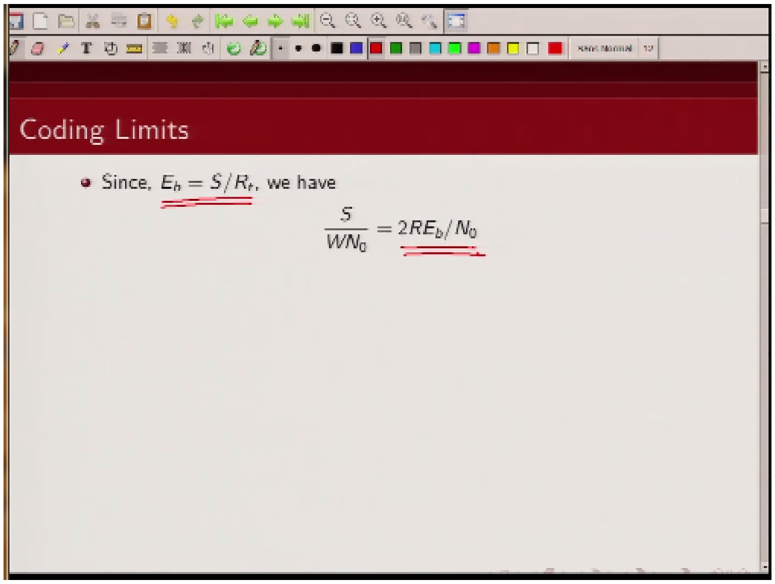

This, fine?

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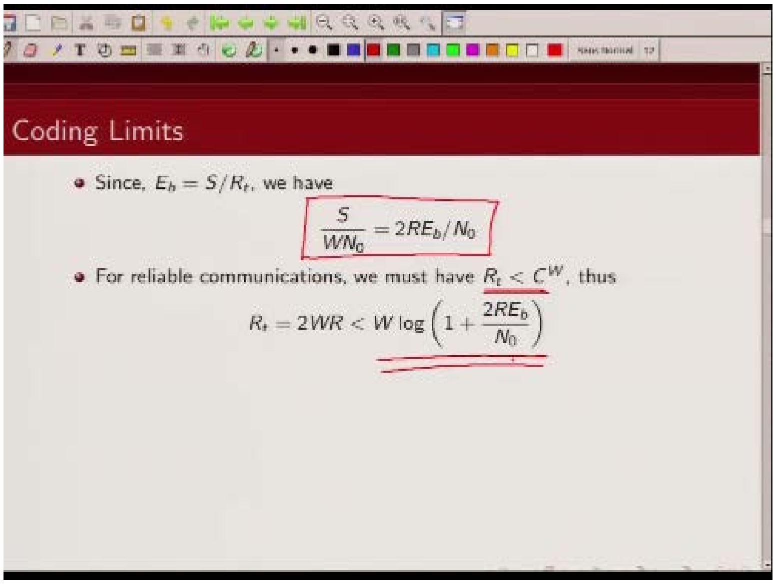

Now for reliable communication we know that our transmission rate should be less than channel capacity and what is channel capacity, that is for the band limited channels W log (1+ S/WNN_0 which is basically given by this expression, so a transmission rate which is 2 times WR should be less then W log (1+ $2RE_b/ N_{0)}$

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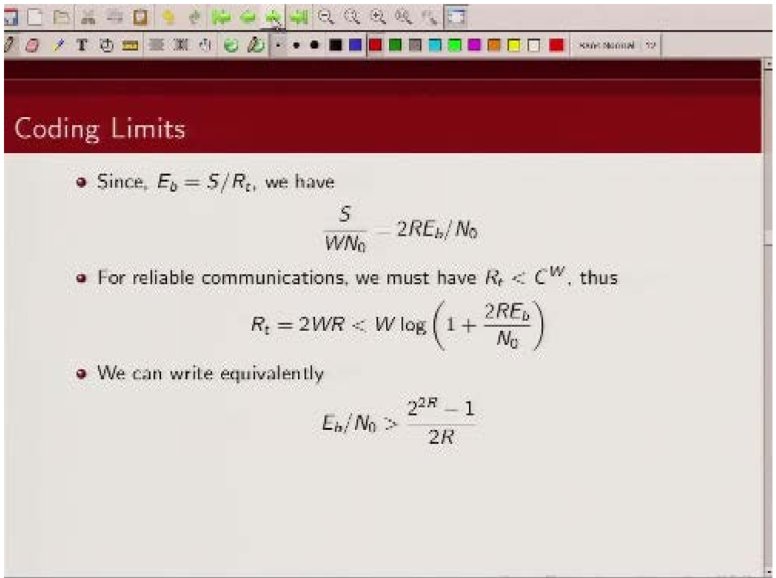

So from here then we can write down

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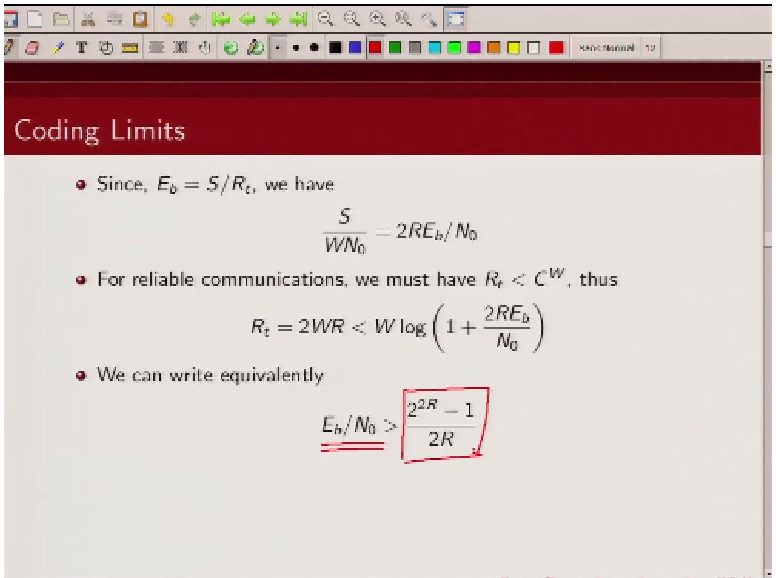

The expression for minimum SNR information bit is a energy per information bit by noise power [indiscernible][00:08:05] so SNR by information bit we can write it as this is should be greater than equal to 2^{2R} -1 / by 2R, now this right hand side term is an increasing function of R

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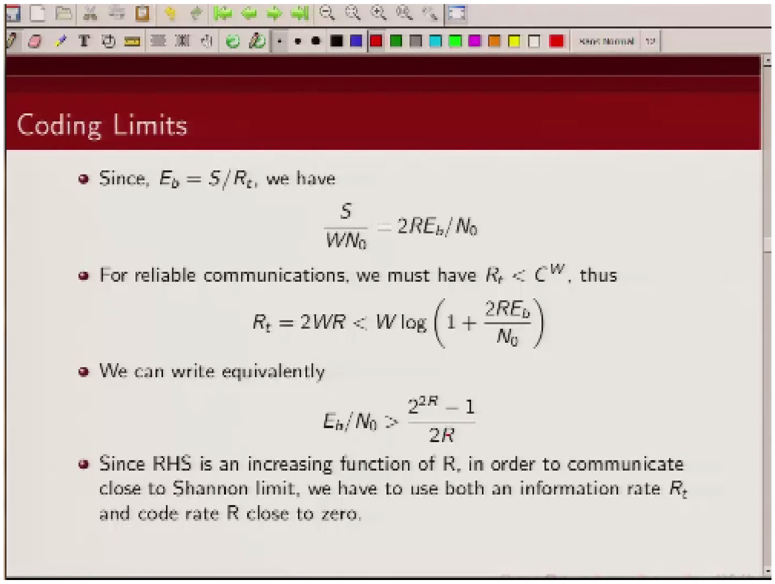

So if you want to communicate close to the Shannon limit then your information rate R_t as well as code rate R should be close to zero. In fact

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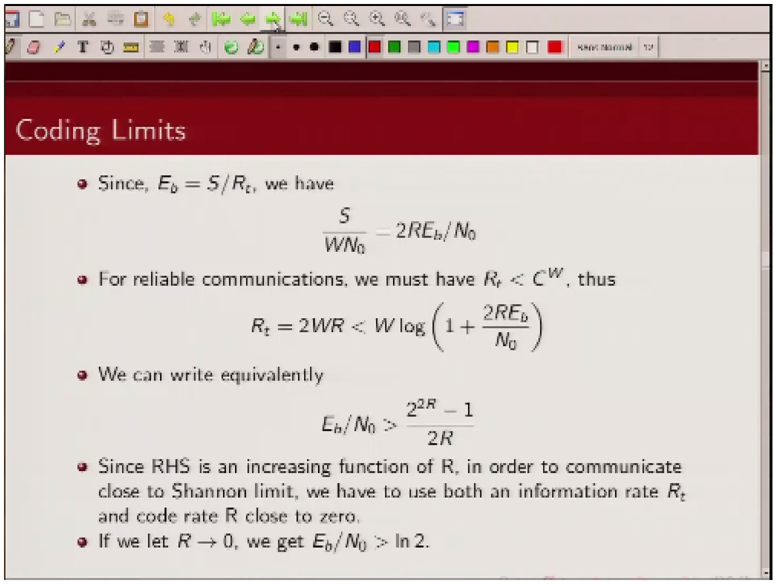

If you let R → 0 what you will see is

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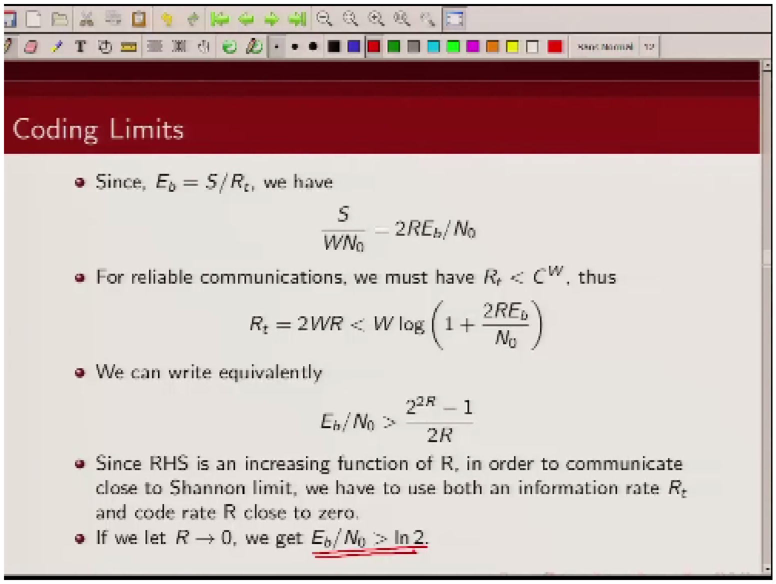

You will get the limit that we had just talked about in the previous problem which was minus 1.6DP and if, let us plug in some practical values of R, let us say R is

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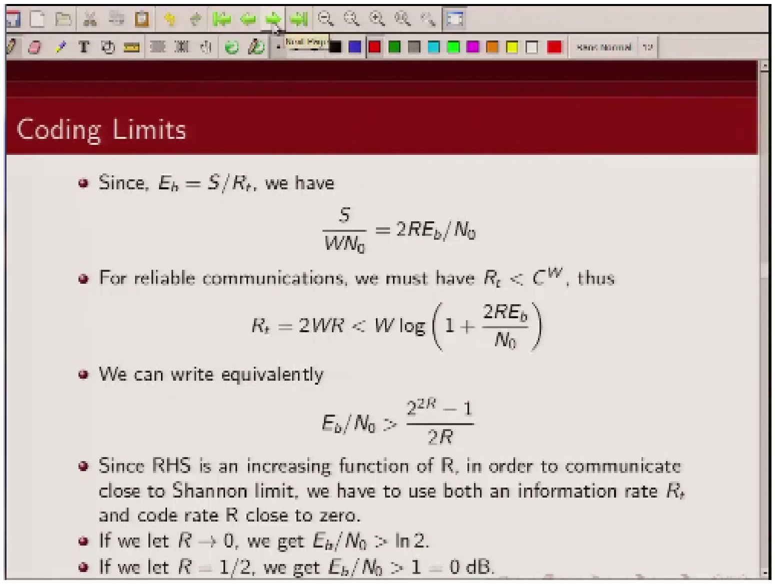

Half , we put value of R to be half then E_b by not should be more than 0 DP, so you can see for any rate which is away from 0 then this minimum SNR required for transmission is also more, okay.

