Estimation for Wireless Communications-MIMO/OFDM Cellular and Sensor Networks.
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Lecture -09.
Wireless Fading Channel Estimation-Mean And Variance Of Pilot/Training Based Maximum Likelihood (ML) Estimate.

Hello, welcome to another module in this massive open online course on estimation for wireless communications. In the previous module, we looked at estimation of the channel coefficient for a wireless communication system, all right. And we had considered the maximum likelihood estimation of the channel coefficient in a wireless communication system and we had been able to demonstrate that the estimate of the channel coefficient H hat where H represent the the channel coefficient, is given as H hat.

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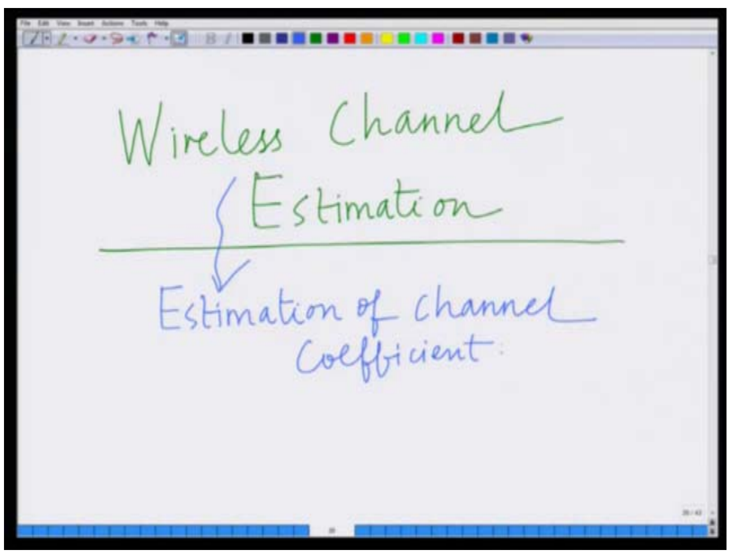

So, we are looking at wireless so, we are looking at wireless channel estimation, in fact we have also said, this is a key process or a key aspect in wireless communication, this is known as wireless channel estimation, that is estimation of the channel coefficient.

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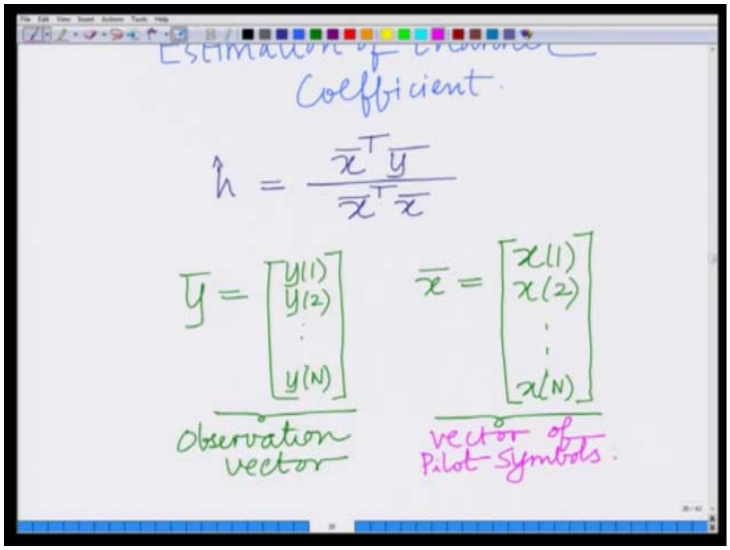

And we have shown that the maximum likelihood estimate of the channel coefficient is given as H hat equals X bar transpose Y bar divided by X bar transpose X bar where the vectors Y Y bar equals the vector, the observations better, that is the vector of observations Y1, Y2 so on up to YN. So this is your this is your observation vector and X bar equals X1, X2 up to XN and this is your vector of pilot symbols. Remember we have said X1, X2… XN are the pilot symbols that are transmitted for channel estimation. So, X bar is the vector of pilot symbols, also we can call this as the pilot vector.

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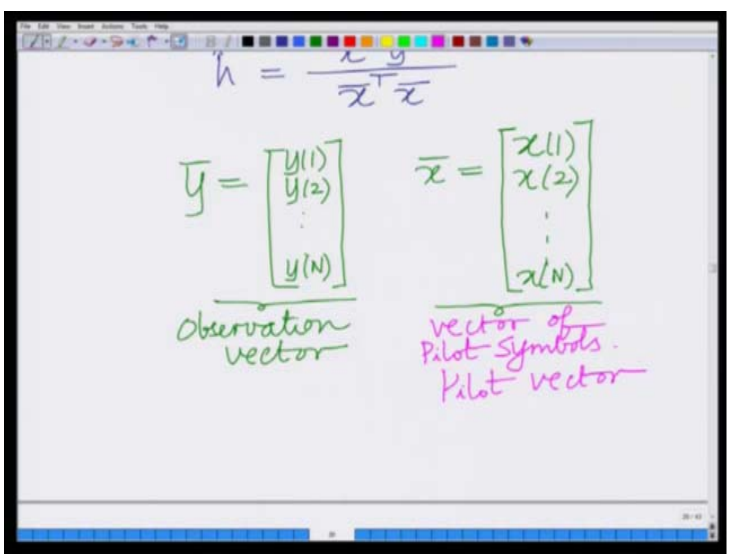

So, we are saying that the channel estimate H hat is given as X bar transpose Y bar divided by X bar transpose X bar where X bar is the pilot vector and Y bar is the corresponding observation vector, that is the output symbols corresponding to the transmitted pilot symbols X1, X2… Up to XN. And you can see that this can also be written as…

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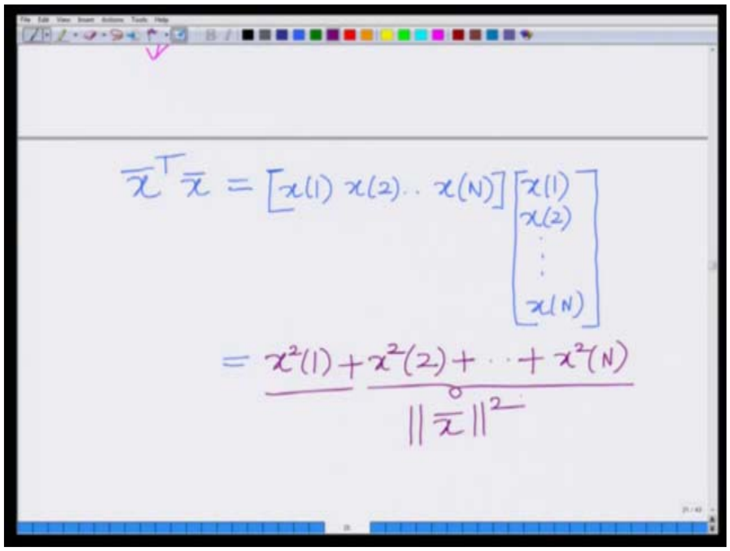

Now, now you can see, I think we have already seen this before that is if I look at X bar transpose X bar, that is nothing but basically your row vector X1, X2 up to XN Times the column vector X1, X2 up to XN which is X bar transpose X bar, this you can see is nothing, is basically summation this you can see is basically X square 1+ X square 2+ so on up to X square of N and this quantity you can see is basically nothing but the norm of the vector.

