Estimation for Wireless Communication –MIMO/OFDM on Cellular and Sensor Networks

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Lecture Number 05

Reliability of the Maximum Likelihood (ML) Estimate - Number of Samples Required

Hello, welcome to another module in this massive open online course on Estimation Theory for Wireless Communication. So far we have looked at the Maximum Likelihood Estimate which is a sample mean. For a simple example of a wireless sensor and with N observations we have looked at the Maximum Likelihood Estimate.

And we have looked at the properties of Maximum Likelihood Estimate. That is the mean and the variant of the Maximum Likelihood Estimate. And we have seen some interesting properties of this Maximum Likelihood Estimate.Today what we are going to do is, we are going to explore this Maximum Likelihood Estimate further.

Let us look at the reliability of this Maximum Likelihood Estimate. So what we are going to start looking at today is another example another simple example. We are going to explore the reliability of this, of this maximum likely hood or basically the ML.The reliability of the Maximum Likelihood Estimate. Namely, what is the number of samples for instance, what is the number of samples required for estimation?

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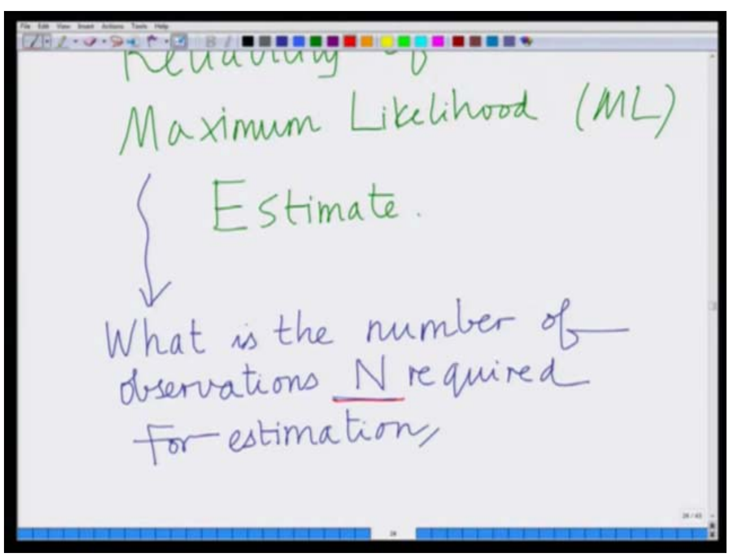

Number of samples or number of observations. Number of observations in required for estimation of a particular accuracy. So what we are going to start addressing today is, what is this number of, how do we decide the number of samples required for estimation? That is how do we how do we how do we choose this number of samples capital N that is required for estimation to achieve a certain degree of reliability?

That is, how do we decide the reliability of this Maximum Likelihood Estimate and it is not easy task because we said that this Maximum Likelihood Estimate h hat is random in nature. So it is, it is quite challenging and we are going to look at how to decide the number of samples N that is required. So basically the approach that you're going to employ is basically one that involves the probability.

Yeah, so we are going to choose, so we're going to address so we can formulate this question in this fashion.

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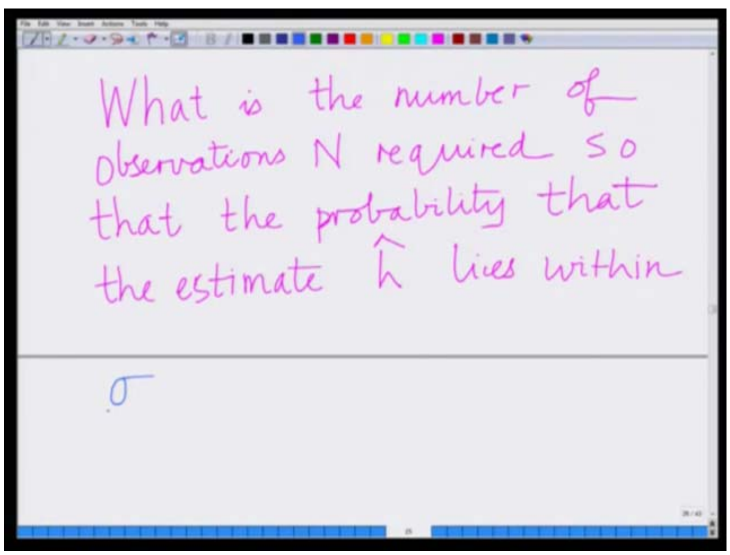

We're going to ask the question; what is the number of observations N that is required so that the probability of the estimate h hat lies within Sigma by 2.