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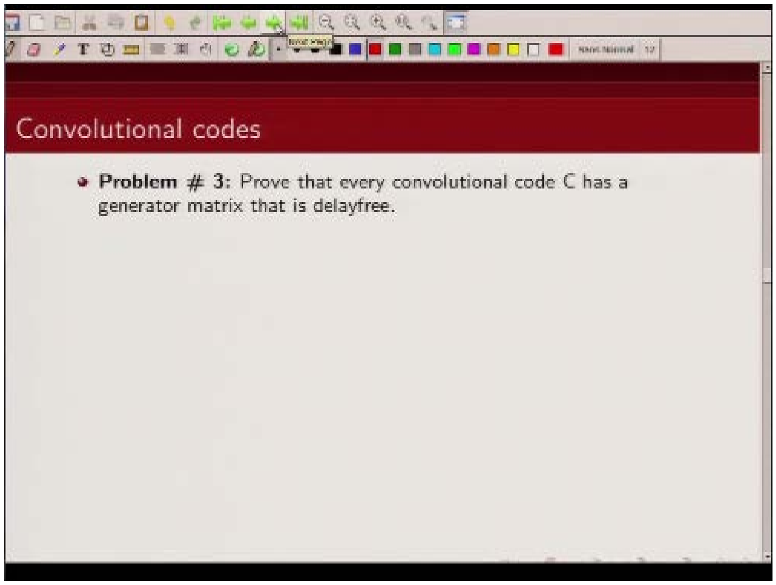

Now the next question that we will solve is to prove that

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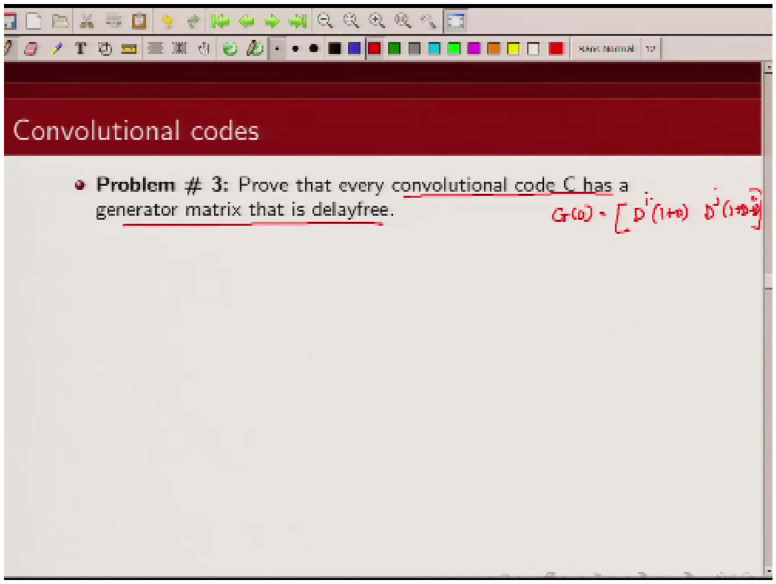

For every convolutional code has a generator matrix which is delay free. Now delay free meanings are basically if we have a generator matrix let us say of the from D^i and then some, something like 1 + D and some D^j 1+ D+D^2 or something like that we can always write an equivalent generator matrix which will be free of these like delay terms, I will talk about that.

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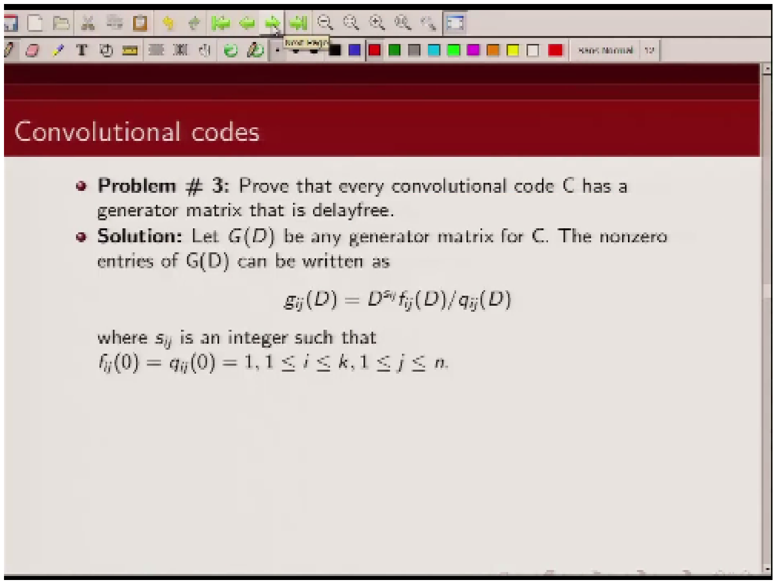

So let us see we have a generator matrix

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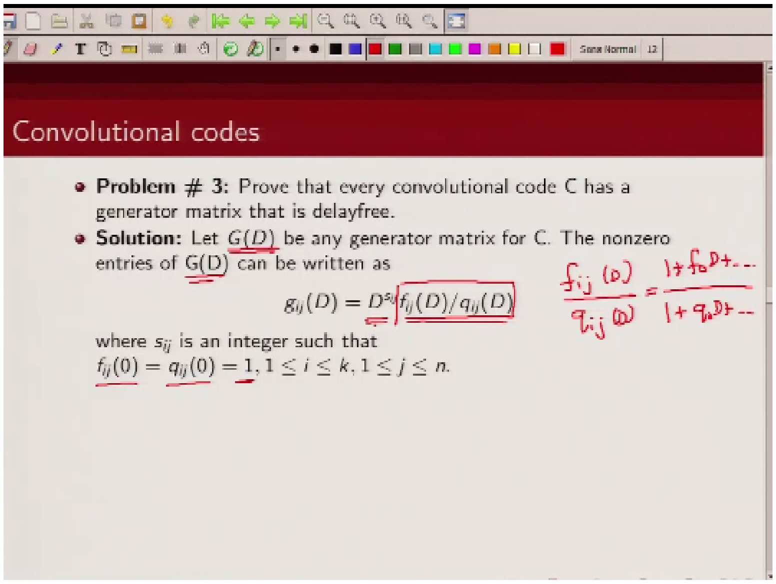

G(D) okay and its non-zero entries can be written in this particular form, so there is some delay term we calling $D^{s}{}_{IJ}$ and then we have this rational function we have this rational function which is $f_{ij(}$ D) /q_{ij}(D) and $f_{ij(}$ D) and q_{ij}(D) is of the form 1+some f_0 D+ …. This is 1+ q_0D+ something, something okay, so that is what I meant if (0) and q_{ij}(0) is 1, so any entry in the generator matrix can be written in this form okay? Now

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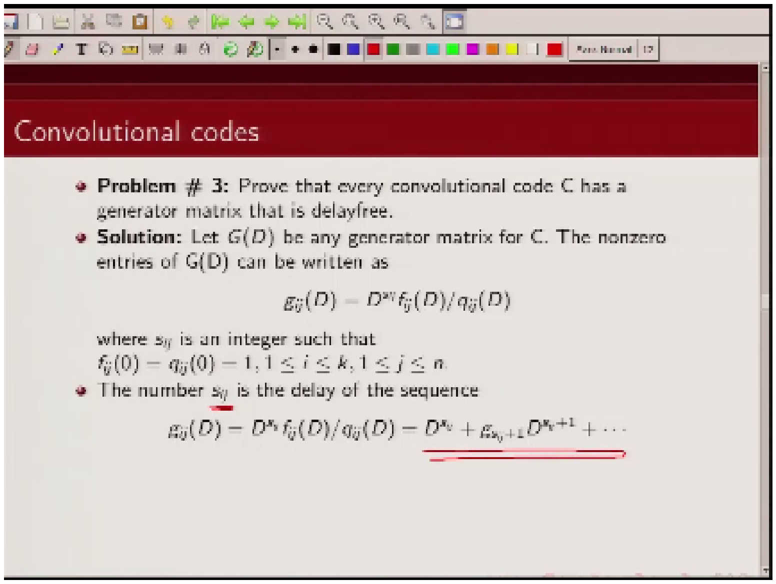

What is this term SIJ; it is essentially a delay term, so if we have some term like SIJ so what you will get is terms of this particular form.

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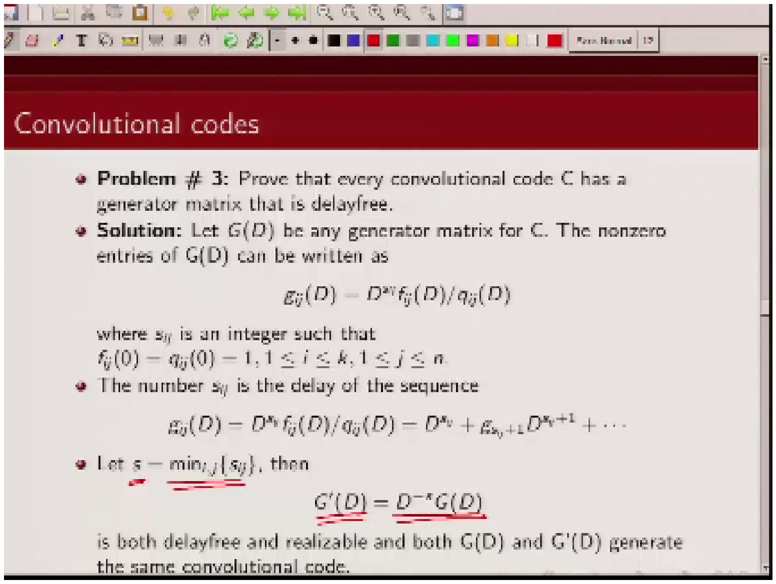

Now if we consider S which is the minimum S_{ij} over all I and J then we can find an equivalent generator matrix which would be G – of D which we can write as D-S GD and this GID will be delay free and it is of course realizable, and it also generate the same set of code words, so whenever we have a generator matrix which has the delay term we can always have - find an equivalent generator matrix which is delay free.