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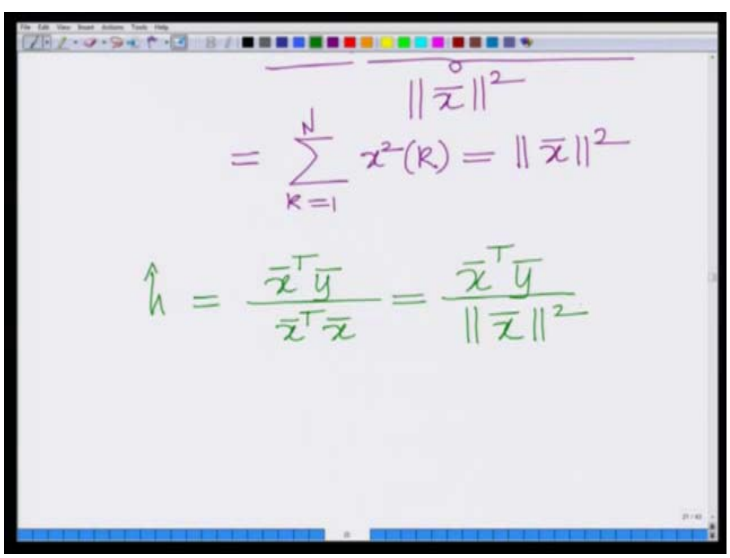

Norm a bar square, where X bar is your pilot vector, this is also basically your summation K equals to 1 to N X square of K and this is basically your norm of X bar square. So, X bar transpose X bar, of course this is the standard result and we have also seen this before that is X bar is a vector, that is a real vector remember, that is what we are considering to begin with. X bar transpose X bar, it is a summation K equals 1 to N X square K which is nothing but the norm square of the vector X bar, therefore I can also write it, the estimate, the maximum likelihood estimate in a compact fashion as X bar transpose Y bar divided by X bar transpose X bar which is also basically your X bar transpose Y bar divided by norm of X bar square. Okay.

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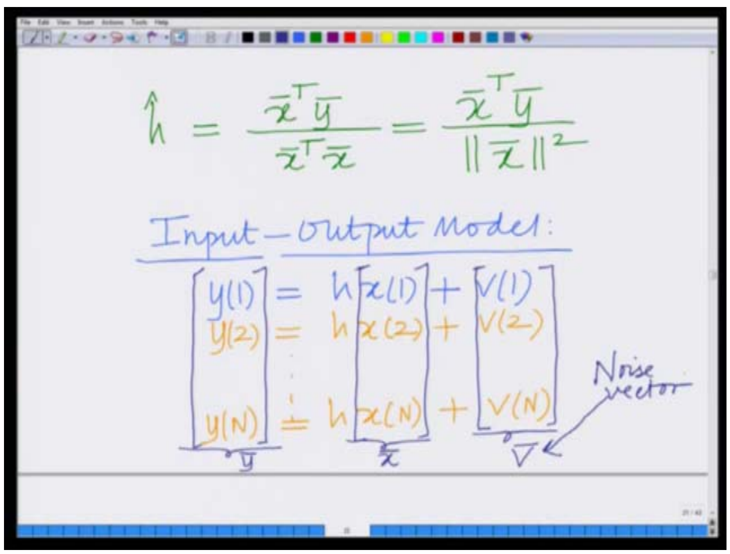

Now, also let us go back to our model, let us go back to our pilot observation, that is the pilot input output model, that is the input output model, our input output model remember, recall is YK equals H times XK + VK where YK is the observation, H is the channel coefficient, XK is the transmitted pilot symbol and VK is the corresponding Gaussian noise samples. Therefore now I can write Y1 equals H times V1 H times X1 + V1, Y2 equals H times X2 + V2, so on YN equals H times XN + VN.

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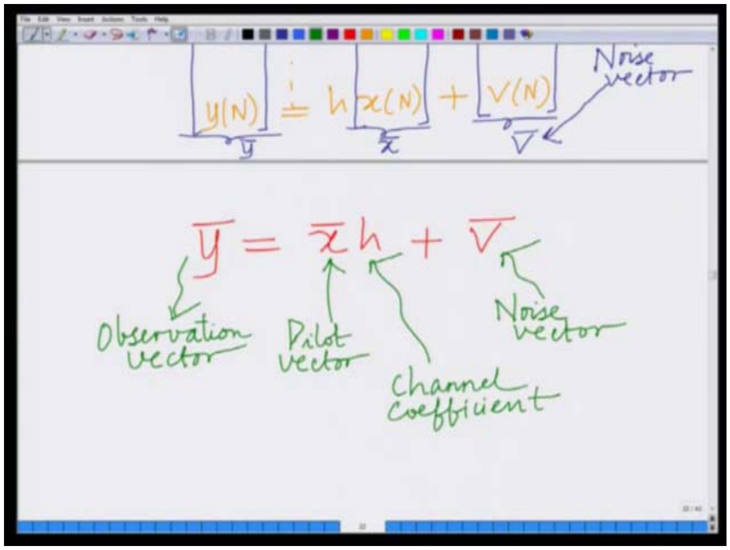

So, we have Y bar, which is basically now vectorising this, to write it in a compact fashion, we have Y bar equals H times X bar class V bar. We have seen that Y bar is the observation vector, X bar is the pilot vector, now we have V bar and this is the noise vector. So, we can write this in a vector fashion that is Y bar equals X bar times H + V bar where of course repeat once again Y bar is the observation vector, X bar is the pilot vector H is the channel coefficient and V bar is the…

So, I have succinctly represented this as in a compact vector, using vector notation I have represented this in a compact fashion as Y bar equals X bar times H + V bar where Y bar is the observation vector, X bar is the pilot vector, H is the unknown channel coefficient and V bar is the noise vector.

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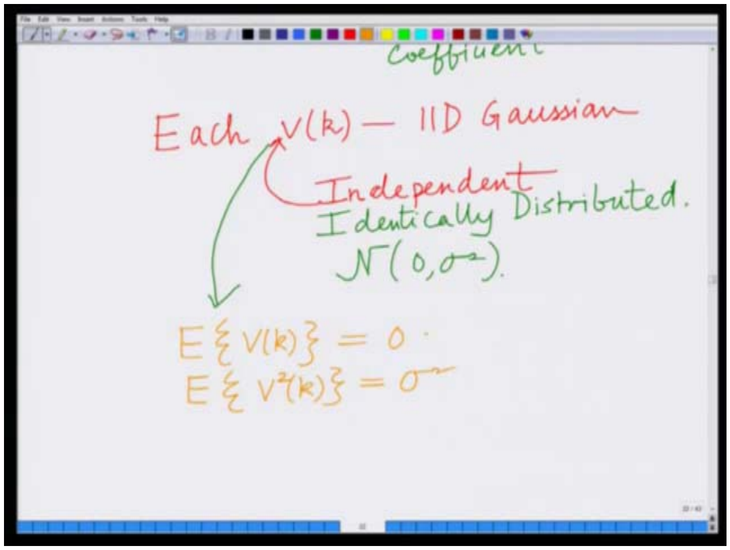

Now, let us look at the properties of the noise vector V bar, so we have already said that each noise VK is Gaussian, further we are going to assume that each VK is IID Gaussian, then this also we have seen before, that is basically independent identical Gaussian.

