Estimation for Wireless Communications-MIMO/OFDM Cellular and Sensor Networks.
Professor Aditya K Jagannatham.
Department of Electrical Engineering.
Indian Institute of Technology Kanpur.
Lecture -06.
Estimation of Complex Parameters-Symmetric Zero Mean Complex Gaussian Noise.

Welcome to another module in this massive open online course on estimation for wireless location. And so far what we have looked at, we have looked at the estimation in a typical wireless sensor network considering N observations Y1, Y2 to YN of a parameter H in the presence of noise, that is considering noisy observations Y1, Y2 YN when the noise samples are IID Gaussian, that is independent identically distributed Gaussian random variables. We have said that the estimate is given, the maximum likelihood estimate is given by the sample mean which is both unbiased and has a variance of Sigma square by N.

Now, one thing that we have done so far is we have looked at the estimation of real parameters which is going to be something which you are going to keep looking at in the subsequent modules also but of course now this can also be extended to the estimation of complex parameters.

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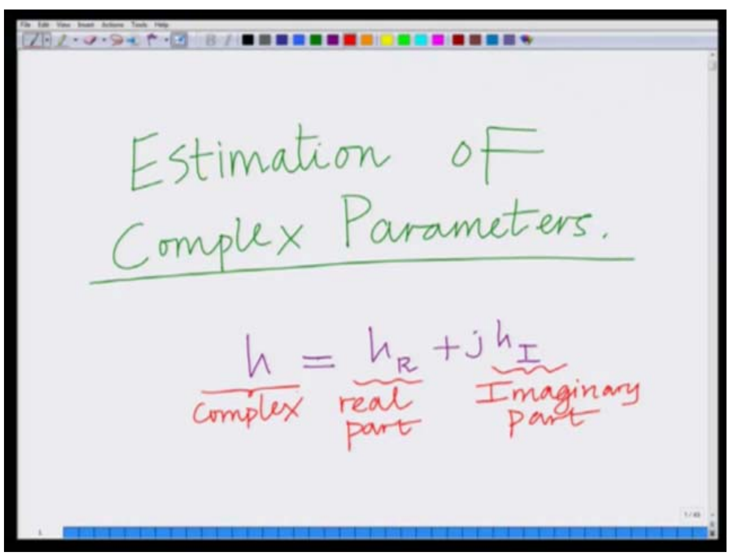

So, I am going to make I am going to talk about the estimation of complex parameters that is how to exchange the results, the estimation of complex parameters. So, we have looked so far a case where H is a real parameter, implicitly we have looked at the scenario where H is a real number.

One can also look at the estimation of a complex number or basically where the parameter that is being estimated is a complex parameter. For instance, I can talk about H which is complex which is HR + JHI, this is basically your real part of H. And this is the imaginary part, as a result of which is this H is complex. So, I have H equals HR + J times HI where HR is the real part and HI is the imaginary part, this is a complex parameter.

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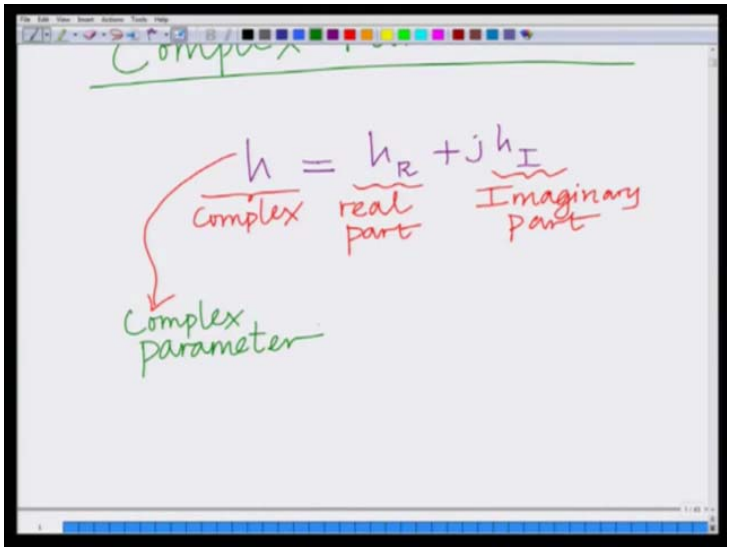

So, this H is basically known as your complex parameter. Yah, so why does this arise the estimation of complex parameter that is also important, it arises in several scenarios, for instance in the estimation of channel in wireless communication systems. The channel can be modelled as a complex baseband channel coefficient.

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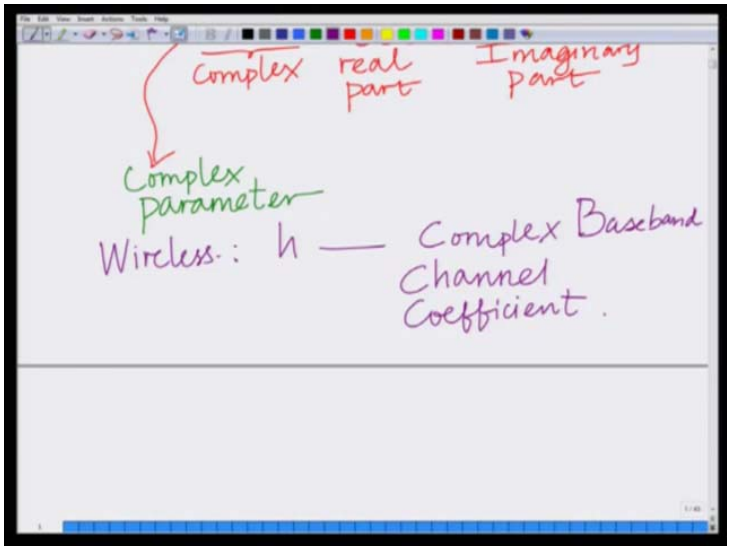

For instance, in the context of wireless communication in a wireless communication system this channel H is a complex the complex baseband, the channel H is a complex baseband channel coefficient, therefore in such a scenario, one needs to estimate a complex parameter, which is the channel coefficient.

And we are going, since this is a course on estimation for wireless communications, we are frequently going to talk about the estimation of complex parameters as well. Now, the same Carey that we have developed so far for the estimation of a real parameter can be extended in a very straightforward fashion to the estimation of a complex parameter. And that is how that is what I am going to illustrate now.