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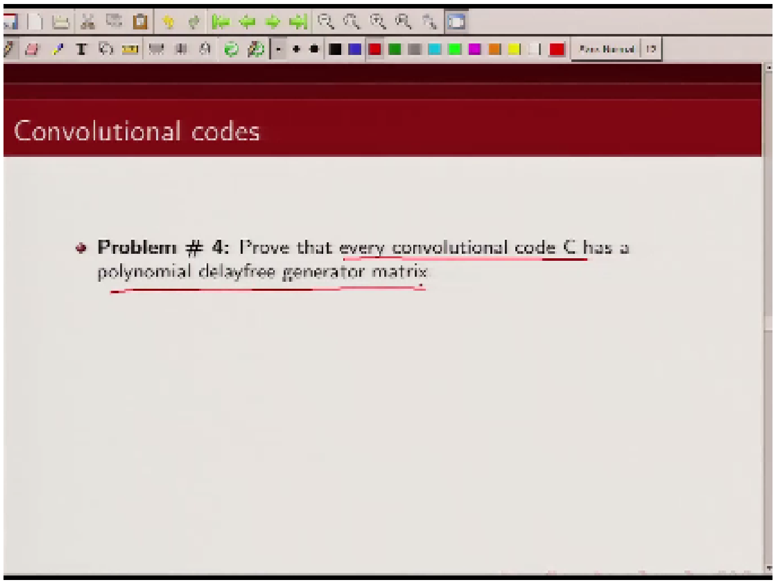

Next question, prove that for every convolutional code has a polynomial delay free generator matrix. So for any convolutional code we have an equivalent polynomial delay free generator matrix.

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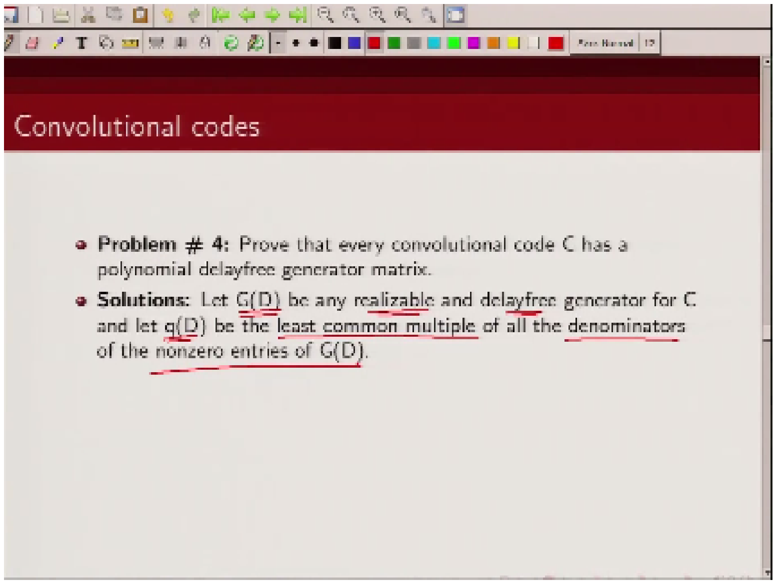

So let us say we have a generator matrix which is given by G(D) and this be any realizable and delay free generator for this convolutional code C, and let q(D) be the least common multiple of all the denominators of the nonzero entries of G(D), so GD basically has a natural form and q(D) is the LCM of the denominator terms of the nonzero entries of G(D).

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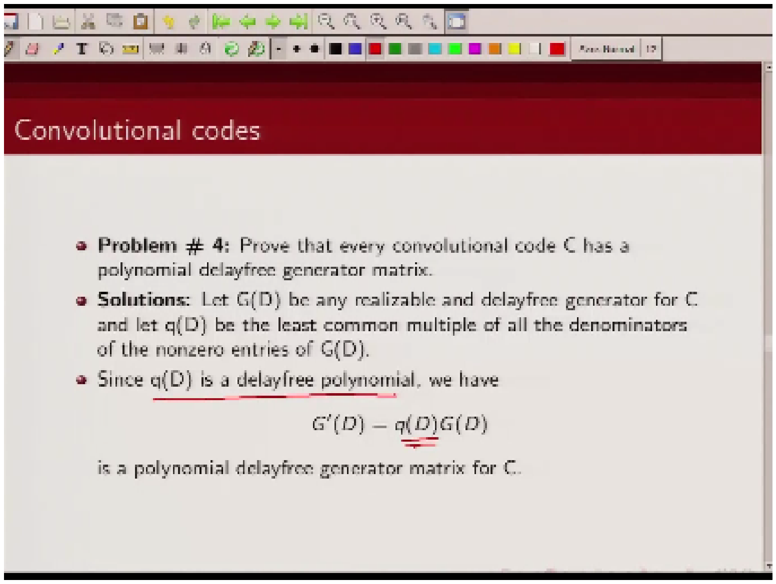

Now if we have such and so we will have a q(D) which is basically delay free so if we multiply our original generator matrix by this q(D) we will end up with a

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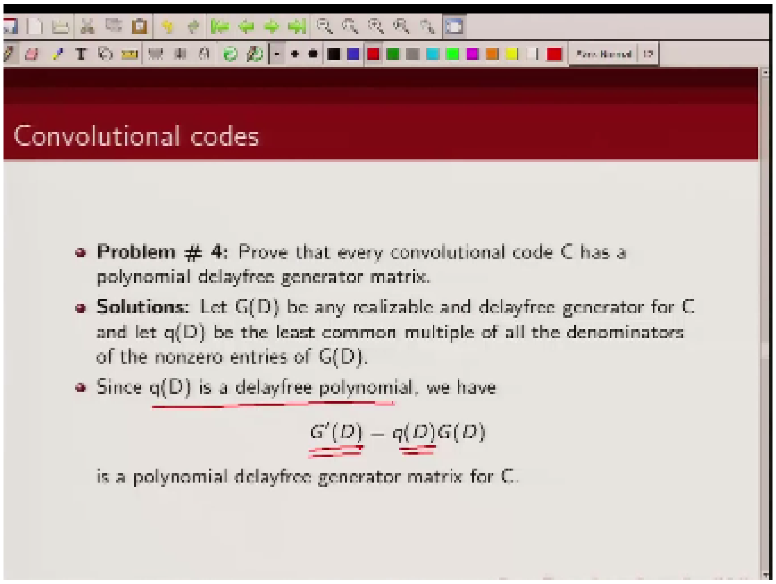

New generator matrix G – of D which will be polynomial and delay free.

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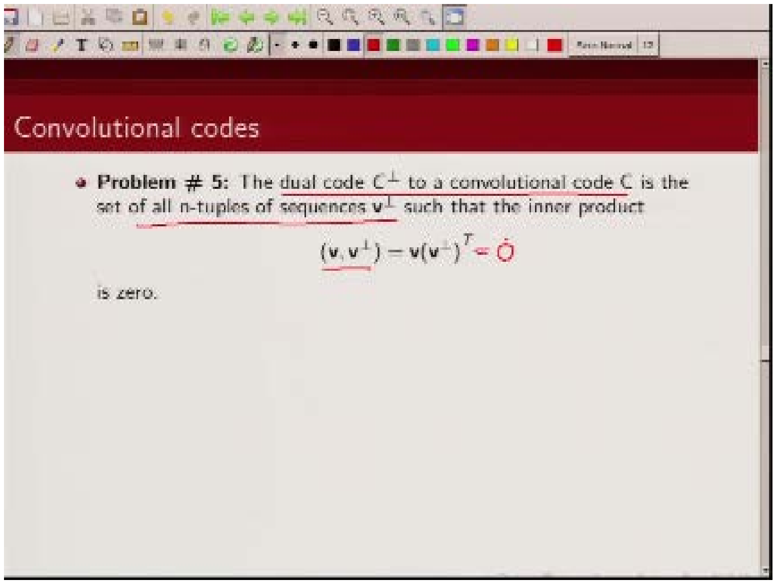

Okay next so let us first define a dual of convolutional code so we define a dual to convolutional code C as a set of all n-tuples of sequence V dual such that the inner product between this which is define as V and this V dual transpose this is essentially zero. So if you have a dual to the original convolutional code, then if you take set of code n-tuples sequence from the original code and the dual code, their inner product will be zero.

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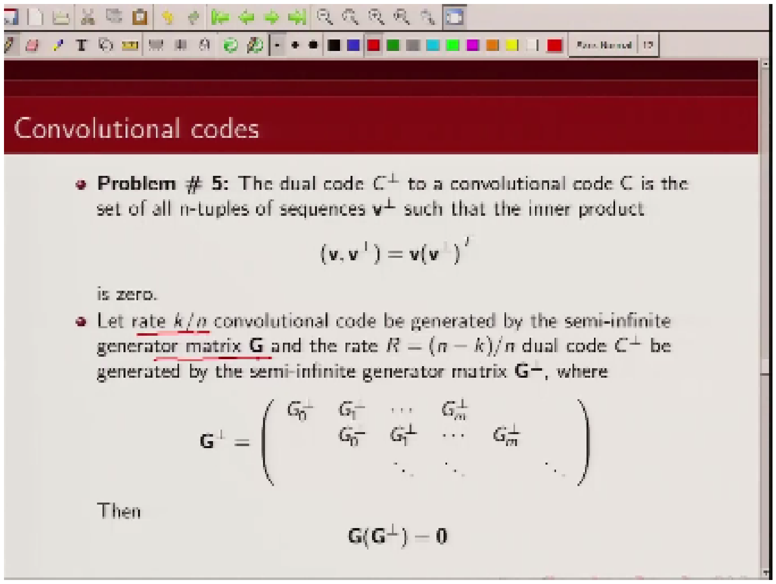

So let us define a rate k/n convolutional code which is generated by a generator matrix G and you know that we can write the generator matrix or convolutional code in a semi-infinite fashion.

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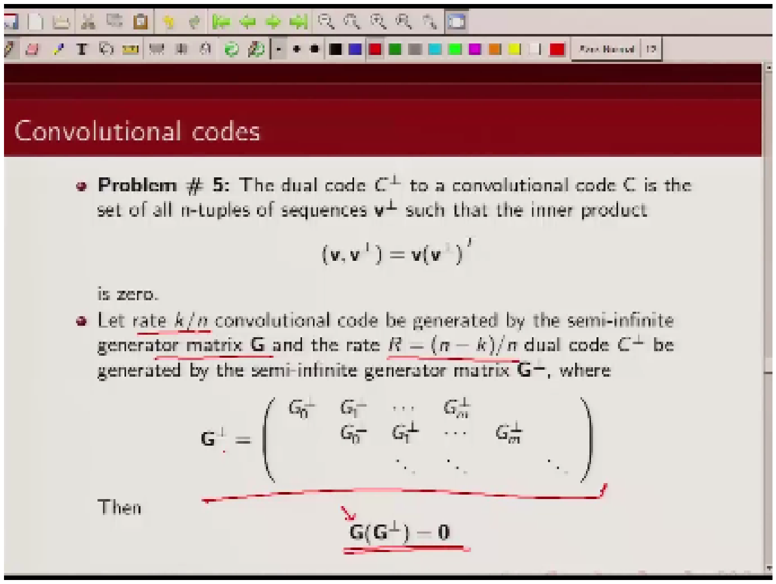

And let us see the dual code has is this, which has rate n – K /n and it is generated by this semi-infinite generator matrix, then show that G, G transpose basically will be zero whereas G is the generator matrix of the original code and this is the generator matrix of the dual code.

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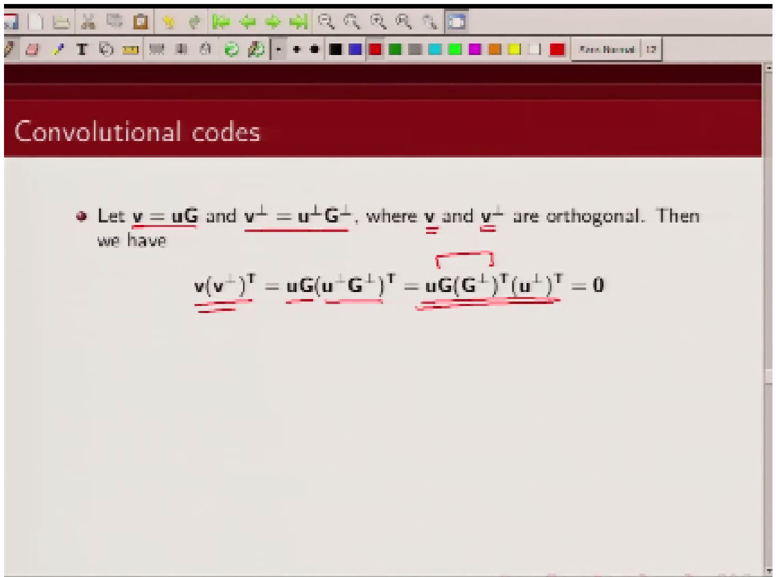

So how does this follows so since V, V transposes zero so what is V, V is U times G and V and V dual is U dual, G dual. Now we know that they will be dual of each other if V, V dual basically is orthogonal so if, if this inner product is zero now if the inner product is zero, so then UG and this transpose should be zero, so this we can write as UG(G) transpose, you transpose. Now this term will be zero only if this term is zero for all u okay.