That is independent identically distributed as Gaussian each with mean 0 and variance Sigma square which implies again to just summarise the expected value of each VK equals 0 expected value of V square K equals Sigma square.

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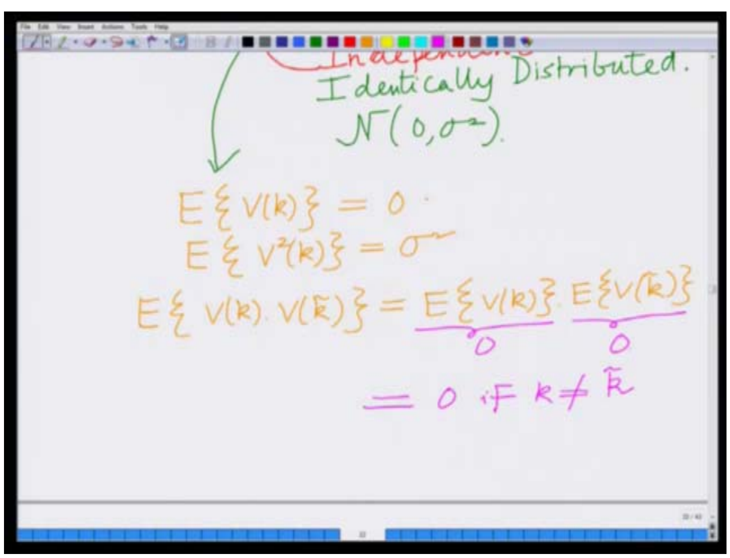

Further, since they are independent, since the reference noise samples are independent, expected value of VK times VK tilde, since they are independent equals expected value of

VK times your expected value of VK tilde, if K nought equals K tilda. And each of these expected values each of these expected value is 0, therefore as a result, the net product is 0 if K nought equal to K tilda. That is the co-relation between 2 different noise samples VK and VK tilde is equal to 0 if K nought equal to K tilde. Since these different noise samples are assumed to be independent.

Naturally, now let us characterise the statistical properties of the noise vector, the 1st property is simple,

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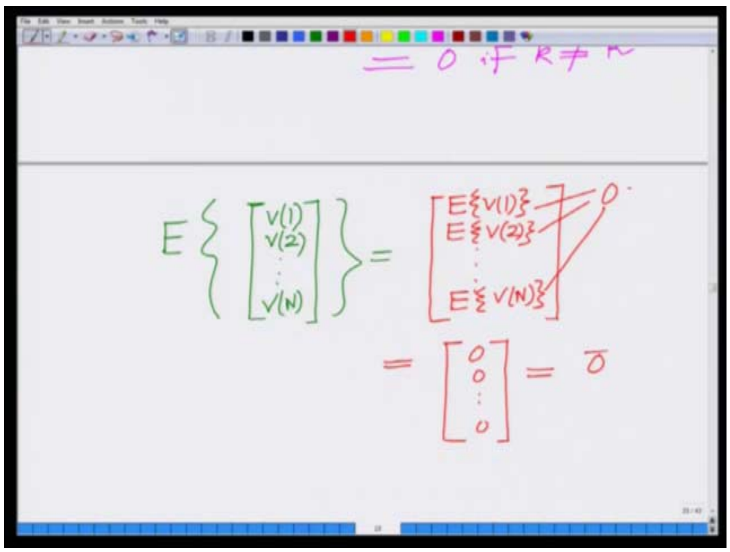

that is if we look at the mean of the noise vector, that is if I look at the mean of the noise vector, that is the mean of V1, V2… Up to VN, now naturally, each component is 0 mean, therefore the mean of the sectors says is 0, that is the expected value of V1, expected value of V2, so on expected value of VN, now each of these is basically equal to, the noise is 0 mean, so basically expected, this is the 0, this is simply the 0 vector.

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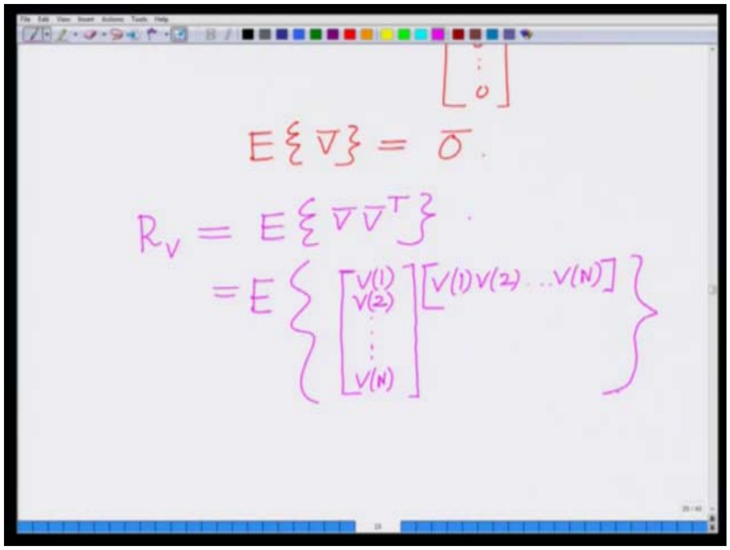

So, we can summarise this as basically the expected value of V bar is simply the… Each noise component is 0 mean, naturally the expected value of the noise vector is the 0 vector. That is simply said as 0. It is understood that it is a vector, since we are talking about a vector, so expected value of the noise vector is 0, now let us look at the variance, of course, since V bar is a vector, we cannot look at the variance of a vector, we have to talk about the covariance of the noise vector and that is defined for a 0 mean vector as follows.