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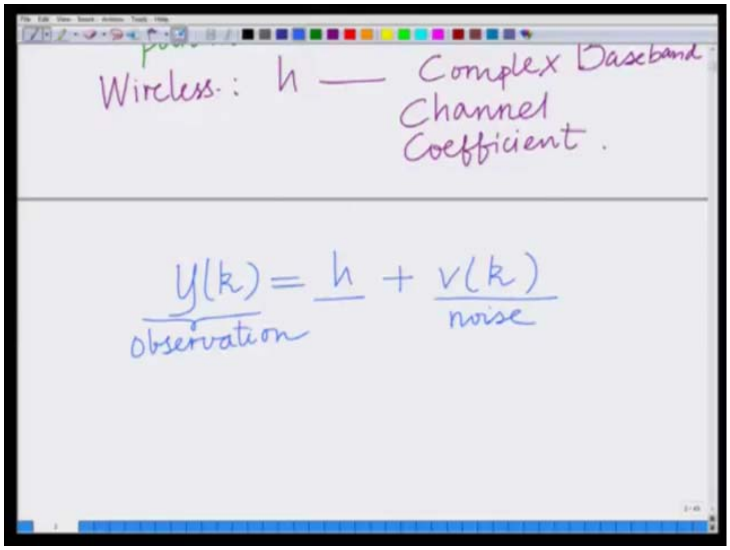

So, let us again consider the noisy observation model for the estimation of complex parameter.

So, let us say I have my noisy observation model, remember the noisy observation model is YK equals H + VK where H is the parameters, V is the noise and Y is the observation. Now, one can extend this to a complex parameter,

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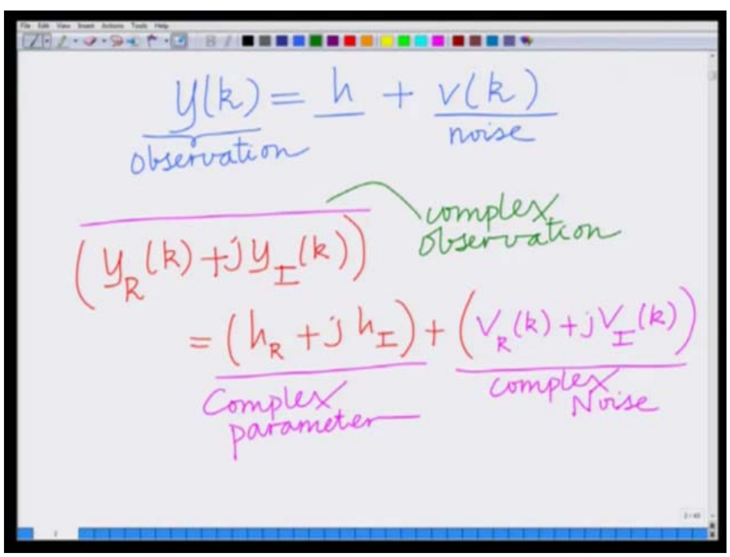

so basically your observation is complex YRK, that is the real part, that is YRK + J times Y I K equals this equals the complex parameter HR + J times H I + the noise which is also complex which is VRK + J times V I K. So, this is your complex noise.

So, this is basically your complex noise, this is your complex parameter, this is the complex observation. This is the complex observation. So, what we have, see remember previously in the noisy observation model, we have the observation equals parameter + noise. Now all we are saying is everything is complex, so we have a complex observation is basically equals your complex parameter + the complex noise.

Now, what we have to do is very straightforward, we can split it into 2 parts, one for the real part one for the imaginary part. So, I can split complex parameter as 2 parts, one is HR, which is the real part and one is H I begin the imaginary part and basically the estimation problem can be split into 2 parts, one for the estimation of a real part and one for the estimation of the complex part.

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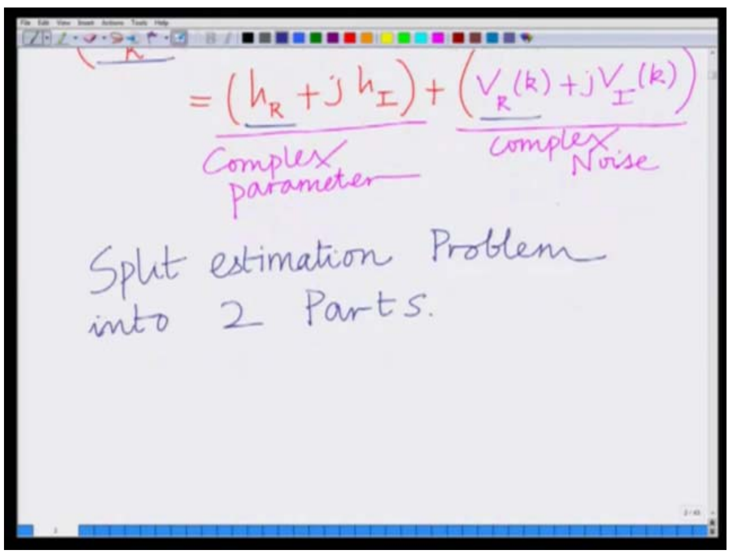

Therefore I can write this problem now as basically if I split it into 2 parts, split the estimation problem into 2 parts. One is if I look at the real part, I have the real part of the observation is equal to the real part of the parameter + the real part of the noise.

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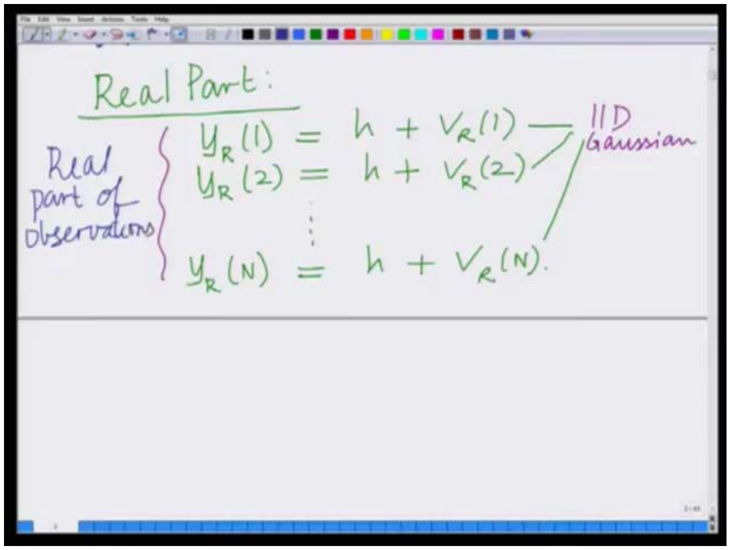

Therefore the real problem will become, if we call this as estimation problem for the real part, I have what do I have, I have YR of 1 equals H + VR of 1, YR of 2 equals H + VR of 2 so on YR of N equals H + VR of N where these noise samples V1, V2, with these noise samples, V1, V2 VN, these are IID Gaussian, these are IID Gaussian, these are the real part of the observation or these are the real, the real part of the observation. These are the observations corresponding to the real part or these are the real part of your observation Y1, Y 2 up to YN, H is the real part of the parameter.