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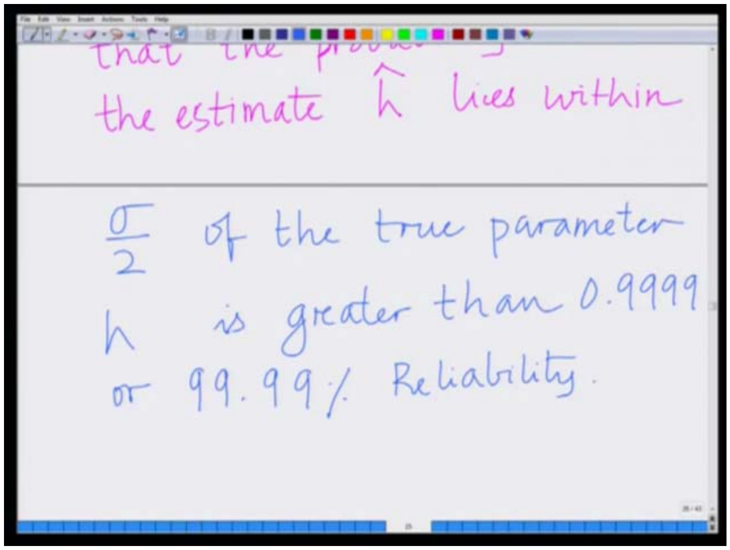

Lies within Sigma by 2 of the true parameter of the true parameter h is greater than 0.9999 or basically 99.99% reliability. One can also call this one can also define this notion, this probabilistic notion as basically reliability. So what we are saying is basically that probability can be used as a measure of reliabilities, since the estimate h hat is random in nature.

And now we are asking this interesting question. What is this number of observations capital N that is required such that the estimate h hat lies within a distance or lies within the radius of Sigma by 2 from the true parameter h with probability that is greater than 0.9999 or basically with 99.99% reliability or 99.99% certainty?

So let us illustrate this pictorially, so what we have is basically we have your estimate which is distributed as a Gaussian random variable.

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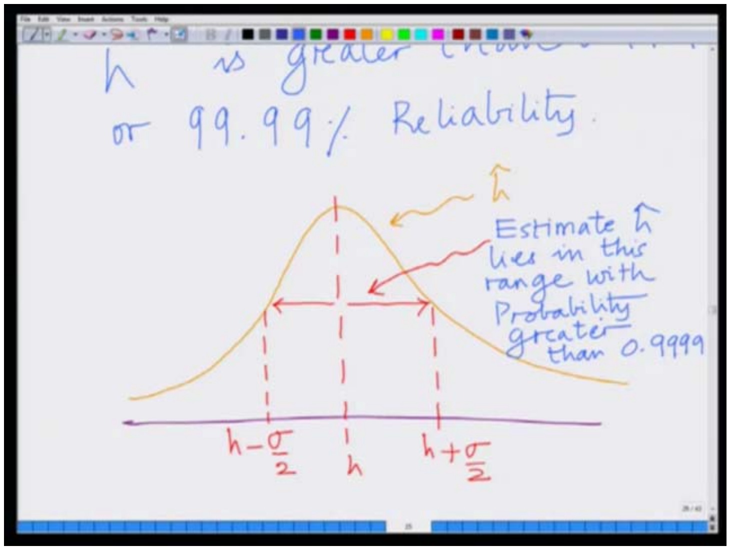

We said the estimate is a Gaussian random variable with mean centered around h, h is the mean. And if we take a Sigma by 2 radius around the mean that is this is h + Sigma by 2 and this h - Sigma by 2, so this is the Sigma by 2 radius around the true parameter h.