The covariance for our 0 mean noise vector is defined as RV, that is the covariance is expected V bar V bar transpose for a scalar we look at the square of the random variables, now since it is a vector, we are looking at expected value of V bar V bar transpose, which I simplify this is expected value of V1, V2 so on VN. V1, V2… VN times the row matrix V1, V2, VN, the cloumn matrix times the row matrix,

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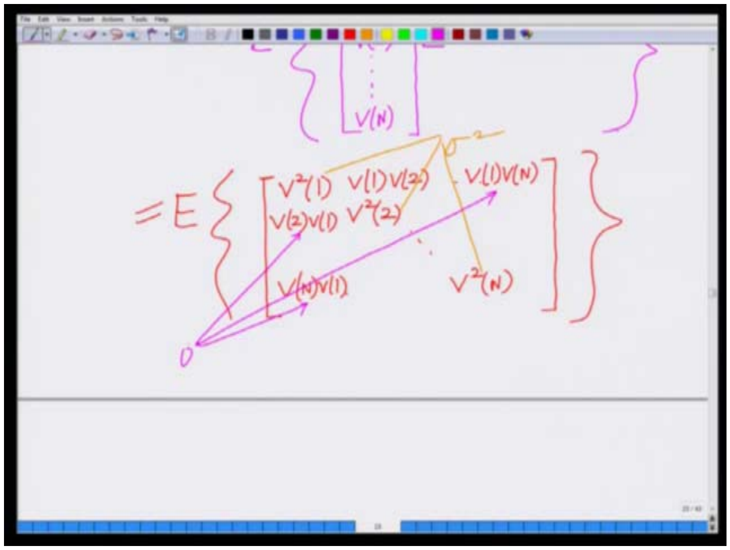

it is sometimes also known as the outer product of the vector V bar V bar transpose which now if you can look at the entries of V bar V bar transpose, if you look on the diagonal, the diagonal, you have the squares of the various elements V1 square, V2 square so on V square N, these off diagonal entries are the cross products V2, V1… V1, V2 so this is for instance V2, V1 V1, V2 so on up to V1, VN, this product here will be VN V1 and so on.

And now if you take the expected value inside, you can see that the expected value of each of the diagonal elements, expected value of V square 1, V square 2, V square N so on is Sigma square, while the expected value of diagonal elements is the different noise elements are independent, the expected values of diagonal elements, the expected, let me draw them in different colors, the expected value of off-diagonal elements, this you can see is clearly is clearly 0 because of different noise samples.

So, these are independent, the only expectation that are going to survive are basically the expected values of the diagonal terms which are basically the squares of the different noise samples,

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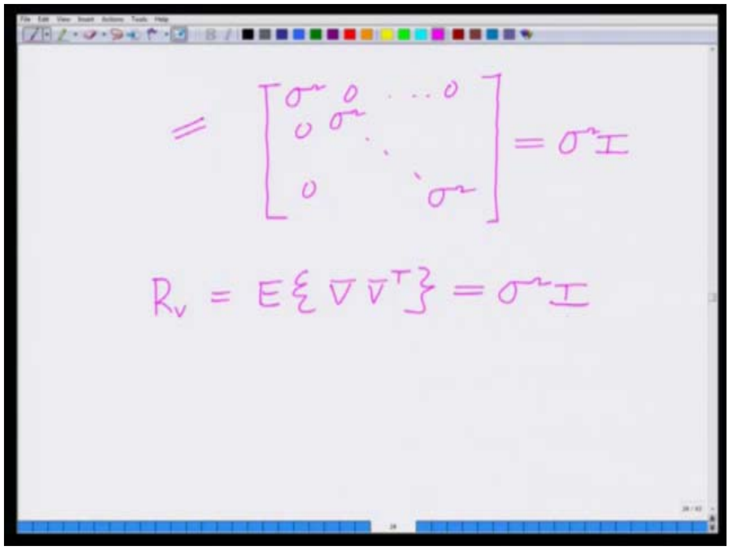

therefore, the covariance matrix is basically the matrix N cross M matrix with Sigma square on the diagonal, each diagonal elements equal to Sigma square, the rest of the entry is 0. So, this is equal to basically your Sigma square I.

So, you call variance matrix RV is equal to expected V bar V bar transpose which is equal to Sigma square times identity. So, we have the noise vector V bar which we have defined, the noise vector V bar is 0 mean, so the expected V bar is equal to 0, the 0 vector and the covariance matrix, expected V bar V bar transpose is equal to Sigma square times the identity, the N cross M dimensions identity matrix. So, Sigma square times the N cross M dimensional identity matrix. And that is also clear because the noise variance is of size N.

But we can also write this explicitly, so this is identity matrix of size N cross M. Now, let us lower the properties of the channel estimate.

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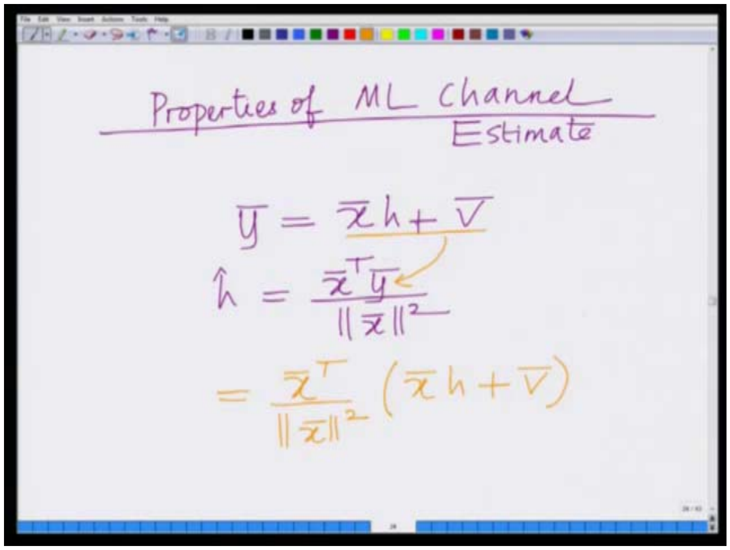

So, let us explore the properties of the maximum likelihood channel estimate, so we will just start exploring, properties of your ML. Properties of the maximum likelihood channel estimate and we know that Y bar equals X bar H + V bar. And we know that H hat equals X bar transpose Y bar divided by Norm X bar square, that is the expression that we have seen for the maximum likelihood estimate. Now I am going to substitute the expression for Y bar over here. So, I will have X bar transpose divided by Norm X bar square times X bar H + V bar, which is equal to again now,

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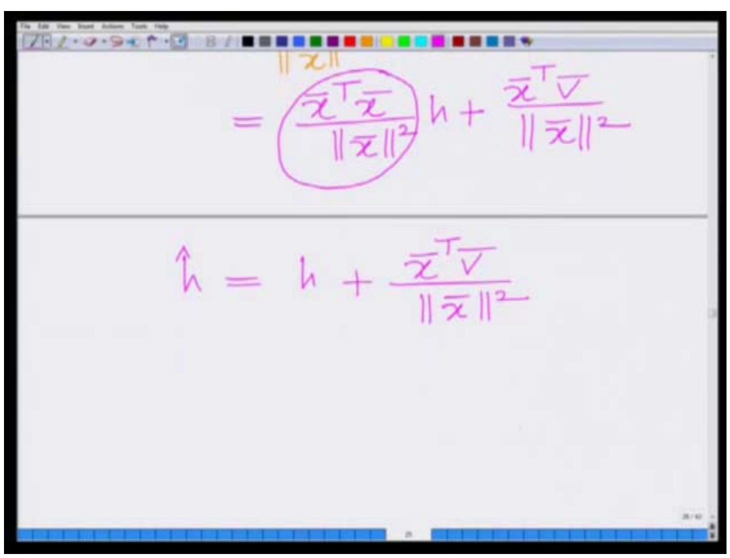

simplifying this, X bar transpose X bar divided by Norm X bar square H + X bar transpose V bar divided by Norm X bar square.