Now if you look at this, this is similar, this has reduced back to the estimation of the real parameter HR for instance, I am sorry; I have to write here HR which is the real parameter,

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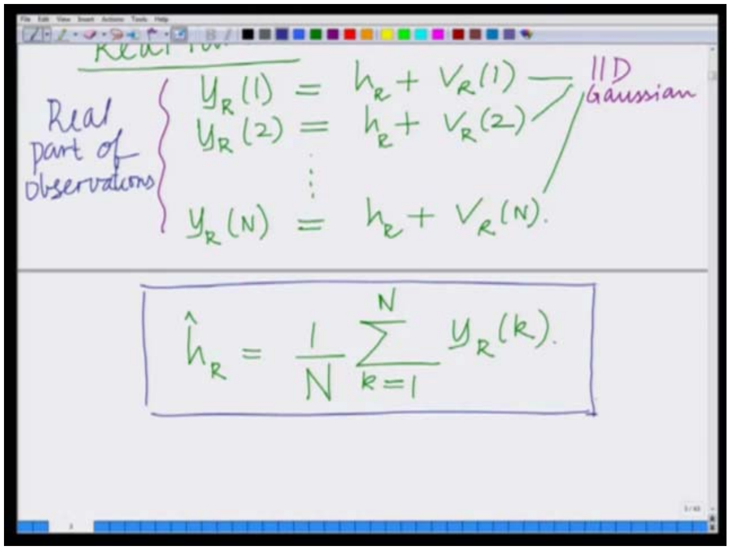

so therefore what I basically have is basically HR or the estimate of the real part HR equals the sample mean of the real part of the observations that is K equals to 1 to N, YR of K, so therefore basically what I have is I have the estimate of the, this is HR hat which is the estimate of the, this is the estimate of the real part.

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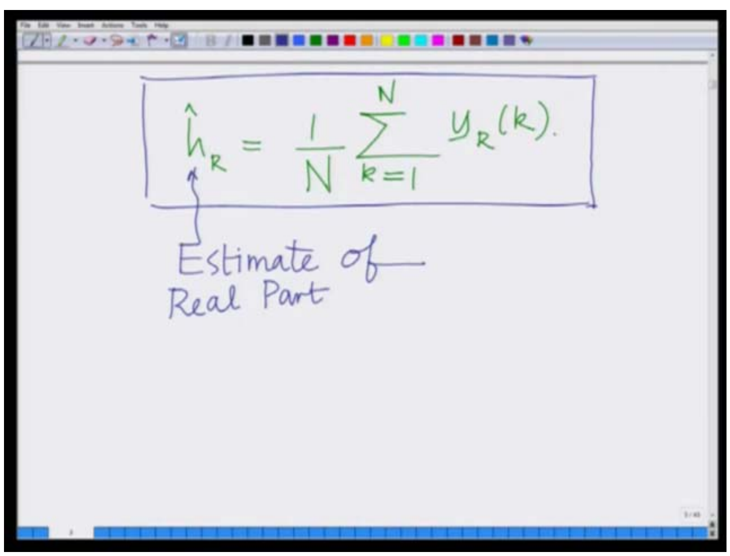

What is this, this is your estimate of the, this is the estimate of the real part.

Now, similarly I can develop a similar optimisation problem for the imaginary part, that is if I look at the imaginary part of the parameter, that is H I, I can look at the imaginary parts of the observations and I can develop a similar estimation problem for the imaginary component of the parameter.

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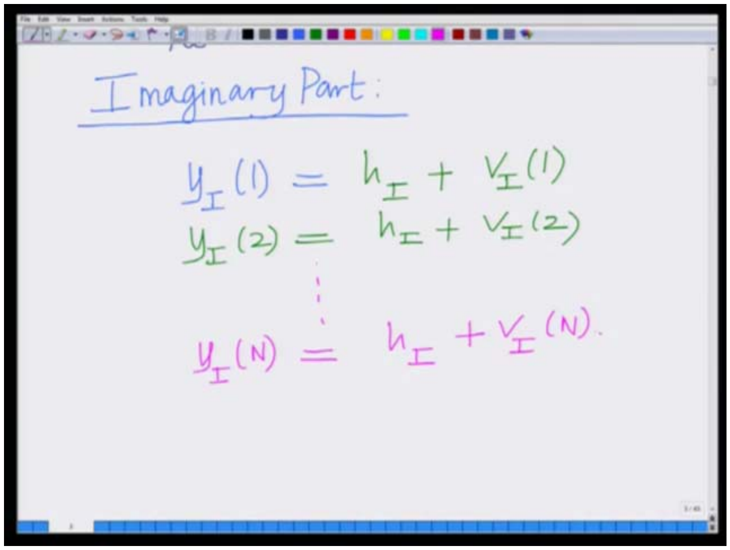

Again that can be written as, now if I look at the imaginary part, I can write this as basically your YI of 1 the imaginary part of the observation is H I + VI of 1 YI of 2 equals HI + VI of 2 so on and so forth YI of N equals HI + V I of N. Once again we have reduced this back to estimation of real parameter.

What is a real parameter, that is H I, remember H I is real that is H equals HR + J HI. H I represents the imaginary part of the parameter H. But H I is real. So, now we have again reduced it to the estimation of another real parameter that is H I. Again, if these noise samples V I1, VI 2 up to V I N are IID Gaussian, then again the estimate of this parameter H I is given by the sample mean of the imaginary parts of the observation.

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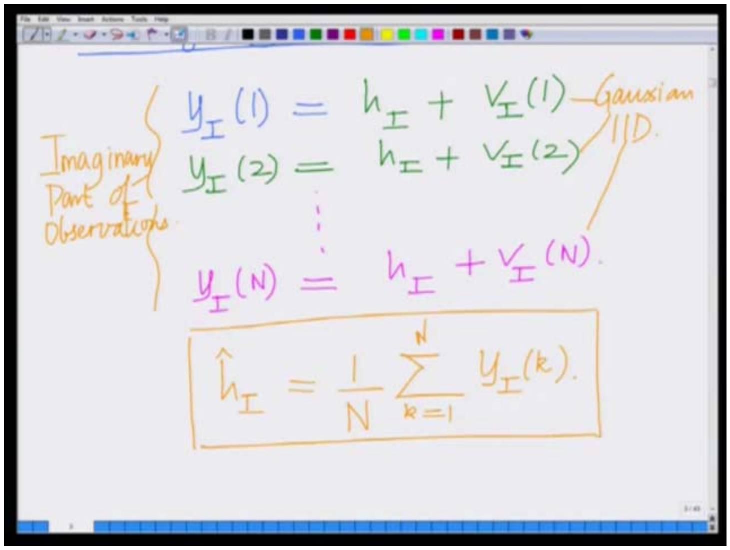

So, basically these are Gaussian again, once again, if these noise samples, these are Gaussian, if these are Gaussian IID, now these of course, these are the imaginary parts of the observations.