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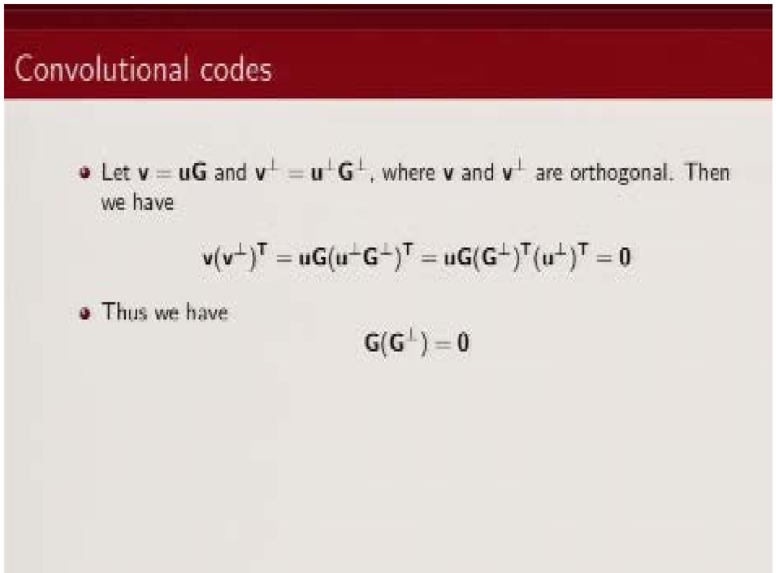

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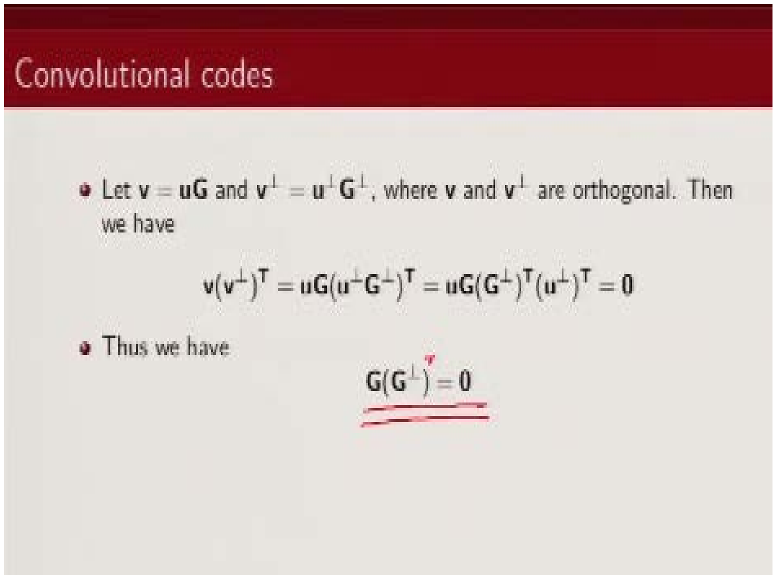

And hence we get this condition that this should be zero, okay

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Convolutional codes

- Let $\mathbf{v} = \mathbf{uG}$ and $\mathbf{v}^{\perp} = \mathbf{u}^{\perp}\mathbf{G}^{\perp}$, where $\mathbf{v}$ and $\mathbf{v}^{\perp}$ are orthogonal. Then we have

$$\mathbf{v}(\mathbf{v}^{\perp})^{\mathsf{T}} = \mathbf{uG}(\mathbf{u}^{\perp}\mathbf{G}^{\perp})^{\mathsf{T}} = \mathbf{uG}(\mathbf{G}^{\perp})^{\mathsf{T}}(\mathbf{u}^{\perp})^{\mathsf{T}} = \mathbf{0}$$

- Thus we have

$$\mathbf{G}(\mathbf{G}^{\perp}) = \mathbf{0}$$

- The convolutional dual code $C^{\perp}$ to a convolutional code C which is encoded by the rate $R = k/n$ generator matrix $G(D)$ is the set of all codewords encoded by any rate $R = (n-k)/n$ generator matrix $G_{\perp}(D)$ such that

$$\mathbf{G}(\mathbf{D})\mathbf{G}_{\perp}^{\mathbf{T}}(\mathbf{D}) = \mathbf{0}$$

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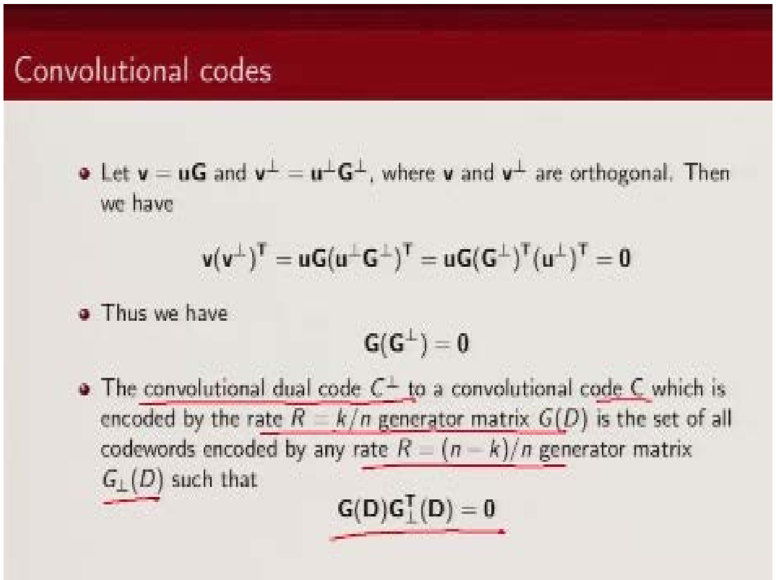

So a convolutional dual to a convolutional code C which is encoded by a rate k/n generator matrix G(D) is set of all code words encoded by rate n-k with generator matrix this such that G(D) and the generator matrix of the dual transpose should be zero, so dual of the code is defined by this.

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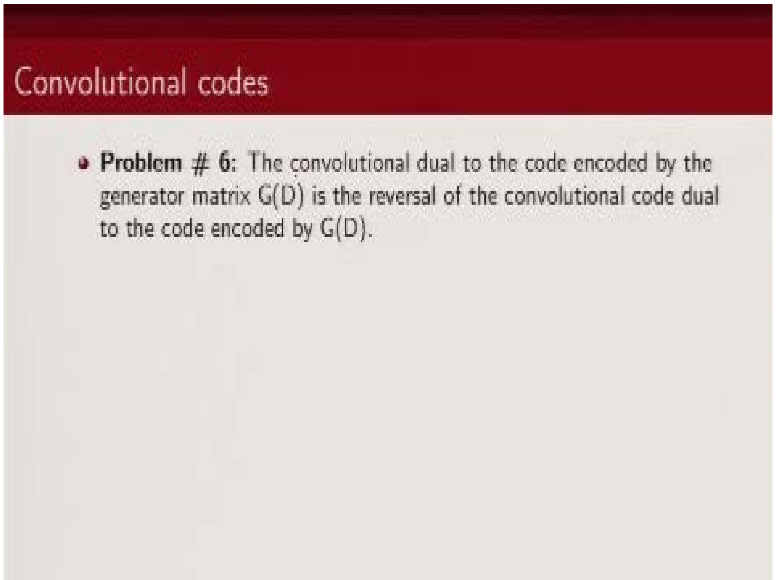

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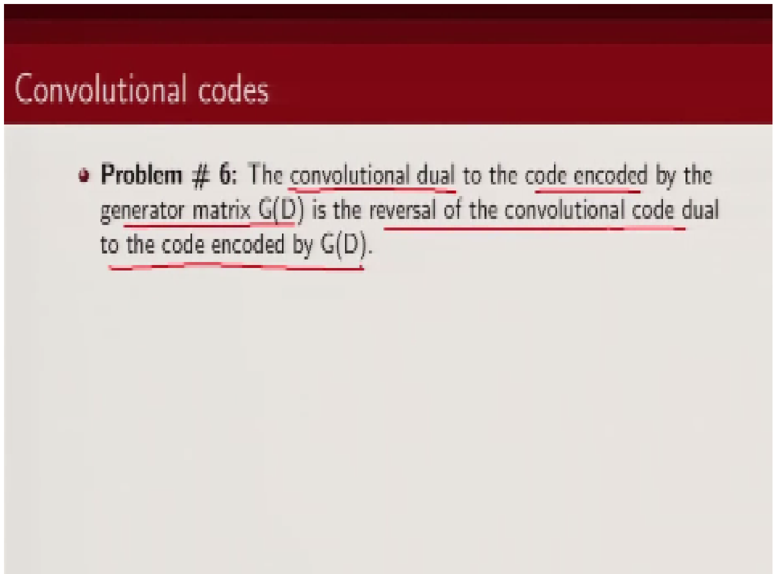

Next, so here we will show that the convolutional dual to the code encoded by the generator matrix G(D) is nothing but reversal of the convolutional code dual to the code generated by G(D).

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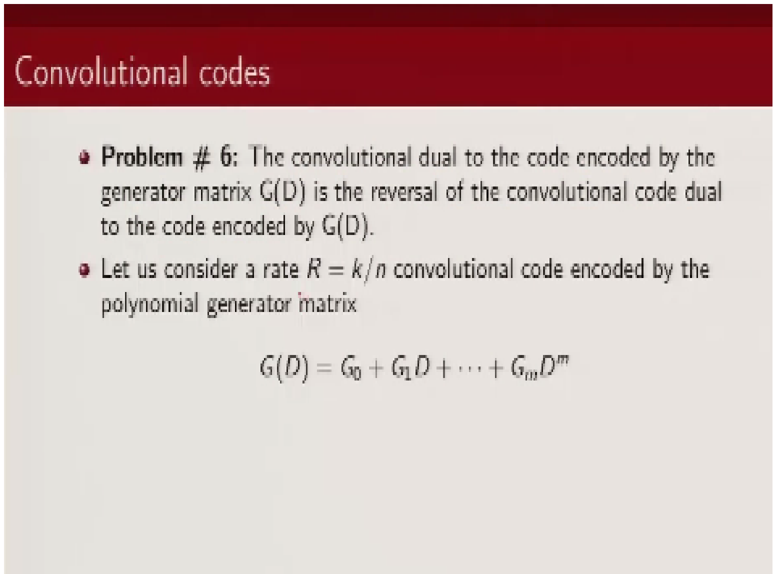

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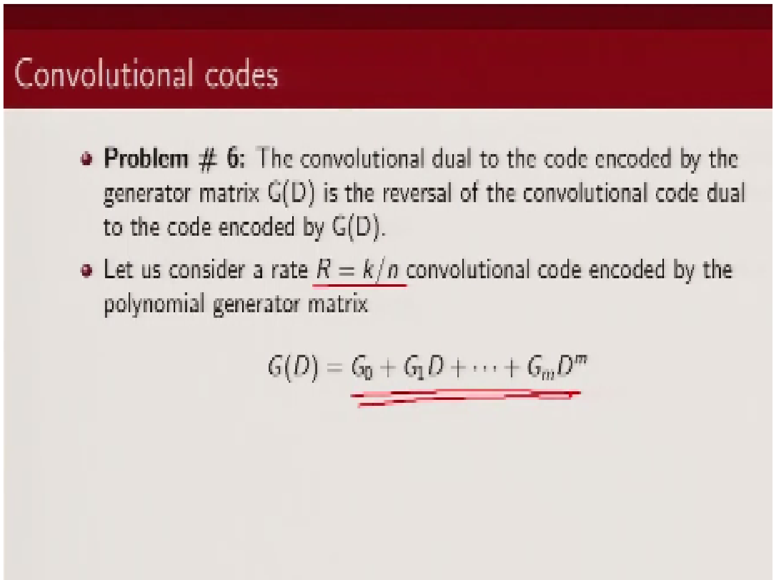

So let us consider a rate k/n convolutional code which is encoded by a polynomial generator. We have already showed that we can find an equivalent polynomial delay free representation of any convolutional code right? So let us see this is our G(D) for our original code C.