And we are saying, what is the number of observation in that is required such that the estimate the estimate h hat lies in this range with 99.99% certainty or with 99 0. lies in this range with probability greater than 0.9999.

So when you are saying what is the probability that this life within a radius that is the estimate h hat lies within the radius Sigma by 2 of the true parameter h with probability greater than 0.9999 or greater than 99.99% of the time okay. And this can be found as follows, so basically we are asking the question.

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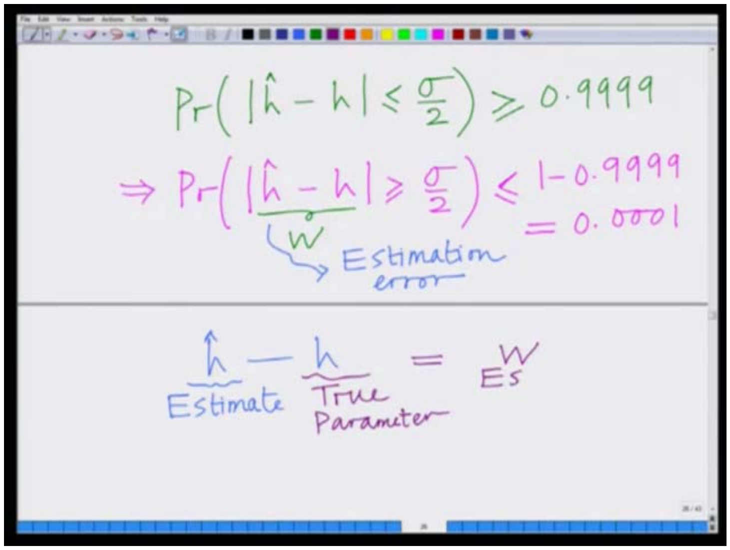

The probability that your estimate - the true parameter that is the radius, that is the estimate minus the true parameter should be less than or equal to radius Sigma by 2 that is, h hat should lie within Sigma by 2 radius of h with probability greater than 0.9999 and that implies basically, now realise that this. Basically what that implies is that implies that the probability that it lies outside this radius.

So, naturally h hat - h is greater than or equal to Sigma by 2 less 1 - 0.9999 that is equal to 0.0001. That is the probability; it lies outside of the radius Sigma by 2. It should be less than 1 - 0.9999 that is 0.0001.

So what we are saying is the probability that it should lie within a radius of Sigma by 2 should be greater than or equal to 0.9999 which automatically mean that the probability it lies outside the radius Sigma by 2 should be less than 1 - 0.9999 that is 0.0001 that is 99.99% of the time it should lie within the radius of Sigma by 2 which means only 0.01 of time it can light outside the radius Sigma by 2.

And now you will realise that this h hat - h, this is nothing but our quantity w which is the estimation error. This h hat - h, h hat is the estimate; h hat is the estimate and this - h. So this is your estimate and this is basically your true parameter and therefore h hat - h is equal to w which is basically the, which is basically the estimation error.

And therefore what we are asking is, we are asking basically, what is the probability that this estimation error that is the probability that this estimation error, magnitude of this estimation error should be greater than or equal to Sigma by 2. Only point with probability 0.001 or

0.01% of the time. And now realise something important, we said that the estimate h is Gaussian with mean h and variance Sigma Square divided by N.

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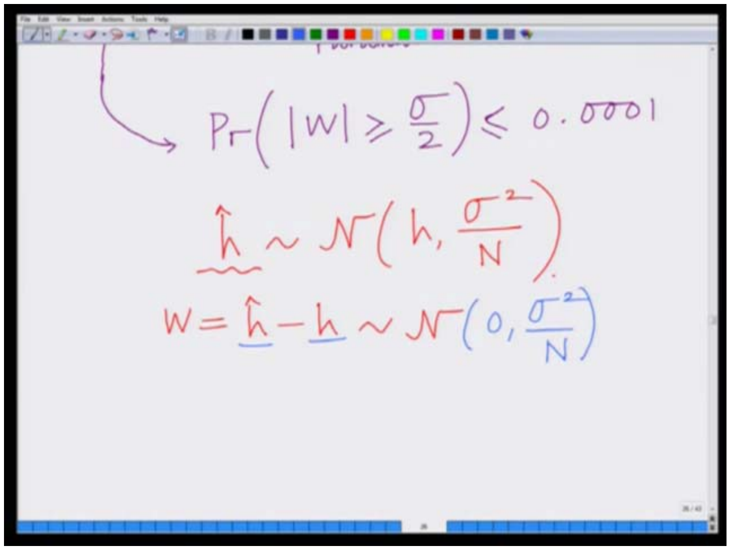