These are the imaginary parts of the observations, then what am I going to have? The estimate of the imaginary part H I hat equals 1 over N summation K equals 1 to N the Sample mean of the imaginary parts, that is H I equals sample mean of YI 1 YI 2 up to YI K,

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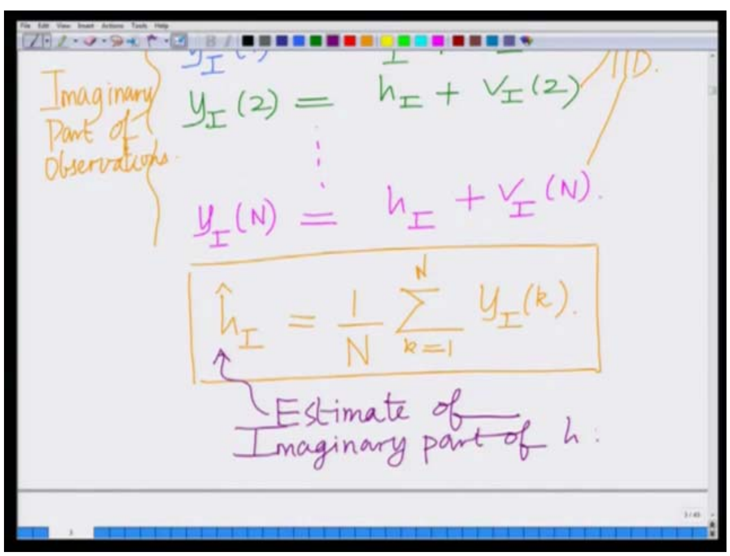

so it H I hat, this is the, what is this, this is the estimate of the imaginary part. This is the estimate of imaginary part of H. And further, now how to get the estimate of the parameter? Simply the estimate of the real part + J times the estimate of the imaginary part.

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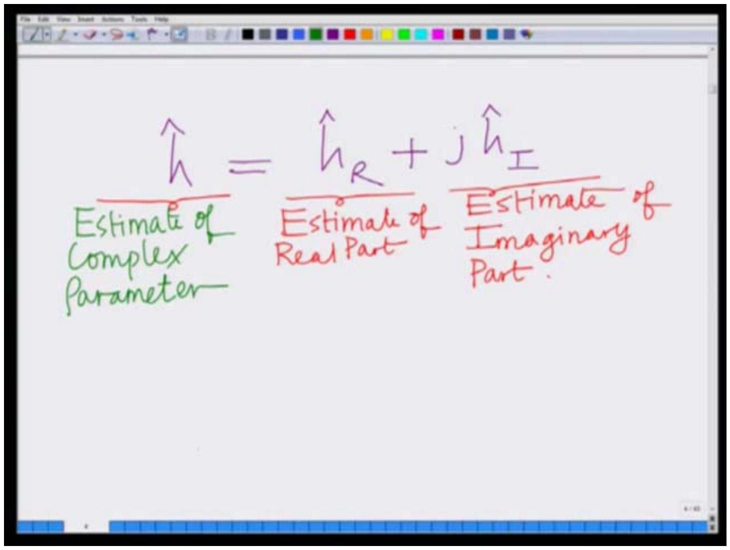

Therefore the estimate of the complex parameter H H hat equals, that is also very simple Naturally, HR hat + J times H I hat. So, this is, what is this, this is estimate of real part, this is estimate of imaginary part, this is estimate of the imaginary part and this is the estimate of the complex parameter.

This is the estimate of the complex parameter, so the estimate of the complex parameter H hat is estimate of the real part HR, that is HR hat + J times H I hat where H I is H I hat is the estimate of the imaginary part. Now, one important property that will help signify the behaviour and properties of this ML estimate significantly is as follows. That is the property of the noise samples which we have not discussed so far, which is the estimate property of V V K which is the complex noise.

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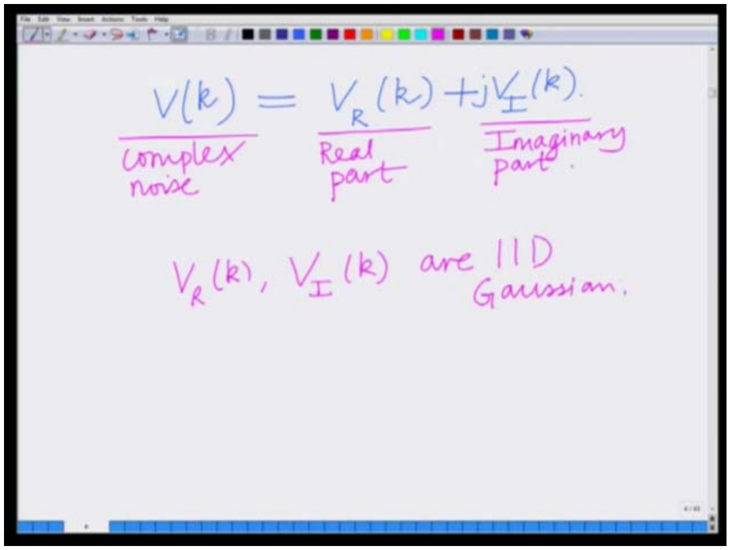

Remember the complex noise VK, we have said VK equals the complex noise which is VRK + J times VI K, so this is your complex noise.

This is the real part of the complex noise and this is the imaginary part.

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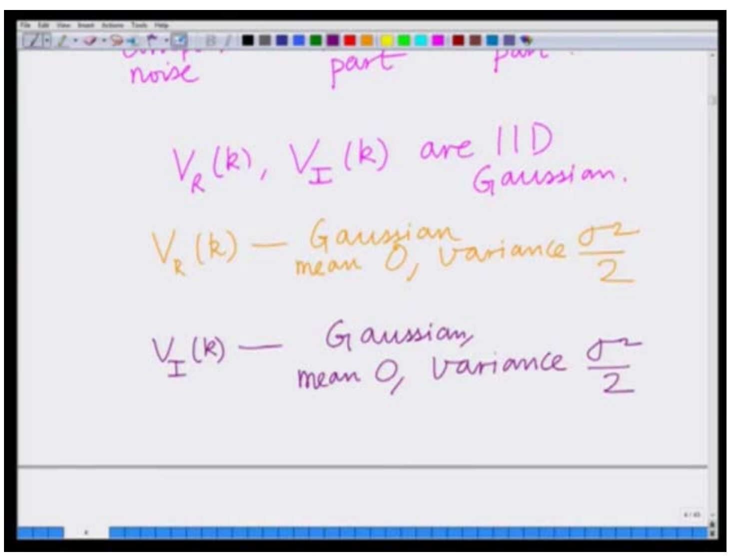

Now what we are going to assume is that V R K and V I K are IID Gaussian that is what does that mean? We are going to assume VRK is Gaussian mean 0 variance Sigma square by 2, that is half Sigma square. Further, we are also going to assume that VI K is Gaussian with mean 0 and variance Sigma square by 2. So, what are we assuming, we are assuming that, well 1st we have said that this VK is complex noise which can be written as VRK + J times VI K.