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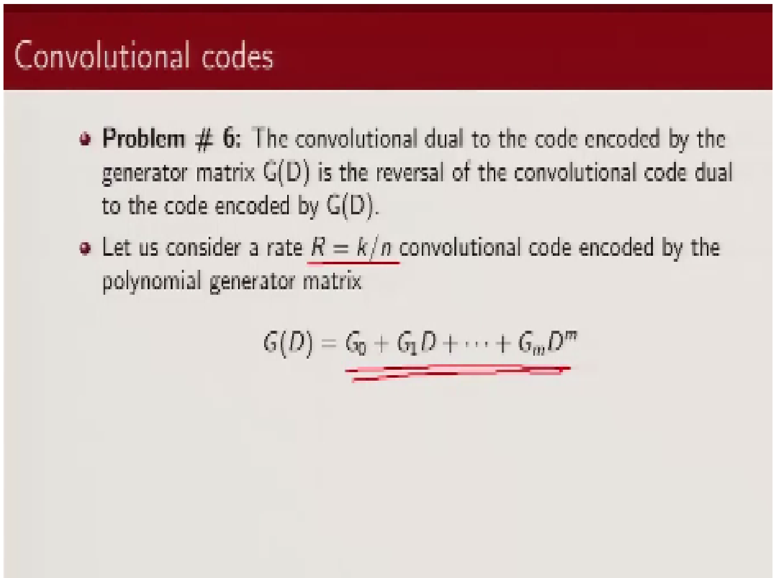

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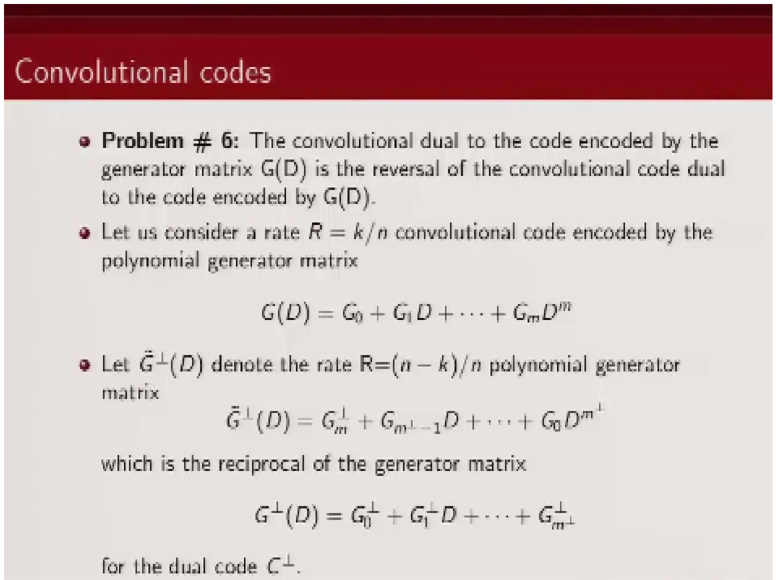

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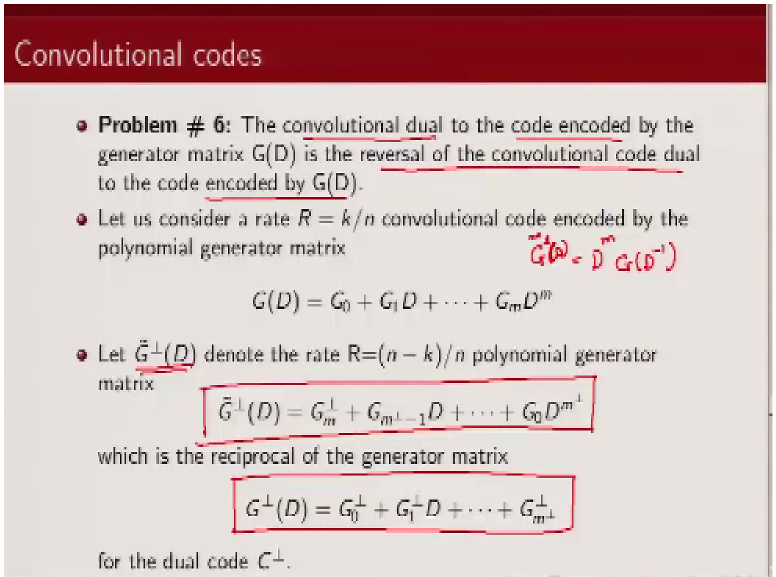

Now we define this as the reciprocal of the generator matrix now how do we come up with the reciprocal, so we do $G(D^{-1})$ and we multiply by the maximum degree which is in this case m, so this is how we, we get the reciprocal, this how we get the reciprocal, okay. Now what I have shown you here is the reciprocal of the generator matrix for the dual, this is a generator matrix of the dual code and this is the reciprocal of the generator metric of the dual code, now note we have to show that the convolutional dual to the code encoded by G(D) is nothing but reversal of the convolutional code dual coded by G(D). So if that is the case then G(D) dot product of G(D) with this should be zero.

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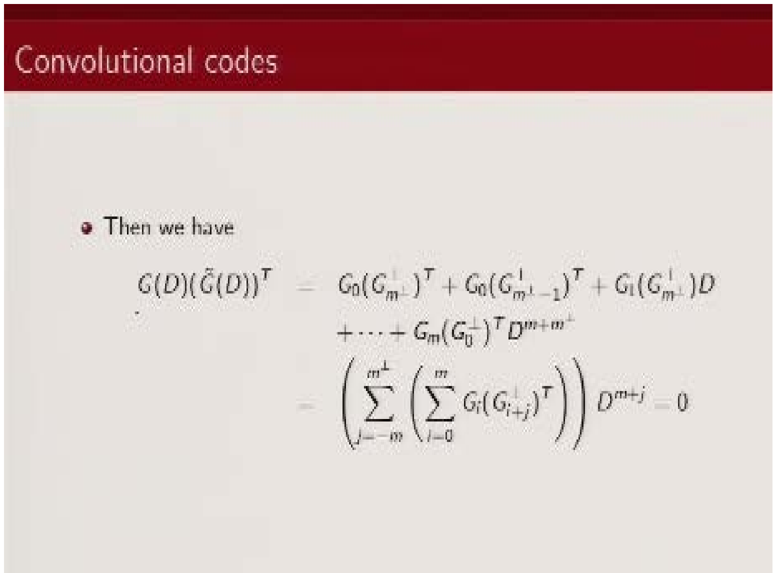

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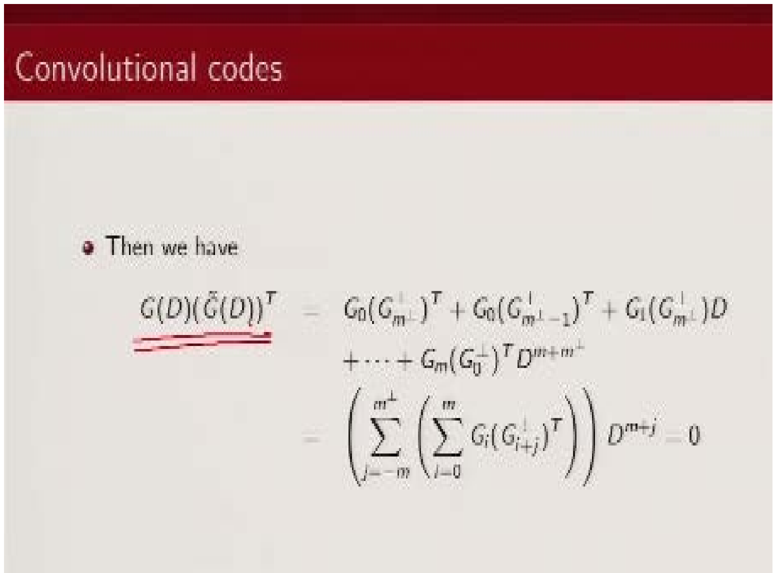

So let us try to find G(D) and transpose of this reversal.

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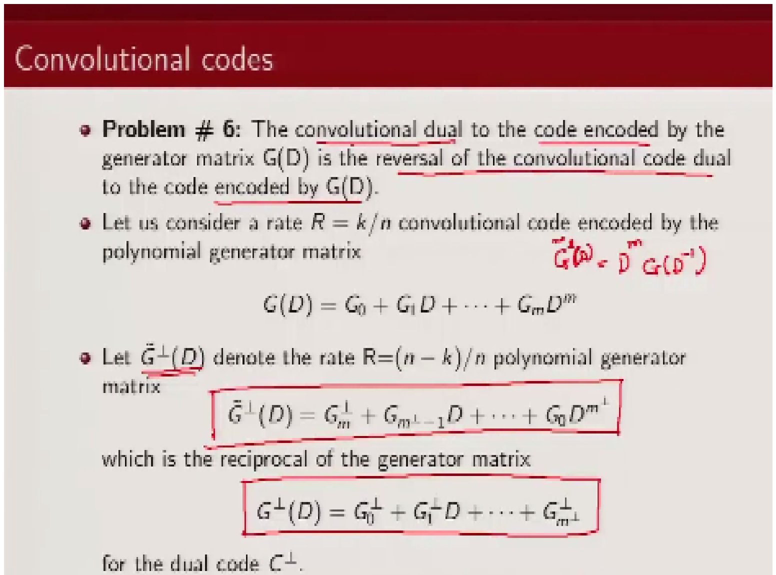

Of the dual matrix, see again pay attention to the question, basically what we are saying is the dual of the code encoded by the generator matrix is given by the reversal of the convolutional code dual so reversal of the convolutional code dual is this right? So if the dual of the original code is given by reversal of the convolutional code then what is the property the this generator matrix should satisfy, this and this transpose should be zero, so we have to show that G(D) and this generator matrix transpose is zero.