Remember, yesterday we derived two important properties of the Maximum Likelihood Estimate in the previous module. That is, we said h hat is an unbiased estimate which means expected value of h hat is h, that is h hat is Gaussian distributed with mean h and it has a variance equal to Sigma square divided by N.

And therefore now if you look at w which is basically from h hat which you from h hat you subtract h which is the mean therefore, naturally this will become a 0 mean random variable. So w is Gaussian distributed with mean 0 and variance Sigma square divided by N. w is the estimation error which is h hat - h. So from h hat you are subtracting the mean.

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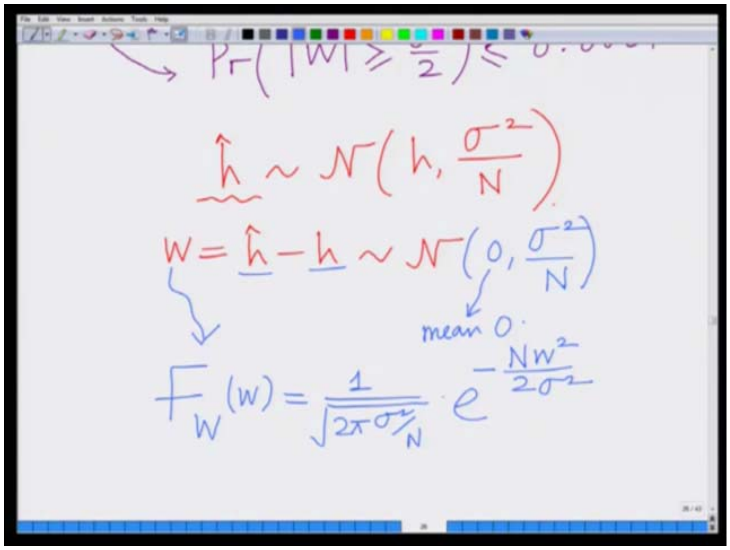

Therefore, w is Gaussian distributed with mean 0 and variance that is if you subtract the mean from a random variable, you get 0 mean random variables say for w which is the estimation error is 0 mean and has a variance Sigma square divided by N.

This is Gaussian in nature, so the probability density function f w of w is given as 1 over square root of 2 pie times the variance that is the Sigma square divided by N e raise to - N w square divided by 2 Sigma square.

So the Probability Density Function of the estimation error is Gaussian with mean 0, variance Sigma square divided by N and therefore, the Probability Density Function 1 over square root 2 pie Sigma square divided by N e raise to - N w square divided by 2 Sigma square.

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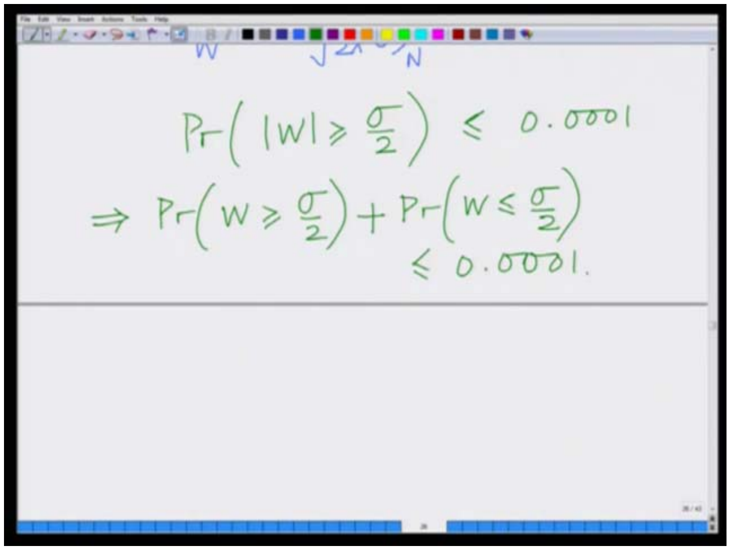

And now therefore we are asking what is the probability, the question that we are asking is what is the probability that mode w is greater than or equal to Sigma by 2 should be less than or equal to 0.001.