VR is the real part, VRK the real component, V I K is the imaginary component. We are assuming that VRK and V I K are IID Gaussian, that is both VRK, that is VRK is Gaussian with mean equal to 0 and variance Sigma square divided by 2. V I K is also Gaussian with mean 0 and variance Sigma square by 2. Further, both of these are independent which means, there also uncorrelated as well.

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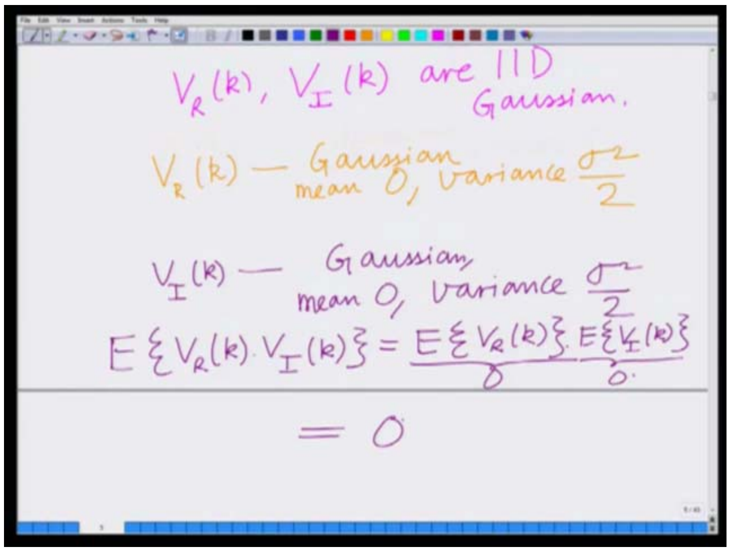

This is Gaussian because independence means uncorrelated which means expected value of VRK times V I K equals expected value of VRK times expected value of V I K. Now both of these are 0, both of these are equal to, this is equal to 0. This is equal to 0.

Which means expected value of VRK times V I K equals to 0, so these 2 components that is the real part and the imaginary part are identical, that is both have mean 0 and variance Sigma square by 2 and further, they are independent, that is the expected value, which means the expected value of the real part times the imaginary part is equal to 0. Such a noise, so both real and imaginary component are identical, Gaussian and both the real and imaginary parts are independent, such a noise is called complex Gaussian noise, further it is a 0 mean symmetric complex Gaussian noise.

So, let me elaborate on that fact a little bit more.

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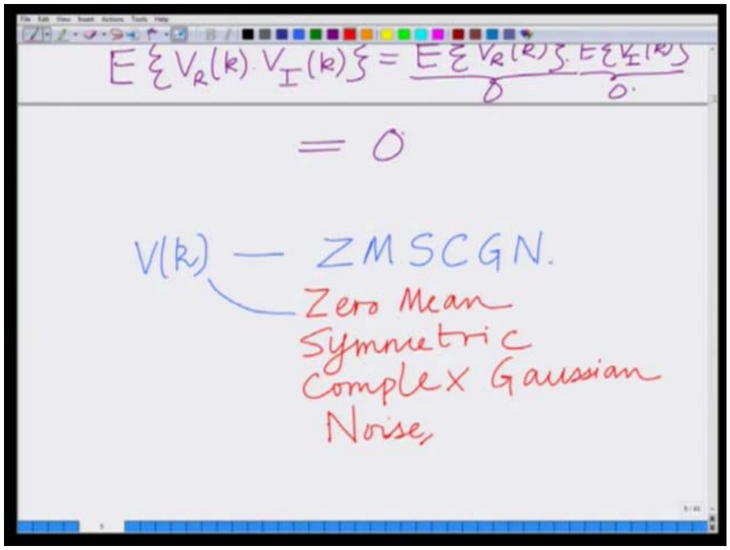

VK is known as a 0 mean symmetric complex Gaussian noise and this is the most important frequent assumption. This stands for 0 mean symmetric, each of these words has a meaning, complex, Gaussian, this is an important property of complex noise that is VK is known as symmetric 0 mean complex Gaussian. What is the meaning of symmetric? It is symmetric because the real part and the imaginary part are basically symmetric, that is they are identical, they have both 0 mean and variance Sigma square divided by 2.

They are symmetric 0 mean ya, that is both the components, the real component and imaginary component have 0 mean, complex that this noise VK is complex in nature, Gaussian, both the components, the real part and imaginary part are Gaussian and this is noise, therefore, this is 0 mean symmetric complex Gaussian noise. Which is basically, to summarise…

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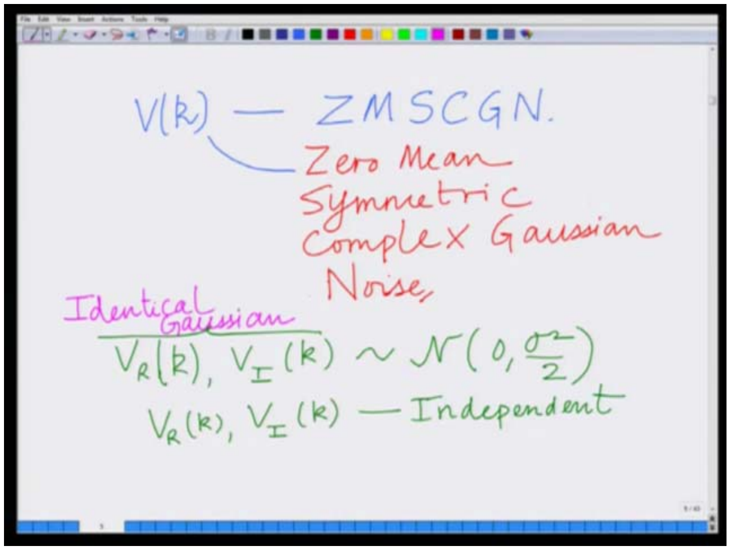

VRK, that is the real component, your VRK and V I K are both 0 mean, variance Sigma square by 2 VRK, VIK are independent. That is, both of these are basically identical Gaussian, both of these are physically identical Gaussian and both of these are independent and this is basically known as symmetric Gaussian or complex Gaussian noise.