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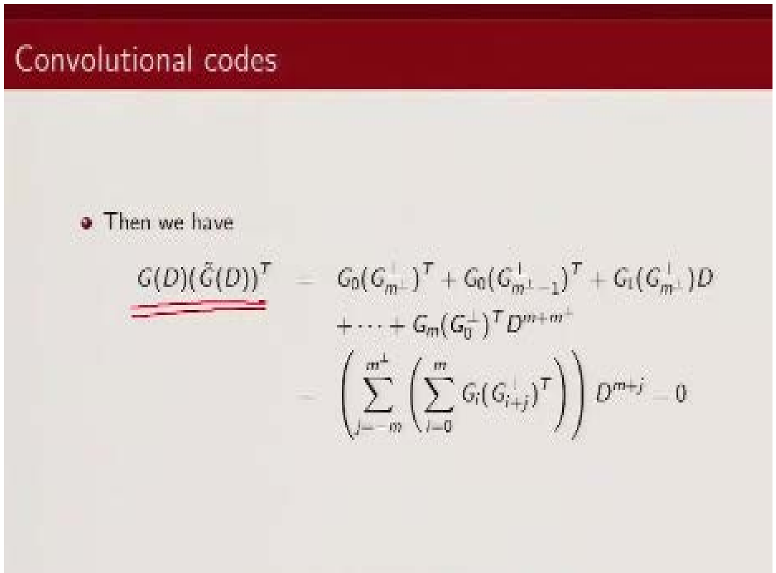

So we do this so this can be written as this term, okay.

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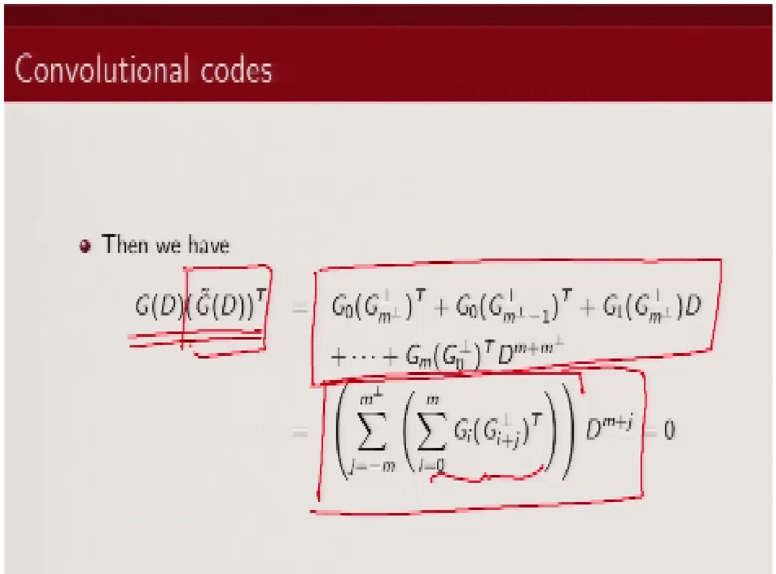

Now further we can write this as double Σ and what are, what is this? This is the dual of generator matrix is the dual of this so this, this G(D) G transpose should be zero, right? So this whole Σ would be also zero, so what we have shown is then the dual of the dual of this code which has generator matrix G.

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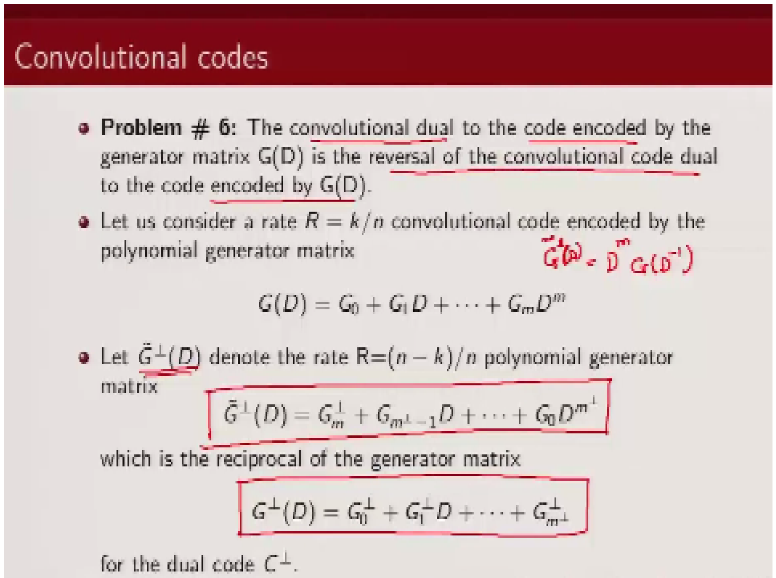

Is nothing but reversal of the convolutional code dual because this generator matrix is nothing but it is the reversal of the generator matrix of the dual code of c okay?

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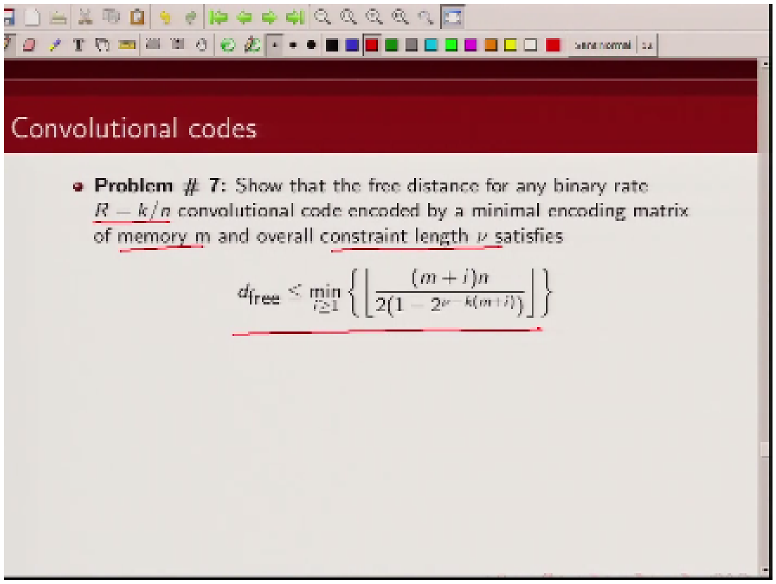

Next we are going to show a bound on free distance of convolutional code, so show that if you have a rate key by n convolutional code whose encoding matrix as memory m and over all constraint length v. Then free distances upper bounded by this.

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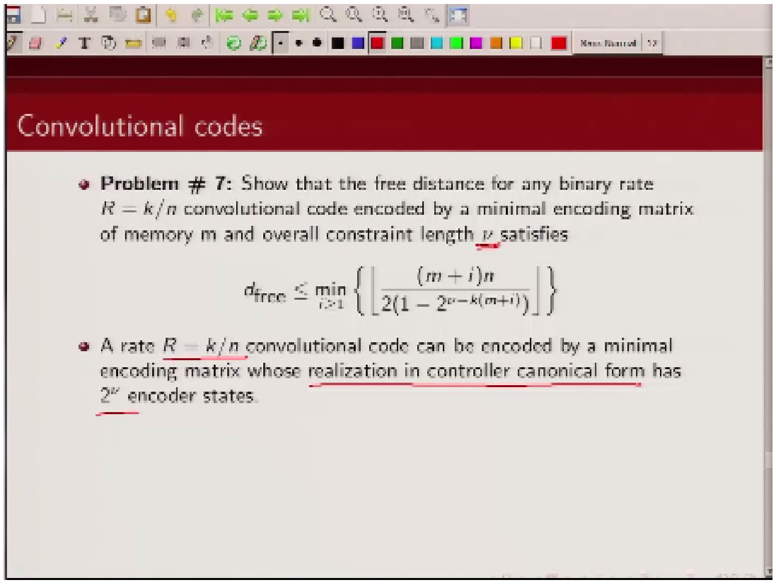

So if you have a rate k/n code we can realize it using controller canonical form using 2^{v} encoder state because this is over all constraint length is v.

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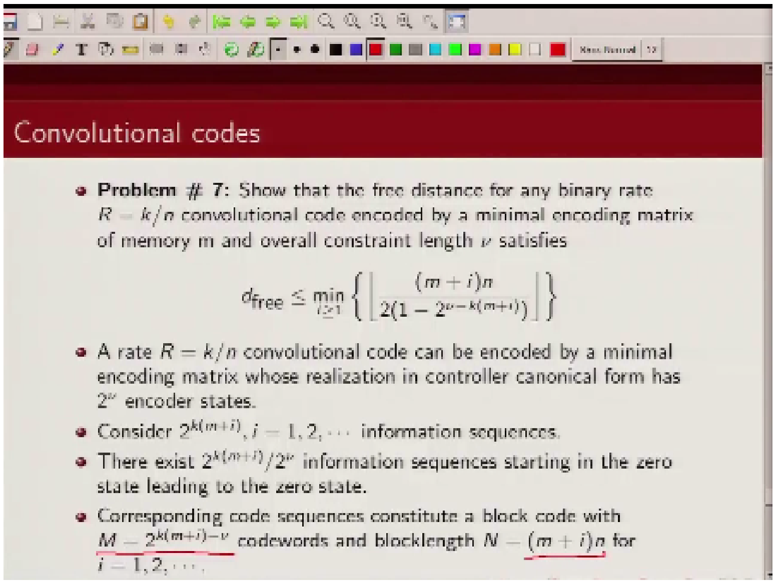

Now if we consider $2^{k(m+i)}$ information sequences. Then there exist $2^{k(m+i)}/ 2^{\nu}$ information sequence starting at zero state leading to all zero state. So if we count number of such code words then number of such code words M is given by $2^{k(m+i)-\nu}$ and what is the length of this code word, length of this code word would be (m+i) n because this is our rate k/n code right? Next we are going to use

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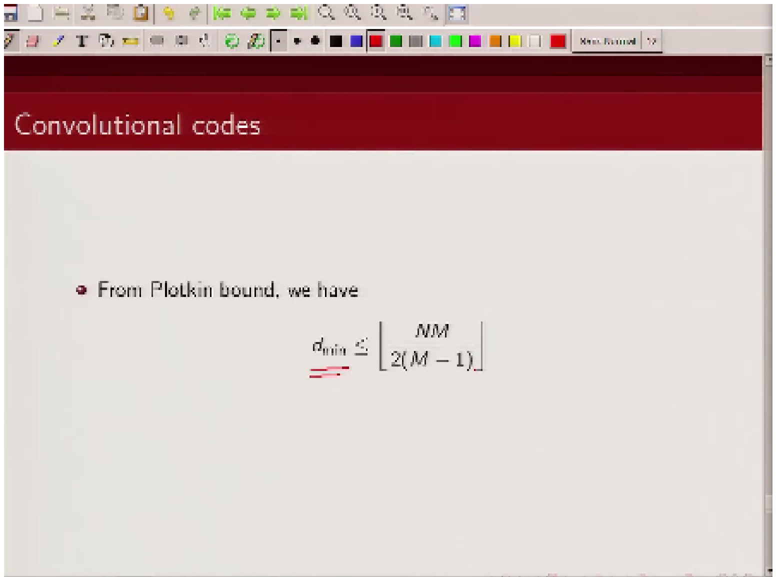

Plotkin's bound, now Plotkin bound says it upper bounds the minimum distance as flower of this is code word length, number of code words 2 into number of code words -1, so in this example

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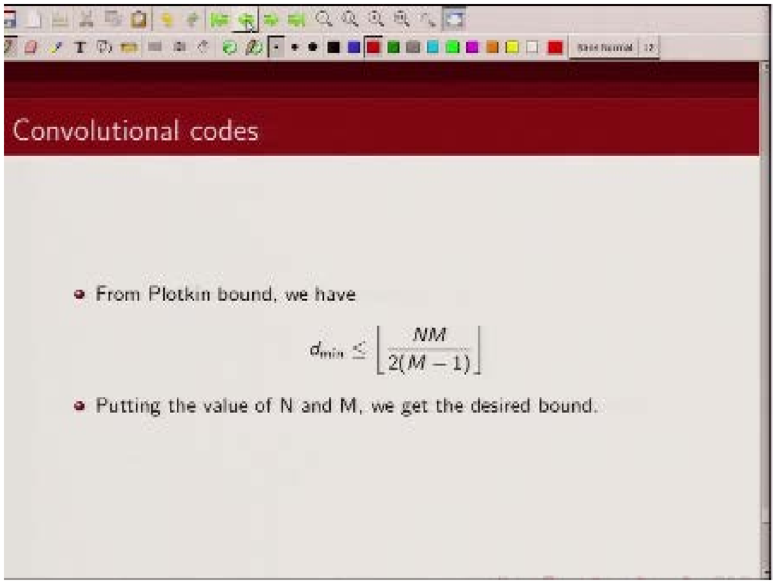