Which implies that the probability w is greater than or equal to Sigma by 2 + the probability w is less than or equal to Sigma by 2, should be less than or equal to 0.001 because magnitude of w is greater than or equal to Sigma 2 when w is greater than or equal to Sigma by 2 or w is less than or equal to sorry there should be a - Sigma by 2 over here.

And now each of these quantities, probability of w is greater than or equal to Sigma by 2.

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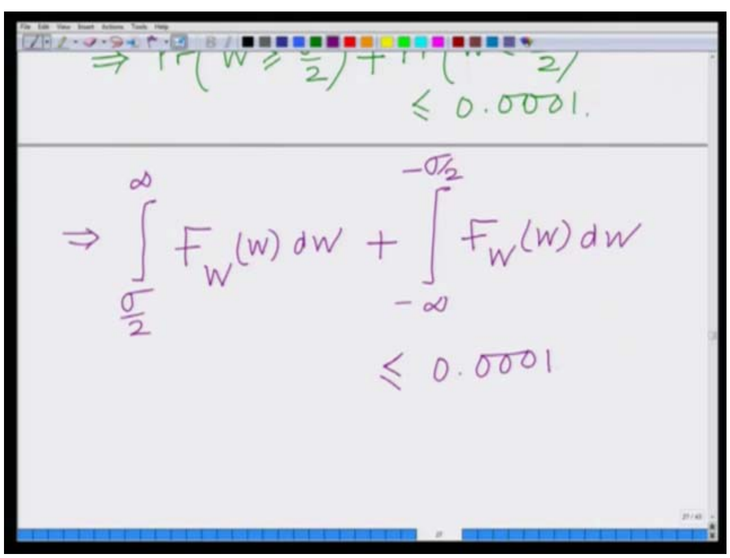

This is nothing but integral, the Probability Density Function of w between Sigma by 2 to infinity, that is as of w W integral between 0 to infinity + the probability that it is less than or equal to - Sigma by 2 is integral - infinity to - Sigma by 2.

The Probability Density Function of w which we are saying should be less than or equal to 0.0001. Now observe that this Probability Density Function is an even function therefore, we have f of w of w is basically equal to f of w of - W, which means this integral both these integrals are equal. The integral from Sigma by 2 to infinity is equal to the integral from - infinity to - Sigma by 2.

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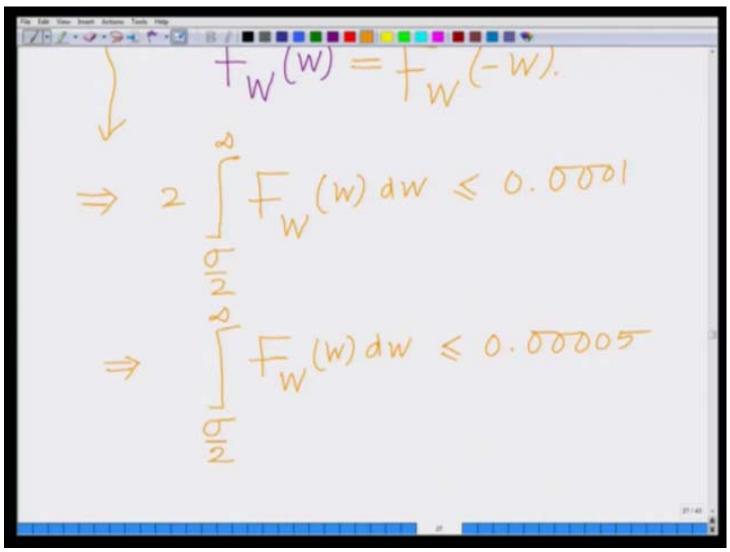

Therefore, this implies basically that twice the integral from Sigma by 2 to infinity of f of w W d w less than or equal to 0.0001 which basically implies that the integral from Sigma by 2 infinity f w of w d w less than or equal to 0.0005 that is 0.0001 divided by 2. And therefore what we have derived is that we have derived the equivalent condition in terms of the estimation error W.