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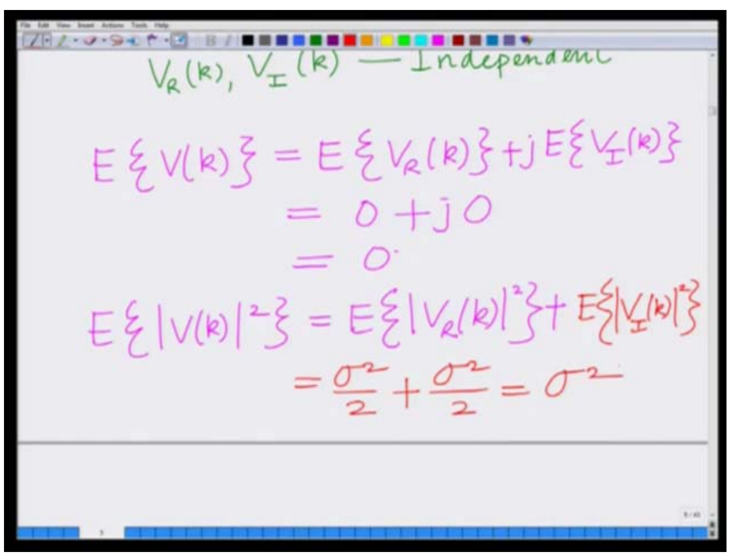

In that case, now you can say that basically expected VK, the complex noise VK is expected VRK + J times expected V IK which is basically 0 + J times 0 which is equal to 0.

That is VK is also 0 mean. Further, the variance of the power, the power of the complex Gaussian noise, expected magnitude VK square is expected magnitude V RK square +

expected magnitude V IK whole square which is equal to Sigma square by 2 + Sigma square by 2 which is equal to Sigma square. So, expected VK, the complex noise VK is 0 mean, the magnitude VK square, expected value, the variance of the complex noise Sigma square in which the real component has half the power, imaginary component has half the power, both of these are Gaussian 0 mean and they are independent. That is the expected value of V IK times VRK is 0, since that is another correlated.

Since they are Gaussian, so automatically mean that they are independent. That is both the components are basically symmetric, Gaussian symmetric, independent and identical. And such a noise is known as 0 mean complex that this 0 mean symmetric complex Gaussian noise because the noise is complex in nature, therefore naturally it is complex Gaussian. Now, with this property, the estimate, the ML estimate, the behaviour of the ML estimate can be simply simplified further.

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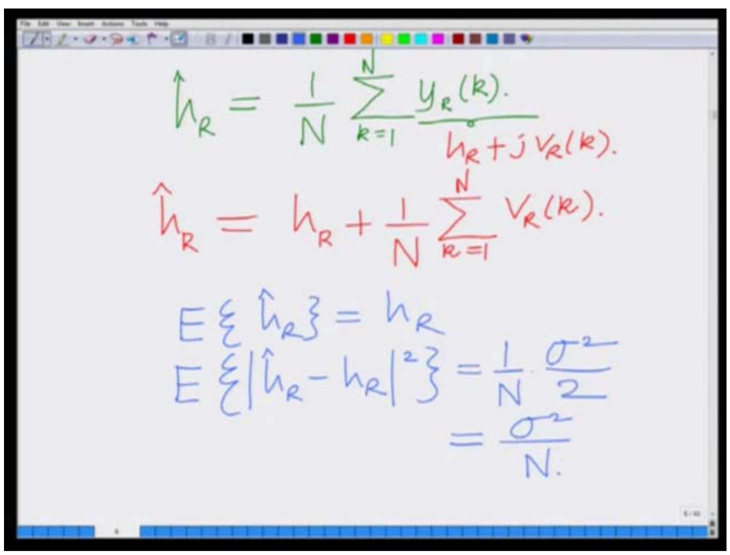

Now, if you look at the ML estimate, now remember that we have said HR hat equals 1 over N, the sample mean of the real part of the observations, that is K equals to 1 to N.

Y Y of the real part K. Now, of course I can simplify this as follows, I can simplify this, if I substitute for every, remember we did this already, ah when we looked at the behaviour of the maximum likelihood estimate, that is a real part of K is HR + J times VR of K, so therefore if I substitute this for YR of K, the design will get H hat, the real part is basically H H, that is the real part of the parameter + the sample mean of the real components of the noise, that is K equals to 1 to N, VR of K, and therefore once again we can show that expected value of HR hat equals HR, that is the real part, the estimate of the real part is unbiased.

The variance of the real part, expected value of HR hat - HR whole square is the variance of 1 by N times the variance of each noise component but the variance of each noise component is Sigma square by 2, so this becomes Sigma square divided by 2.

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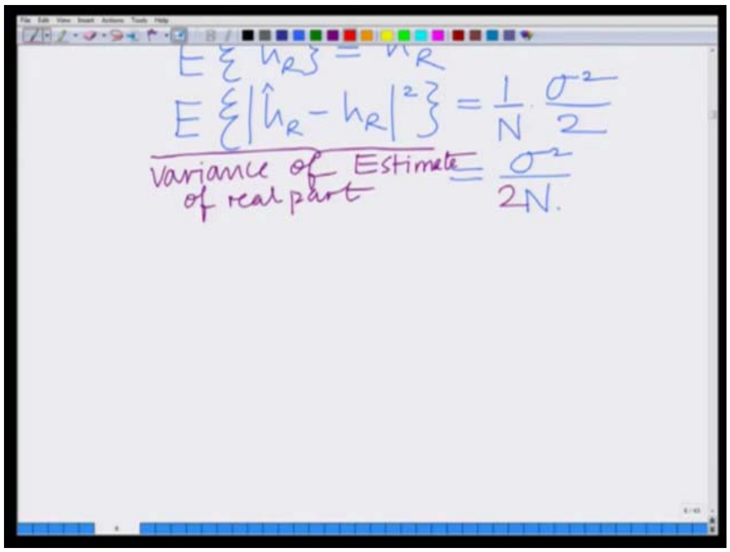

So, the variance of the real part, that is a variance, that is a mean squared error, that is the variance of the real part of the error, the variance of the real estimate estimate, variance of estimate of real part is Sigma square divided by 2N.

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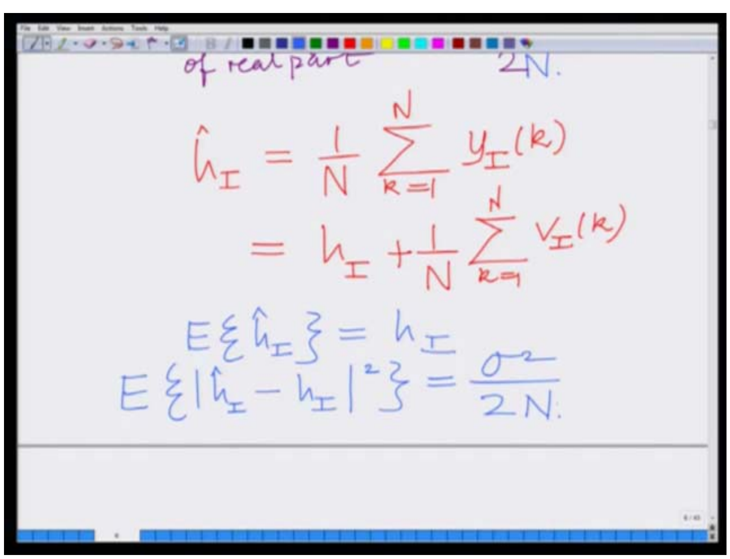

Further again, now we can extend this again to the imaginary part, that is H I hat, remember I wrote this as 1 over N sample mean of the imaginary part of the observations, that is K equals to 1 to N Y of imaginary part of the Kth observation Y I of K, which now you can simplify as H I + 1 over N summation K equals to 1 to N V I of K.