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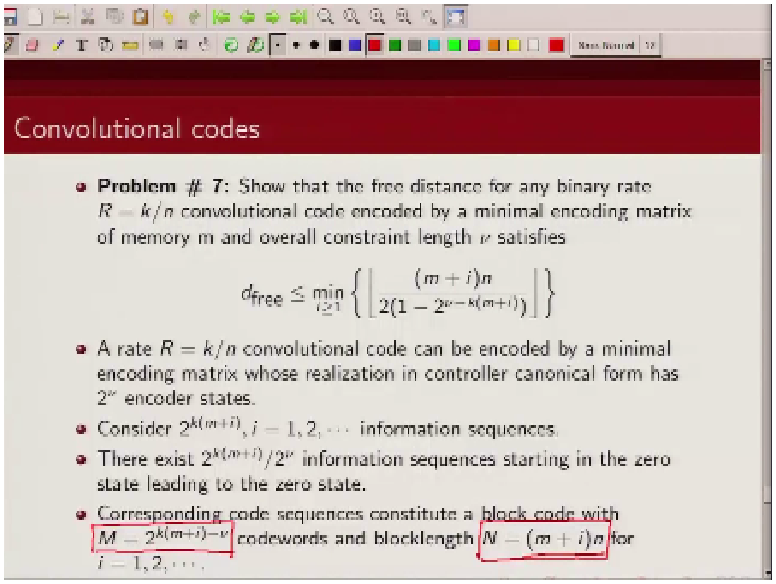

What is our N, our N is this, okay. This is our N and what is our M, our M is this. So if you plug in this value of M and N in our Plotkin bounds.

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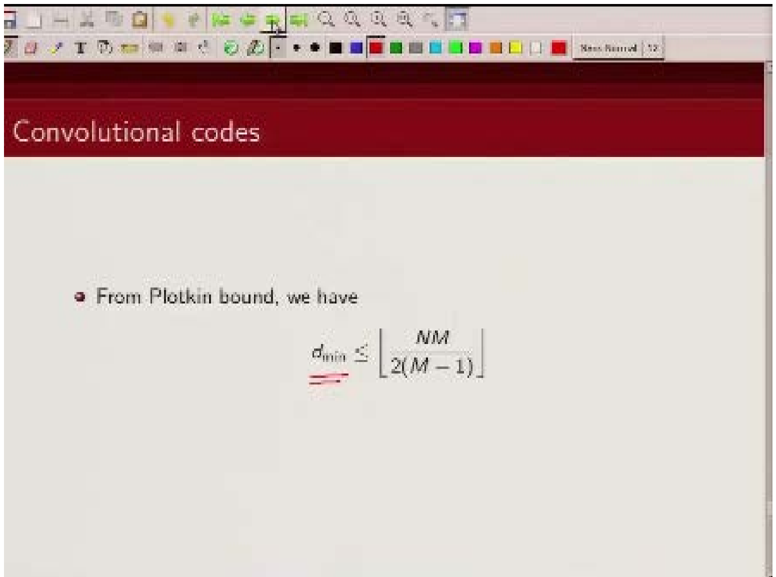

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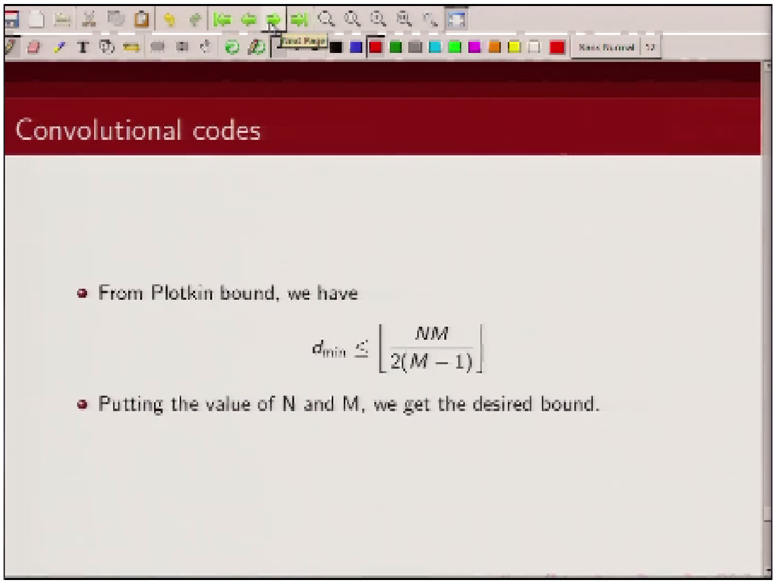

What we will get is our desired bound.

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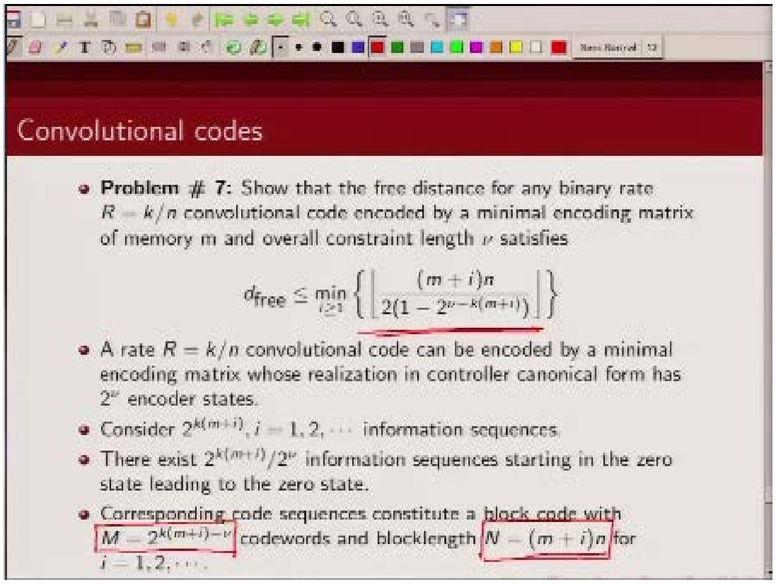

Which is this result, okay?

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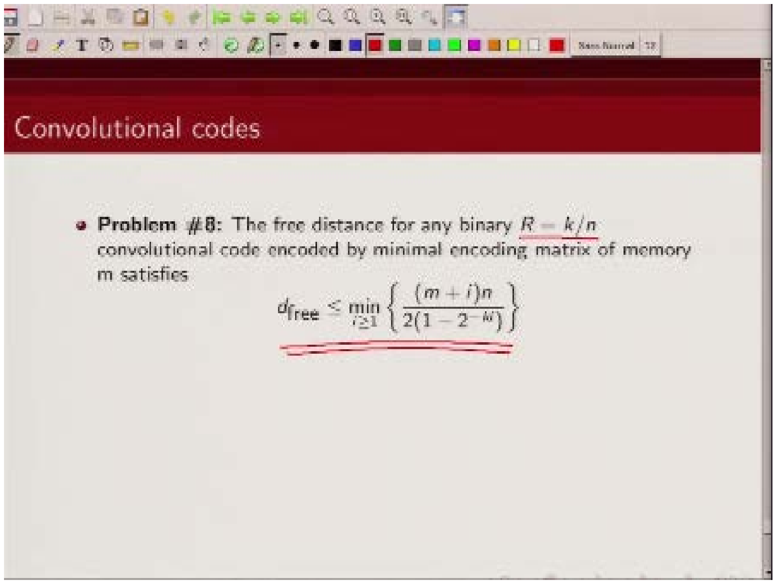

So the next problem that we will solve is to show that free distance of a rate k/n convolutional code is upper bounded by this. Now this result can be obtained from

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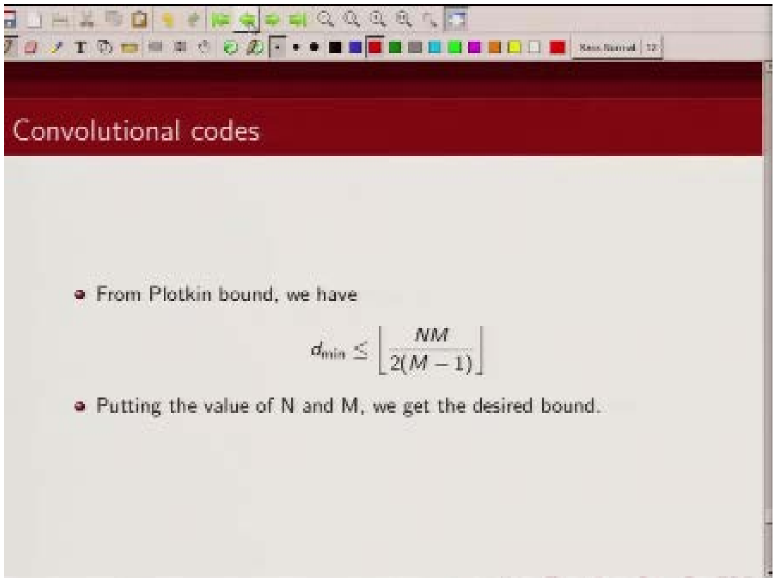

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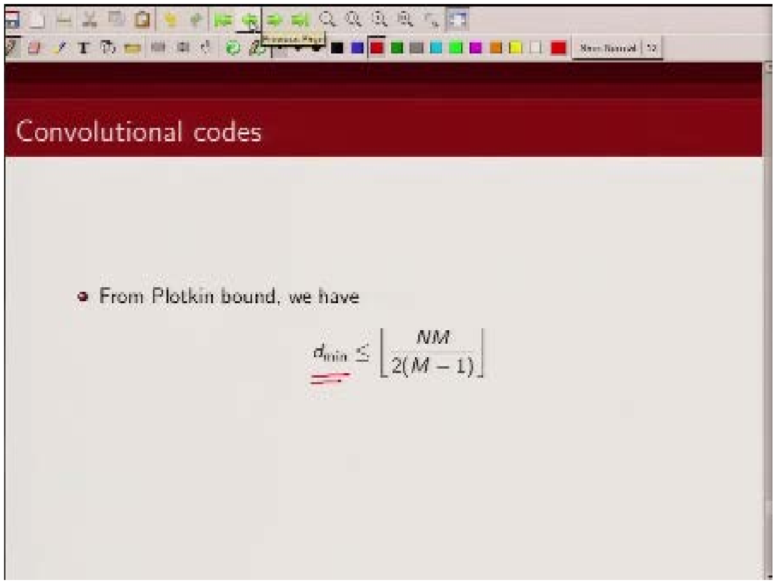

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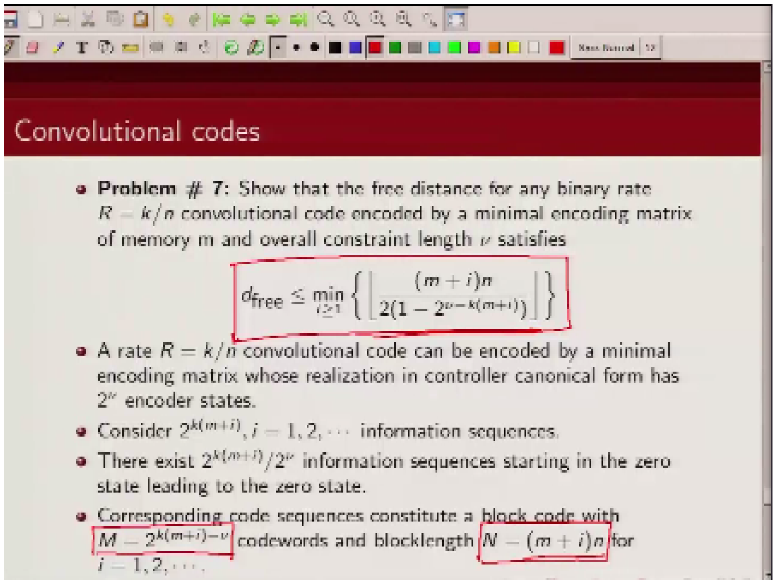

The bound that we have derived earlier, this bound okay?