We have said, for the probability that basically the estimation that the h hat should lie within a Sigma by 2 radius of the true parameter h with probability greater than or equal to 99.99% that is 0.9999 that is equivalent to the condition that the estimation error should lie between that is equal to their that is equal to the probability that the magnitude of the estimation error should lie outside.

That it should be greater than or equal to Sigma by 2 with probability less than or equal to 0.0001 which is basically reduced to this integral condition that is the integral evaluated between Sigma by 2 to infinity f of w d w should be less than or equal to 0.00005 that is the condition that we have derived.

Now let us simplify this integral, we know the Probability Density Function, we are given the Probability Density Function lets us substitute this Probability Density Function of this random variable w which is the estimation and we know that we have derived here Probability Density Function.

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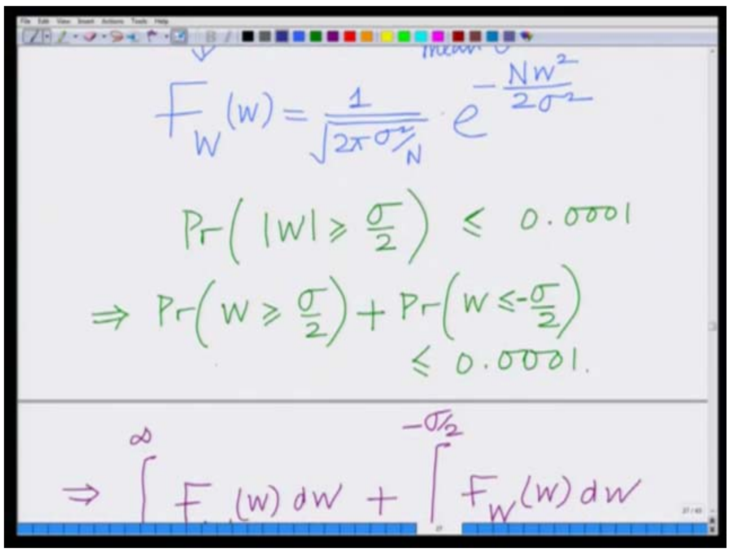

That is 1 over square root of 2 pie Sigma square divided by N e raise to - N W square divided by 2 Sigma square.

Therefore if we substitute this over here.

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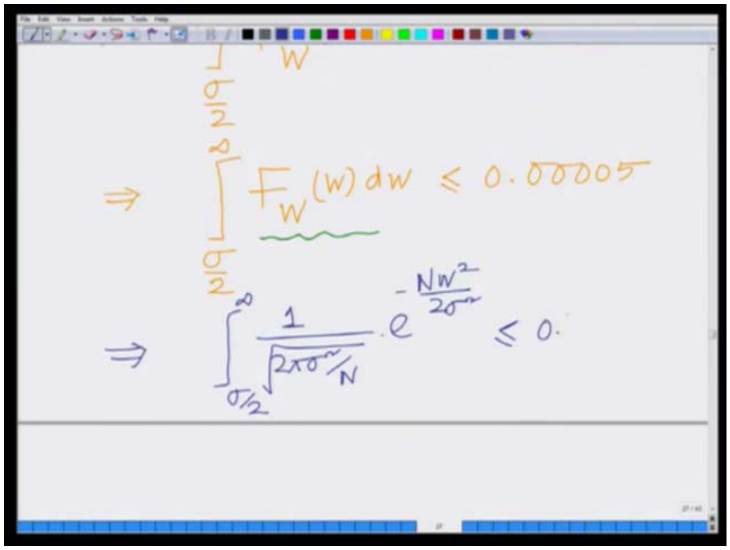