And therefore it follows that the imaginary part is also unbiased, the estimate of the imaginary part, that is the expected value of H I hat equals H I and further the variance of imaginary part, the estimate of the imaginary part, expected value magnitude of H I hat - H I whole square equals Sigma square divided by 2 N. And now if you look at, it is important at this point to look at the errors, look at the errors,

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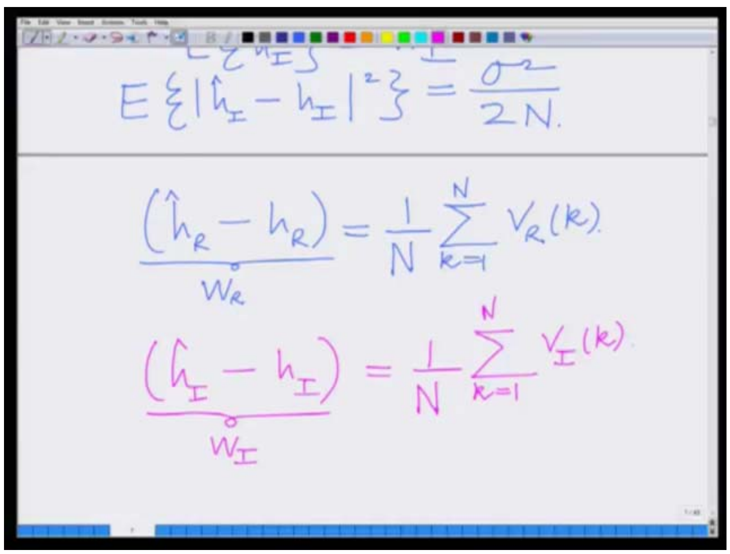

HR hat - HR, that is the error in the real part, HR hat - HR, let us call this as WR, that is the estimate, and the error in the real part, that is 1 over N summation K equals to 1 to N VR of K.

And the estimate of the, the error in the imaginary part, that is H I hat - H I, if I call this as W I, that is the error in the imaginary part that is equal to 1 over N summation K equals 1 to N V I of K. Now look at the interesting property. We have the different VI K and VRK because we are assuming 0 mean symmetric complex Gaussian noise, V IK and V RK, the real and imaginary parts of noise VK are independent. And further these different complex noise samples are independent anyway. Therefore what we have is basically all these noise components that is VR1, VR2 up to VRN and V I 1, V I 2 up to VIN are independent, are all mutually independent, in fact independent identical Gaussian.

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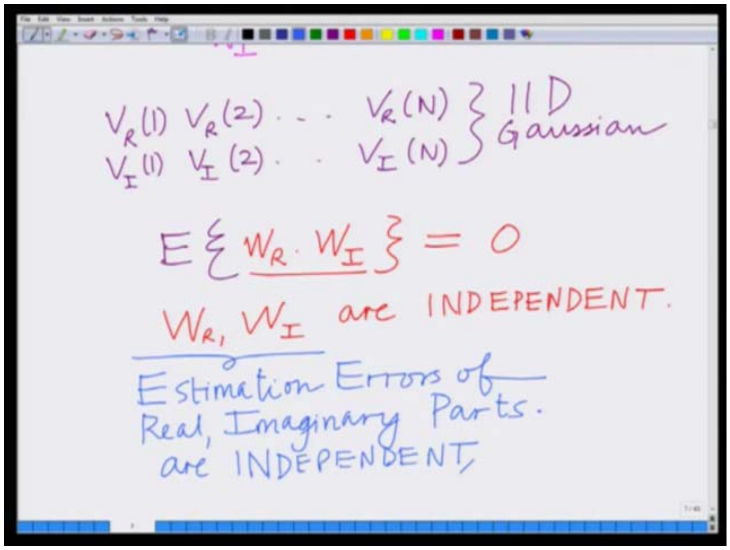

Now if you look at all the noise samples taken total, that is we have VR 1, VR 2 up to VRN, we have VI 1, VI 2 up to VIN, all of these are in fact IID Gaussian, these are IID Gaussian, all of these are IID Gaussian with mean 0, variance Sigma square by 2 and therefore now if you look at the co-relation between the real the the real and imaginary, the errors in real and imaginary parts, that is WR and WI, WR depends only on the real components of the noise, W I depends only on the imaginary components of the noise, therefore the co-relation between the errors of the real part and the imaginary part is 0.

And that is an important property, that is expected value of WR times WI equals 0, further, these are Gaussian, which means it also follows that WR, W I are independent. And this is a subtle, a very subtle property a very subtle property but a very important property which says that if you look at if you look at basically the estimate of the real part and the estimate of the imaginary part separately.

Once you have decoupled this problem into the estimation of the real part and the estimation of the imaginary part, the estimate of the real part has the estimate of the real part has a variance which is Sigma square by 2N, the estimate of the imaginary part has a variance which is Sigma square by 2N, more importantly, it says that if you look at the error in the estimation of the real part and the error in the estimation of the imaginary part, these 2 estimation errors are basically uncorrelated.

And now by virtue of because the noise is Gaussian, they are also basically independent. Although it is not generally true that there always independent, generally it is true that they are uncorrelated. If they have symmetric noise, they are uncorrelated but here specifically we have an example where the noise samples are Gaussian, so it also follows that basically they are independent. So, the estimation errors in the real part and the imaginary part of the parameter are independent.

So, these are WR WI, remember these are the sales these are the estimation errors, these are estimation errors that is estimation errors, the estimation errors of the real and imaginary parts are basically, the estimation errors of the real and imagery parts are basically independent. And that is an important property. I can treat this as basically the estimation of the real part and the estimation of the imaginary part and the estimation of the real and imaginary parts are independent. Now, if I look at the estimation error in the complex parameter as a whole, that is, remember, if I look at expected value of…