Now if you look at this**,** this is X bar transpose X bar which is Norm X square divided by Norm X bar square. So, this is one and so therefore net I have H hat equals H + X bar transpose V bar divided by Norm X bar square. Okay. So, what we have, we have demonstrated that simplifying this expression for the maximum likelihood channel estimate, we have shown that H hat, the channel coefficient estimate of the channel coefficient H hat equals H + X bar transpose V bar divided by Norm X bar square where X bar is the pilot vector and V bar is the noise vector.

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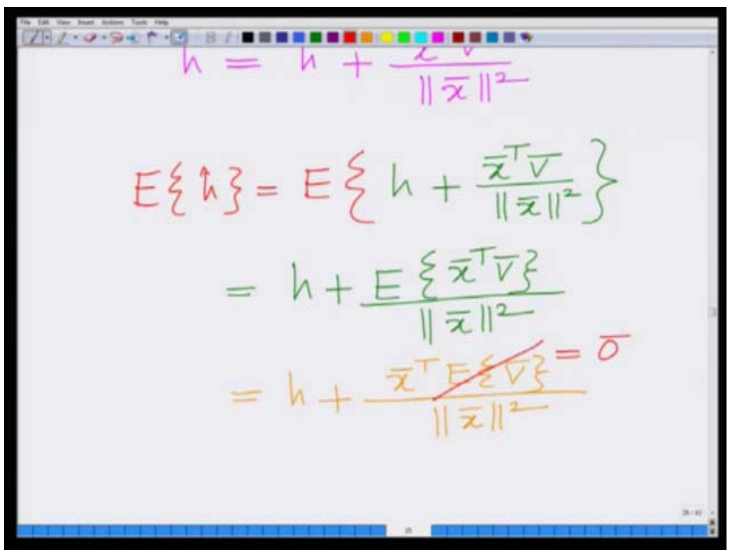

Now, it is easy to see that if I look at the mean of the noise, mean of the estimate, that is expected value of H hat, that is nothing but your expected value of H hat + X bar transpose V bar divided by Norm X bar square, I am sorry, this is H, not H hat. Now H is a constant, so therefore this is H + expected value of of course X bar transpose V bar divided by Norm X bar square which is equal to now H + X bar transpose expected value of V bar divided by Norm X bar square but now we have already said that the noise vector is 0 mean, so this quantity is the 0 vector, so X bar transpose 0 bar is nothing but 0.

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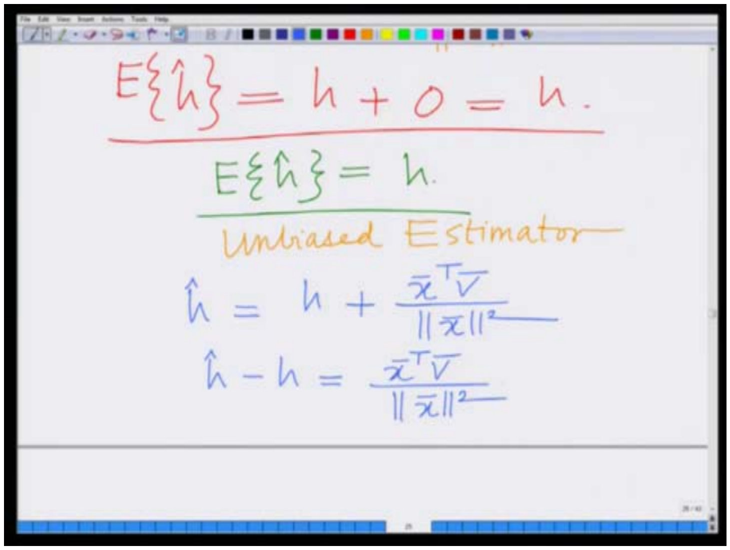

So, I have net, very interesting property, that is H hat expected value of H hat equals H +0 which is equal to H. So, summarising what we have is something that we have seen already in the context of your sensor network where expected value of the estimate of the parameter is the parameter itself that is expected value of H hat is equal to H and therefore such an estimate is known as an unbiased estimate. That is expected value of H hat is equal to H, therefore such an estimate is known as the unbiased estimate. So, this is basically an unbiased estimator.

This is basically an unbiased estimate or this maximum likelihood estimate of the channel coefficient is an unbiased estimator. Now, let us look at the variance, now let us look at the variance of the channel estimator. So, let us now look at the variance of the channel estimate, now we know that H hat equals H + X bar transpose V bar divided by Norm X bar square which means H hat - H or H - H look at H hat - H if we look at H, H hat - H, H hat - H, that is equal to X bar transpose V bar divided by Norm X bar square.

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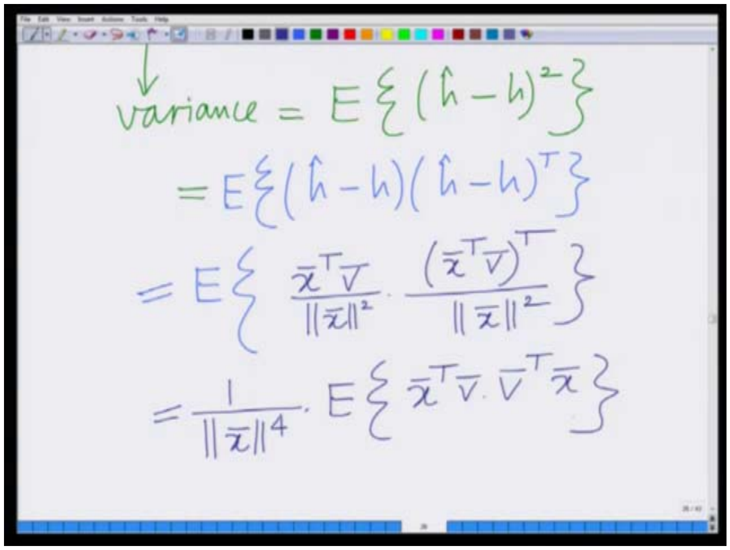

Now, the variance is basically the expected value of, the variance of the estimate is expected value of H hat - H whole square which is basically, I can write this, now this is a scalar quantity H hat - H, so this is equal to its transpose, so H hat - H square, I can write it as H hat - H times its transpose. So, H hat - H, so what we are saying is this quantity, channel coefficient is basically a scalar quantity. Scalar quantity means it is basically a number. So, therefore, expected value of H hat - H all square is basically expected value of H hat - H Times itself which is expected value of H hat - H times H hat - H transpose.