We have the condition Sigma by 2 to infinity 1 over, we have the condition Sigma by 2 infinity integral Sigma by 2 to infinity 1 over 2 pie Sigma square by N e raise to - N w square divided by 2 sigma square less than or equal to 0.00005. Now let us use the substitution

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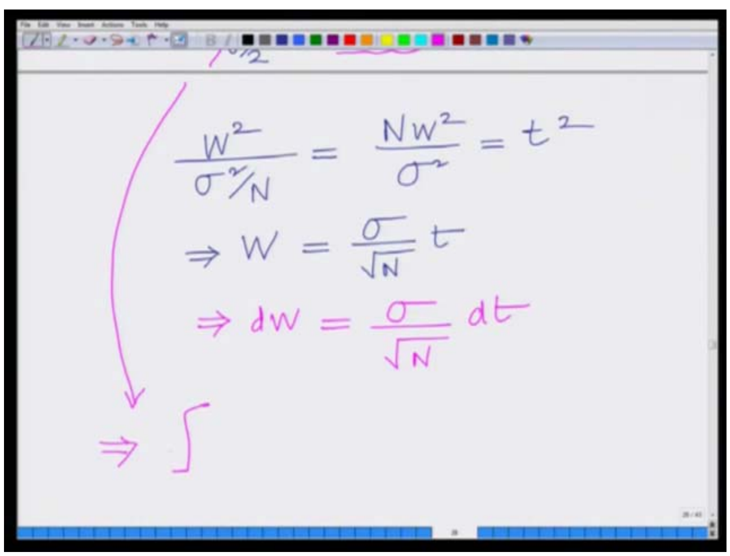

Now let us use the substitution W square divided by Sigma square divided by N equals N w square divided by Sigma square equals t square, which basically implies that w is equal to Sigma divided by square root of N t, which also basically implies that your d w equals Sigma divided by square root of N d t.

And now if you substitute this in this integral, we have, you can see the limits; this integral condition becomes integral Sigma by 2 w.

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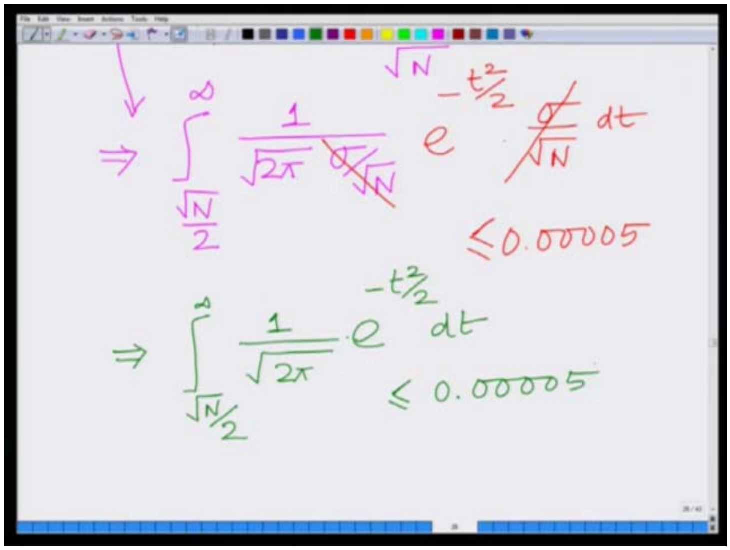

We have t is basically w divided by Sigma divided by square root of N, so this becomes square root of N divided by 2 to infinity. Now f 1 over 2 pie Sigma square 1 over or 1 over 2 pie.

If I bring the Sigma square over N out that is Sigma divided by square root of N times e raise to of course N w square divided by Sigma square, this 2 square so e raise to - t square divided by 2 and d t is Sigma divided by d w Sigma divided by square root of N d t.

And therefore, we have this Sigma divided by square root of n cancelling and what we have is basically this integral should be less than or equal to 0.0 or this integral should be less than or equal to 0.00005 and now if you look at this integral, we have this integral, square root of N over 2 to infinity 1 over square root of 2 pie e raise to - t square by 2 d t or equal to 0.00005 and therefore now if you look at this.