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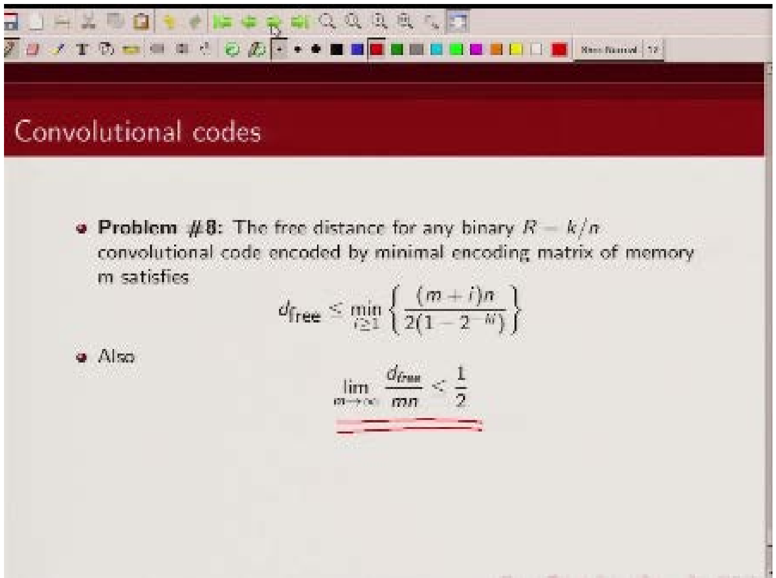

And also we will show that this relation holds.

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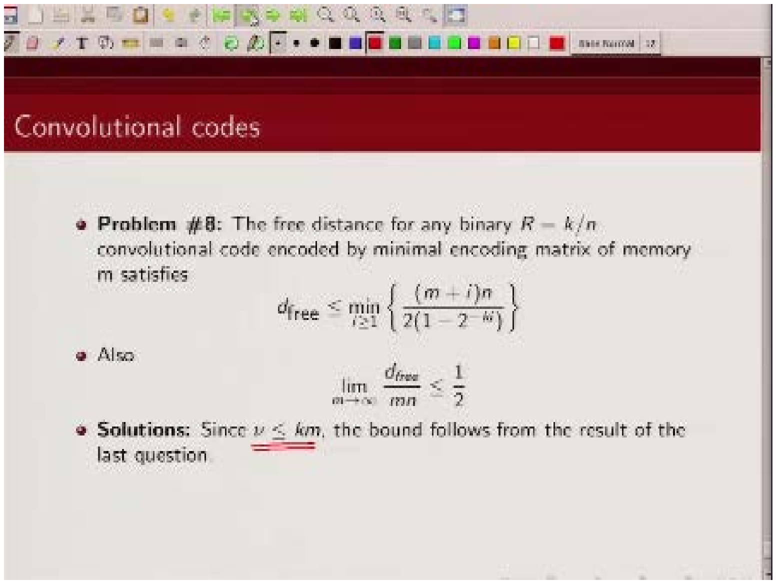

So since overall constraint length is less than k times m, then if you go back to this expression.

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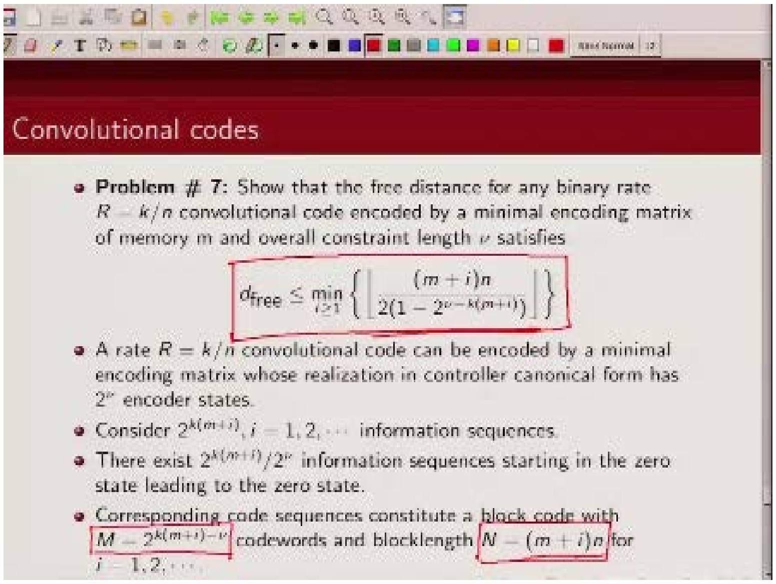

v-km is actually less than zero so we can then upper bound this by just to restore –ki. So if we do that we get this expression okay?

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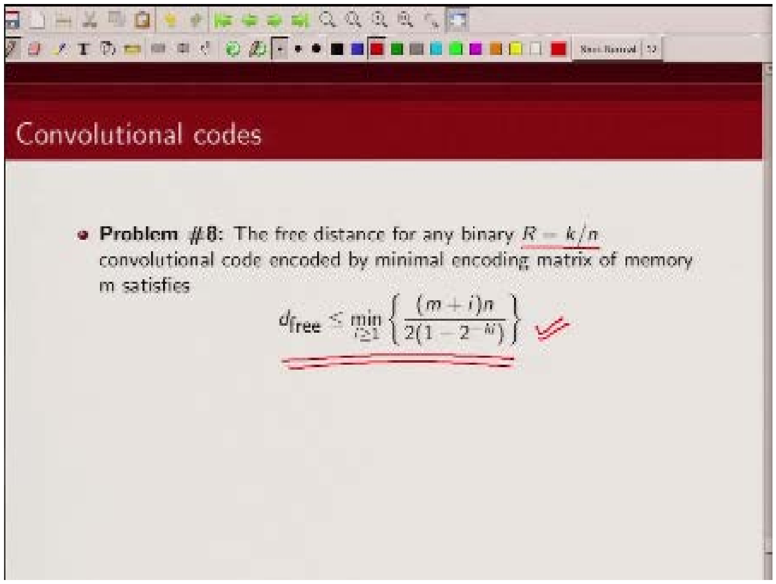

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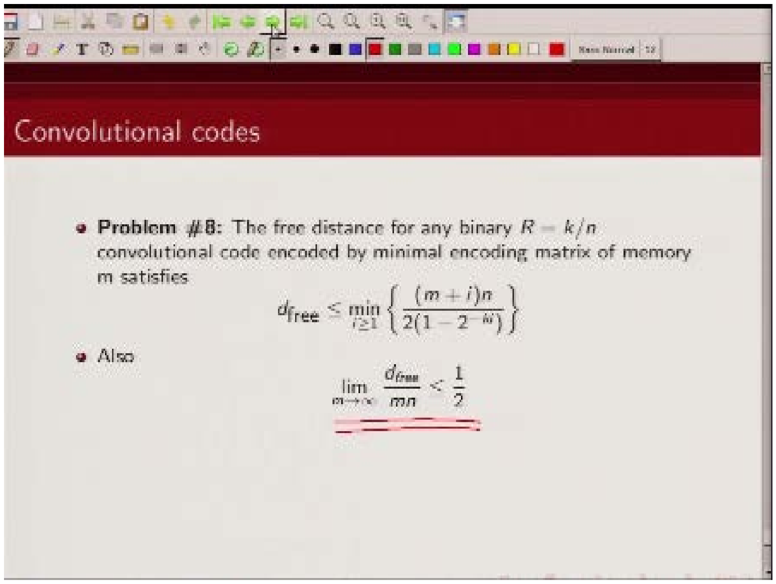

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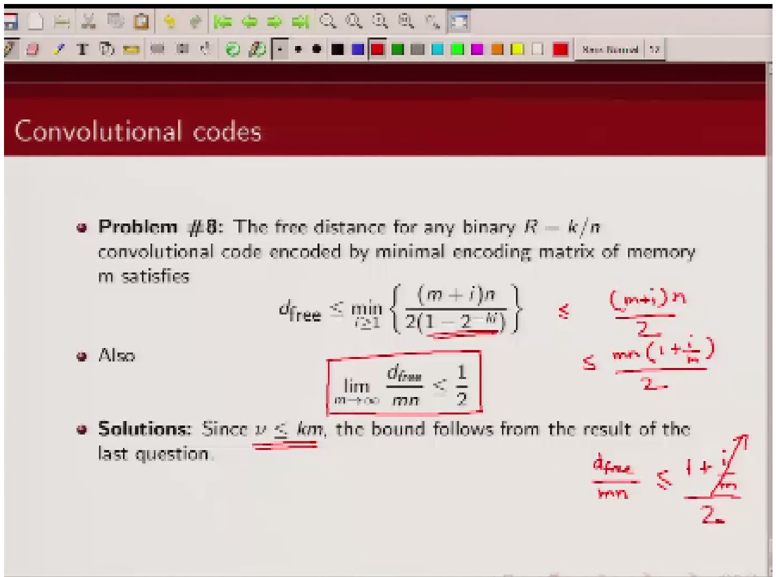

Now how do we get this expression, so we know that this term is less than 1 and so this we can just upper bound by (m+i)n/2 and if I take m and n out this will be (1+i/m) /2. So dfree by m of n will then be so if I do further dfree of / mn would be upper bounded by 1+i/m /2 and if we let m go to infinity this will go to 0 so that will be upper bounded by half. So that is the proof.

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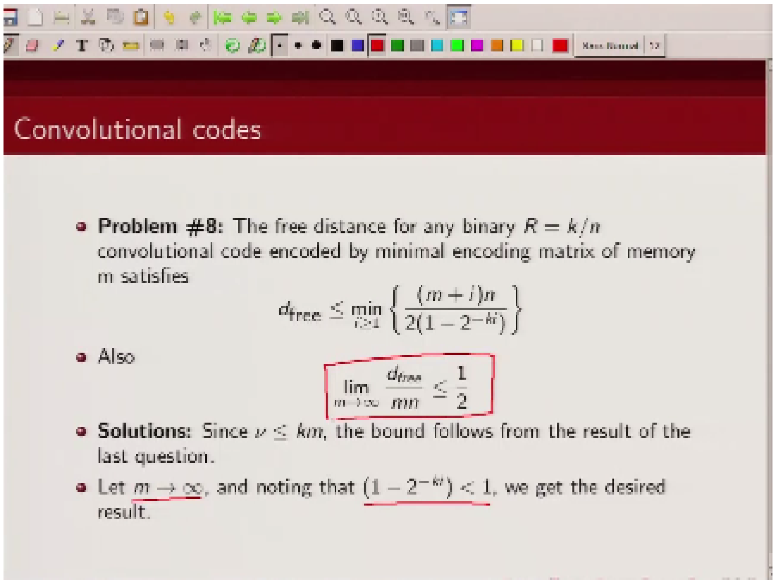

So you let m go to infinity and since this is less than 1 what you will get is this, thank you.

<u>Acknowledgement</u>

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