Because the transpose of a scalar quantity or the transpose of a number is basically the number itself. But this will help us, this trick will help us greatly simplify the variance of the estimate as follows. And now therefore we can write this as substituting for H hat - H from above, you can see H hat - expected value, H hat - H is X bar transpose V bar divided by Norm X bar square times X bar transpose V bar transpose divided by Norm X bar square. Which is equal to, now bringing this Norm X bar square outside, I have Norm X bar 4 expected value of X bar transpose V bar times X bar transpose V bar transpose is V bar transpose X bar which is now equal to,

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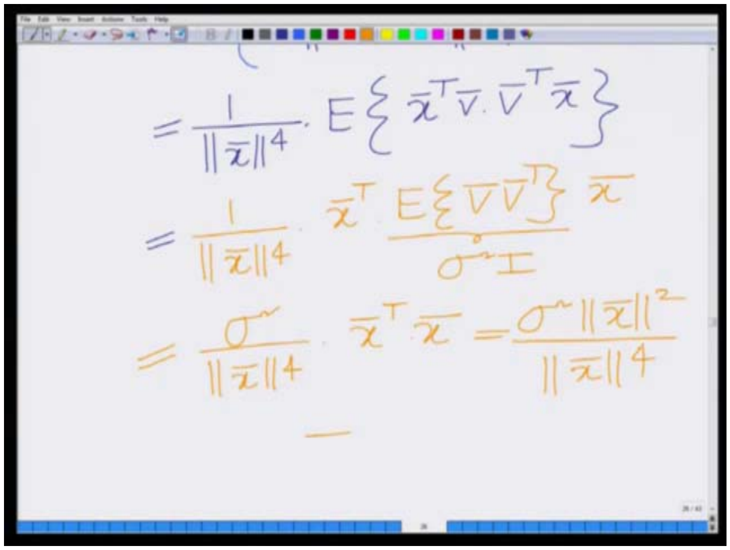

if I move the expected operations, the expectation operation inside, that is 1 over Norm X bar to the power of 4 times X bar transpose expected value of V bar transpose X bar…

Now, this is interesting because this is your Sigma square I which we have already shown is the noise covariance, therefore this is 1 over Norm X bar power 4, X bar transpose Sigma square times identity, the Sigma square will come out, so I can write Sigma square over here, X bar transpose identity X bar is simply X bar transpose X bar which is Sigma square norm X bar square divided by Norm X bar power 4.

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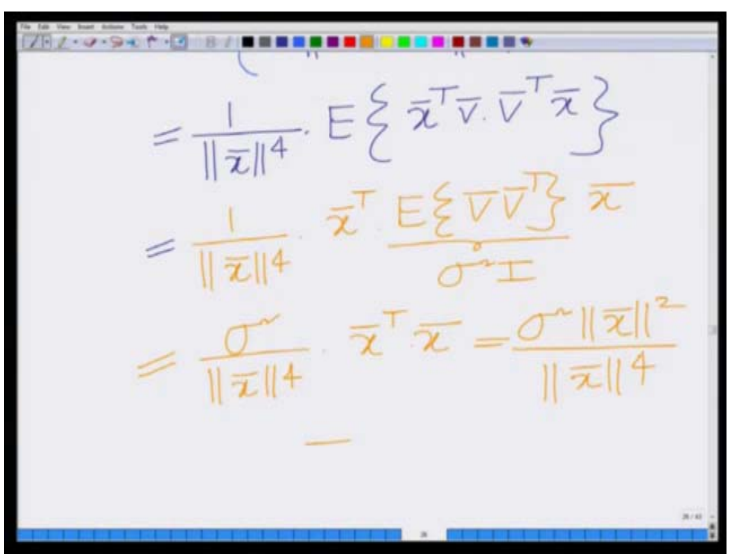

This is basically your Sigma square divided by Norm X bar square.

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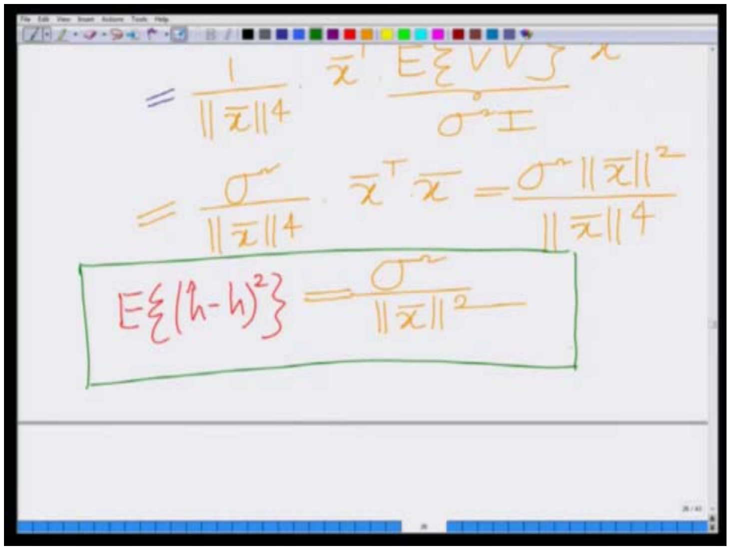

And therefore we have an interesting expression for the variance, that is the variance expected value of H hat - H whole square is basically your Sigma square divided by Norm X bar square. Sigma square divided by Norm X bar square. So, we have derived the expression for the variance of the estimate, we have shown that the variance of the maximum likelihood estimate of the channel coefficient is Sigma square divided by Norm X bar square where X bar is the pilot vector. Alright, so this is the expression for the variance of the channel estimate and further now, we can characterise the distribution of the channel coefficient, of the channel estimate H hat, if you go back and look at H hat, for instance you can see H hat is basically your H + X bar transpose X bar V bar which is a linear combination of Gaussian.

So, this is a linear combination of an therefore it follows that H hat is in turn is Gaussian, that is H hat, you can see H hat is basically this Gaussian X bar transpose V bar shifted by the mean which is H. So, this is the mean of H hat, therefore it follows that H hat is inside Gaussian,

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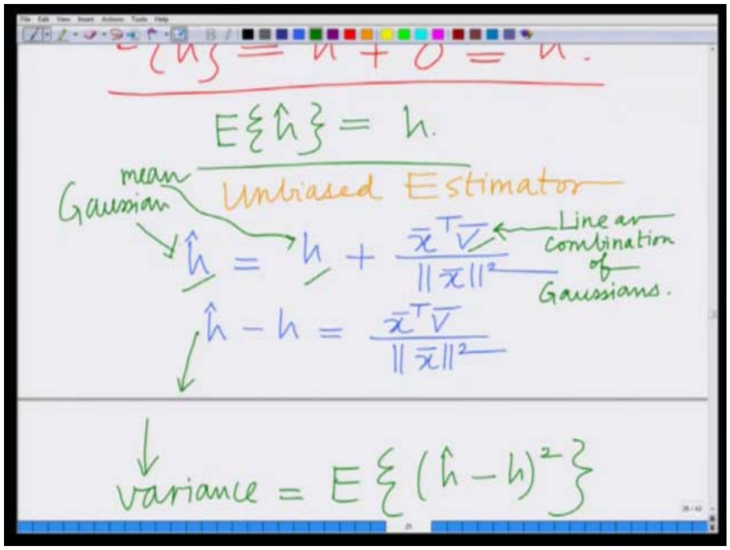