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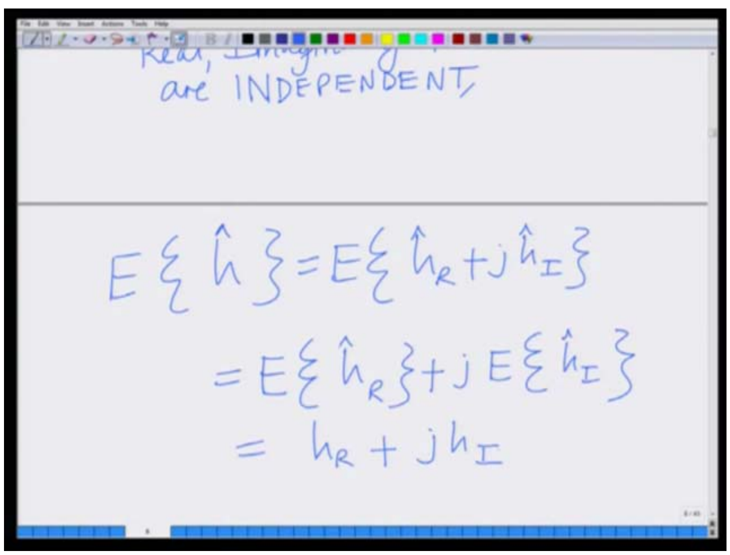

Remember we have already said that if I look at the expected value of the estimate of the complex parameter, that is the expected value of estimate of HR hat + J H I hat which is the expected value of estimate of HR hat + J times the expected value of estimate of H I hat, of course this is equal to HR + J times H I,

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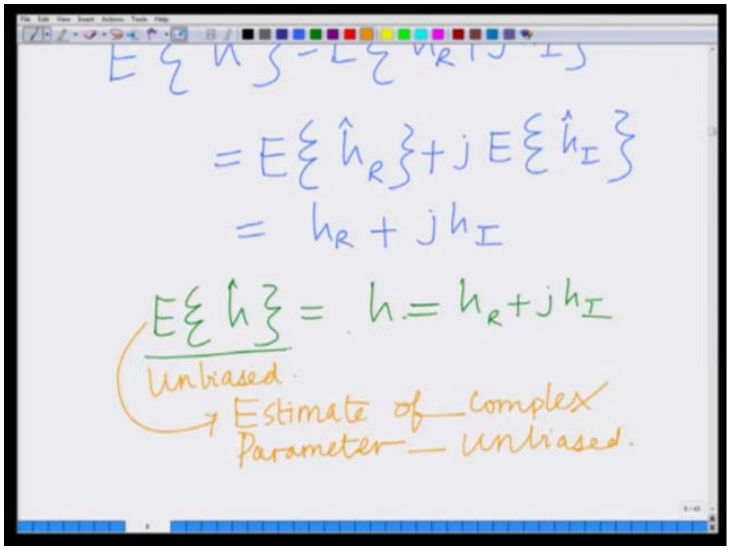

so therefore the estimate of the complex parameter is unbiased. Therefore estimate of the complex parameter equals H hat equals H which is the complex parameter, that is equal to HR + JHI. So, this estimate is an unbiased estimates, so this is basically unbiased, the maximum likelihood estimate of the complex parameter is unbiased.

The estimate of the complex parameter is unbiased, so this is the estimate of the complex parameter, this is unbiased.

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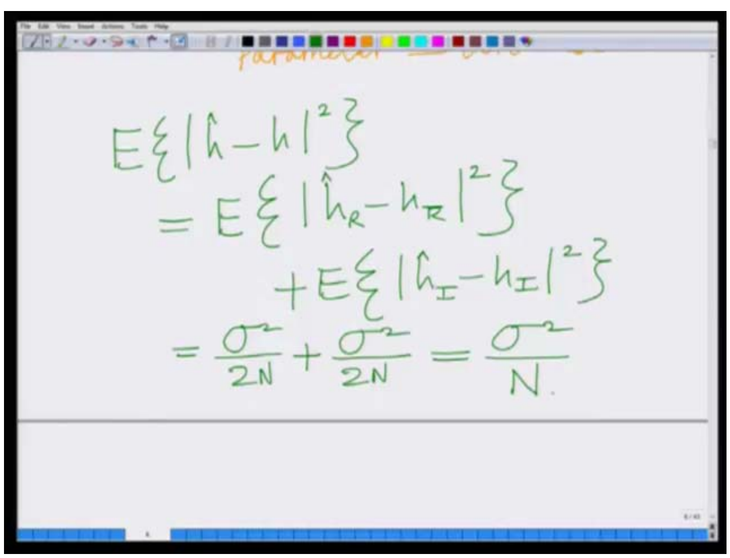

Further, if you look at the variance in the estimate of the complex parameter, that is expected value of magnitude of H hat - H whole square, that is basically expected value of magnitude of H hat real - H hat H real whole square + expected value of the variance of the imaginary parts, that is H hat I - H I square. Now, each of these components is Sigma square by 2 N, as we have shown earlier.

This is Sigma square by 2N, this is Sigma square by 2N, so that we have Sigma square divided by N. That we have Sigma square divided by N. And therefore what we have seen in the following interesting thing, that is the real part has a variance that is given by Sigma square divided by 2N, the imaginary part has a variance that is given by Sigma square by 2N, the error variance of the total estimation of the complex parameter is Sigma square by N. Another way of stating is as all those, if I have a complex parameter H and if I estimate it, I can use the same maximum likelihood estimate that I have derived for the estimation of the real parameter, correct. So, I have maximum likelihood estimater that is the that is the sample mean, ya.

And what we have shown is basically if we assume, in addition if the noise samples are complex Gaussian, 0 mean and symmetric, which means the noise samples have real and imaginary parts which are independent, complex Gaussian, identically independent Gaussian and identically distributed, in that scenario, where the complex noise is such that each the real part and the imaginary part each have 0 mean and variance Sigma square by 2, the error, the error variance of the in the estimate of the real part is Sigma square by 2N, the error variance in the estimate of the imaginary part is Sigma square by 2N.

The error variance in the estimate of the entire complex parameter is Sigma square by N. Or in other words, if the error variance in the estimate of the complex parameter is given by Sigma square by N, the error variance in the estimate of the real part and the error variance in the estimate of the imaginary part is basically half of that and further it is also important to note that the estimation errors in the real part and the imaginary parts are basically uncorrelated. Similarly in this case of Gaussian, they are independent. Yah, this is an important property that the estimation errors of the real part and the imaginary part are uncorrelated or basically independent for this specific case of Gaussian.

So, that is how basically we can extend the same results, we are going to, we are going to use this property ah, when we talk about the estimation of complex parameters, we have mentioned there in the example in the beginning of this module that the estimation of the complex parameters arises in the context of wireless communication where you are estimating the complex time coefficient. So, once we derive for instance an estimator for the real part, the same can be used to estimate the complex to estimate a complex parameter noting that basically the estimate, the noise is 0 mean complex Gaussian symmetric, the estimates, the errors in the estimate of the real part and imaginary part are basically independent and also basically the errors in the estimate of the real part and the imaginary part have half the variance of half the error variance or basically the estimate of the total complex, that is the error variance in the estimate of the complex parameter.

And this is an important property to keep in mind when we extend the results for real parameters to the complex parameters. So, with this insight or brief note about the estimation of complex parameters, we will stop this module, we will cover we will continue with other aspects in subsequent modules. Thank you very much.