we have shown that the mean of H hat is H

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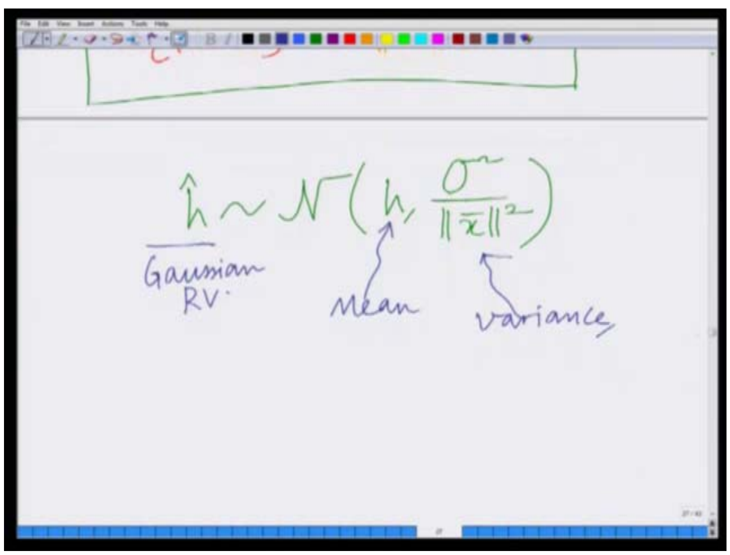

and so therefore it follows that H hat is indeed Gaussian with mean and H and variance Sigma square divided by Norm X bar square.

So, it follows that H hat is, this is a Gaussian random variable which is this mean and this, with this mean and this, so it follows that H hat is a Gaussian random variable with this mean and variance. H hat has a mean of H which is the true unknown channel coefficient and the various is Sigma square divided by Norm X bar square and this is also intuitive.

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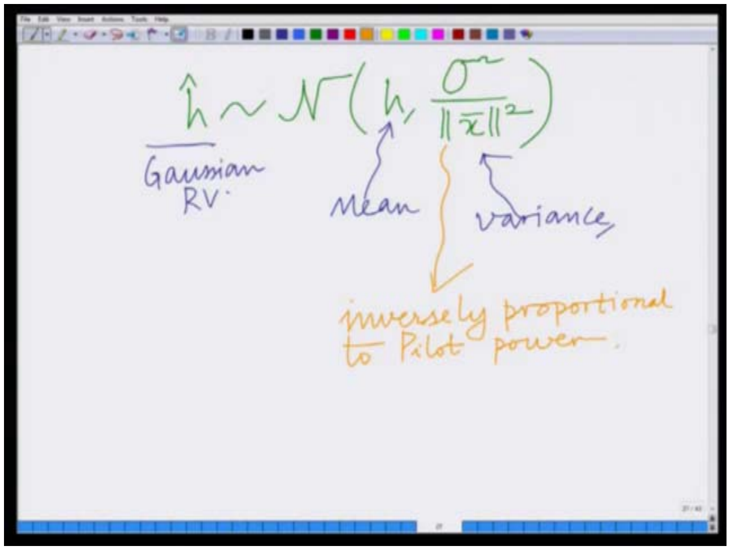

This shows as the energy, this is inversely proportional to the pilot energy and directly proportional to the noise power. The noise power part is obvious, as the noise power Sigma square increases, obviously the estimation error increases and also it shows that as the energy in the pilot increases, that is Norm X bar square increases, the variance of the estimate decreases.

Which is also very intuitive, that is if you transmit more and more pilot power, naturally your estimate will be better and better. So, let us note that also, this is inversely proportional, that is variance is inversely proportional, various is inversely proportional to the pilot power.

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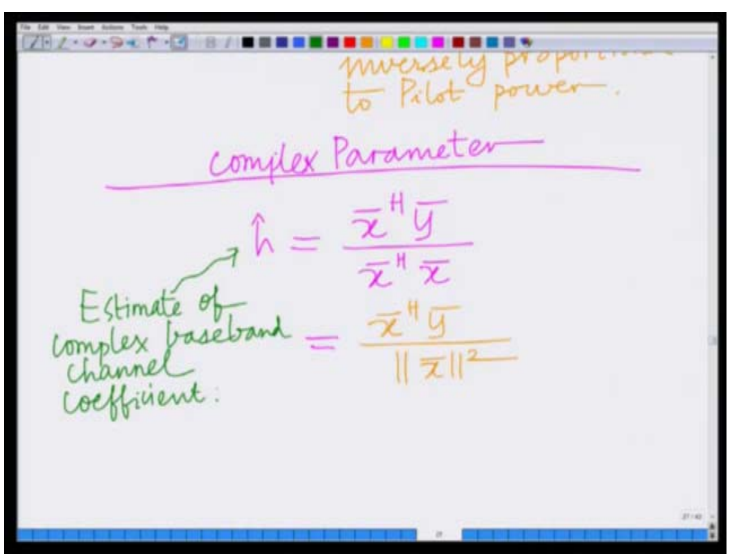

Now, we have also seen that for a complex, how to extend this for complex parameter, just again, because we have said that frequently the channel coefficient is complex in nature, we have said the estimate of the complex parameter is simply, replace the transpose while Hermitian, that is X bar Hermitian Y bar divided by X bar Hermitian X bar, which is basically nothing but again X bar Hermitian Y bar divided by X bar Hermitian X bar is once again Norm X bar square and this we have said is estimate for complex baseband channel coefficient.

That is estimate for a complex parameter estimator of your complex baseband, this is the estimate of your complex baseband channel coefficient. That is when the parameter channel coefficient H is basically complex in nature. That is, you have to replace the transpose by the Hermitian, this is X Hermitian Y bar divided by Norm X bar square.

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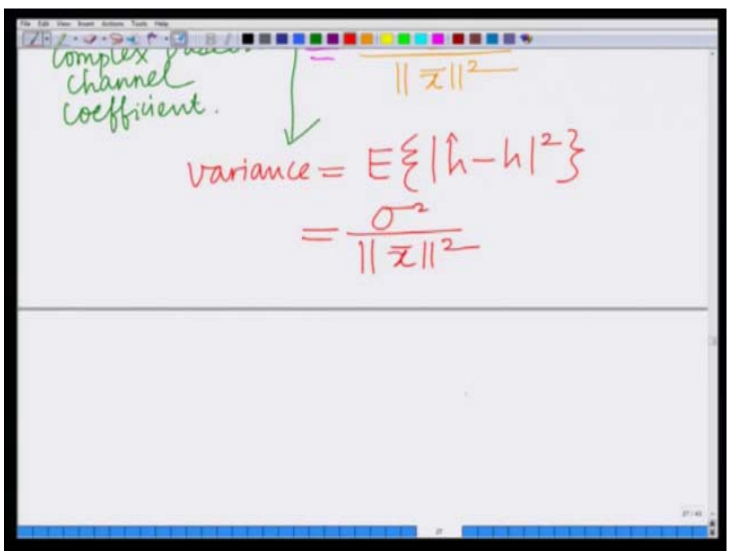

And now, what is the variance, so again you can derive the variance, we said variance will remain, the variance of the complex parameter that is variance is basically expected value of, no, you cannot simply use the square, I have to use magnitude of H hat - H whole square, the magnitude of H hat - H whole square divided equals Sigma square divided by Norm X bar square. Yah.

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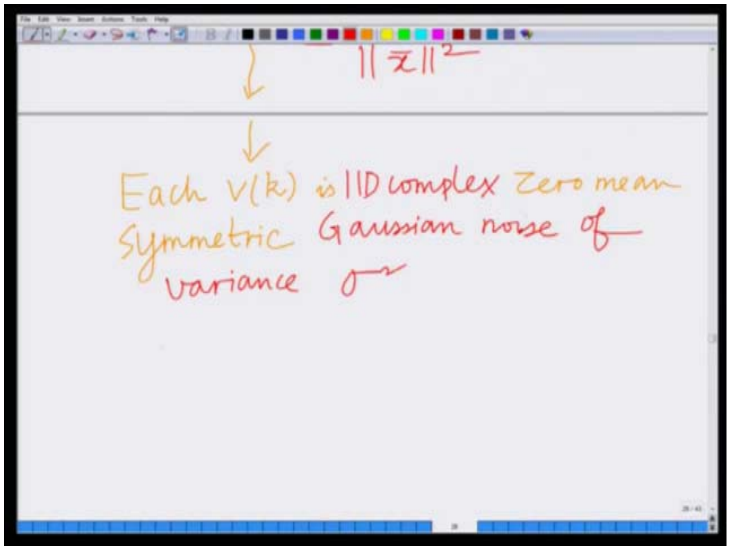

Where we are assuming, where we are assuming that each VK, this is important, each VK, noise VK is complex 0 mean symmetric, remember, this is an important point, symmetric, which we have talked about symmetric Gaussian… Symmetric Gaussian noise of variance Sigma, this complex 0 mean symmetric Gaussian noise, each VK is IID, in fact let us also emphasise that the variance VKs are IID, independent IID complex 0 mean symmetric Gaussian noise of variance.

So, in the noise vector V bar is a complex symmetric Gaussian, complex circle and symmetric Gaussian noise vector, which means each VK is basically IID complex Gaussian complex Gaussian symmetric 0 mean random variable of variance Sigma square. Which means the real and imaginary parts are 0 mean independent and the variance Sigma square by 2 each. Then we can show, also show that expected, that is basically the variance of the real and imaginary part,

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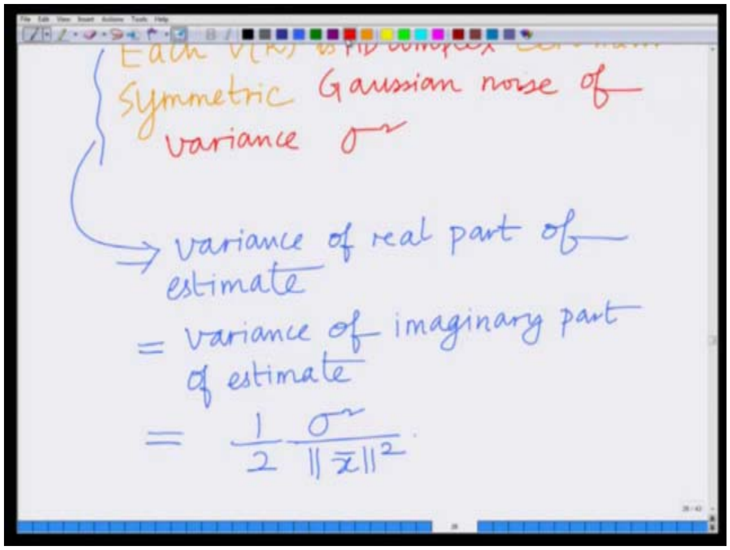

then it follows that, variance of real part of estimate is equal to the variance of imaginary part of the estimate which is equal to half Sigma square divided by Norm X bar square.

So, what we are saying is basically it is, it can be extended to a complex parameter H, that is complex baseband channel coefficient H in the straightforward manner. 1st the channel estimate H hat is X bar Hermitian Y bar divided by Norm X bar square and further, the variance of estimation of the complex parameter H hat is basically Sigma square divided by Norm X bar square, that remains unchanged and the real and imaginary parts, that is the real part and the imaginary part have an estimation error of half of the net estimation error, that is half Sigma square divided by Norm X bar square and also one can notice that the estimation errors of the real and imaginary parts are uncorrelated.

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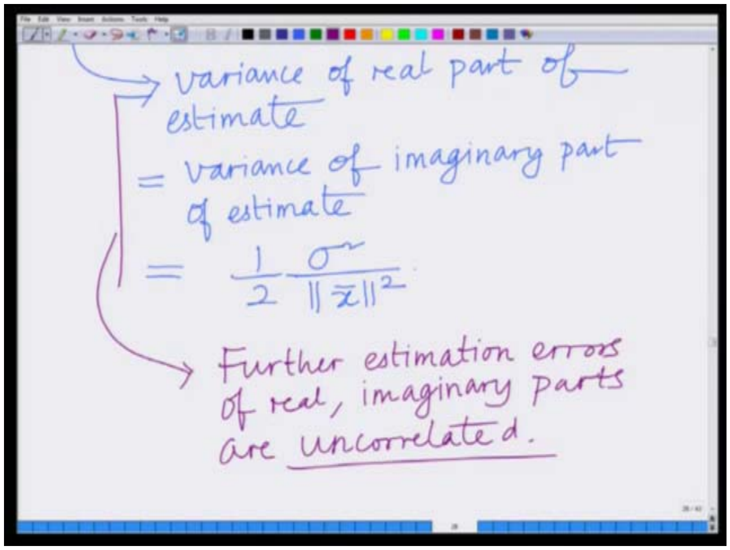

And that is the last point, that is estimation errors of real and imaginary parts, further… Of real, imaginary parts are uncorrelated. Okay, that is a brief note for the estimation of the complex baseband channel coefficient H. So, in this basically module, what we have seen is basically, we had started with the maximum likelihood estimate of the channel coefficient we have developed in the previous module and explored its various properties. We have shown that… Basically we derived covariance of the noise vector V bar and based on that we have shown is basically that the noise estimate is missing, that the estimate of the channel coefficient is again basically, it is an unbiased estimate estimator, which means expected value of H hat, the estimate is equal to H, which is the true unknown, underlying channel coefficient.

Further, the variance in the estimate of the channel coefficient H is given as Sigma square divided by Norm X bar square, Norm X bar square denotes the total energy of the pilot signal and therefore the variance is decreasing with increasing energy of the pilot signal or the pilot vector. So, we will stop here and continue with other aspects in the subsequent modules, thank you very much.