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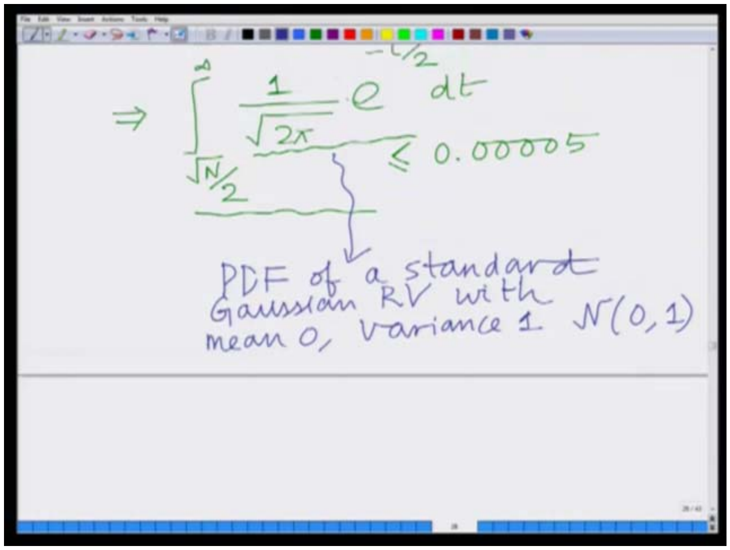

This is a well-known standard integral, so what we have is this condition that is integral square root of N over 2 to infinity 1 over square root of 2 pie e raise to - t square by 2 d t should be less than or equal to 0.00005. Now this integral is a standard integral, if you look at this Probability Density Function that is 1 over square root of 2 pie e raise to - t square divided by 2.

That is the Probability Density Function of a standard normal variable or standard Gaussian variable with mean 0 and variance 1. So if you look at this, this one over 2 pie e raise to - t square by 2, this is the PDF of a standard. This is a PDF of a standard currency and RV with mean 0 variance 1 that is N 0, 1, which implies and therefore this integral has a standard definition.

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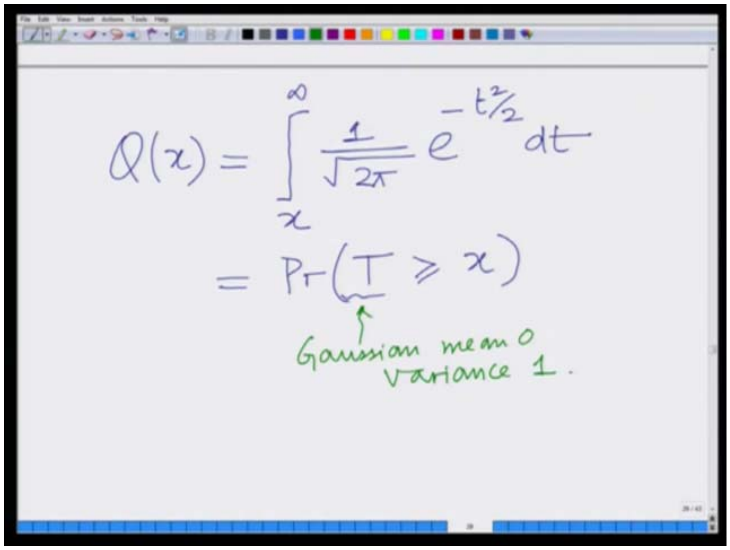

This integral is known as q of x. That is q of x is defined as a Gaussian q function, which is integral x to infinity 1 over square root of 2 pie e raise to - t square by 2 d t, which is the probability which is basically the probability that your standard Gaussian random variable t is greater than or equal to x.

This is the probability that your standard Gaussian random variable t which is Gaussian, mean 0 variance Gaussian with mean 0 and variance 1 is greater than or equal to x, this is the q function, q function is defined.

q of x is the probability that the Gaussian random variable with mean 0 and variance 1 is greater than or equal to x or basically lies between the interval x to infinity which is a integral from x to infinity 1 over square root of 2 pie e raise to - t square divided by 2 d t.

And therefore now if you look at this integral.

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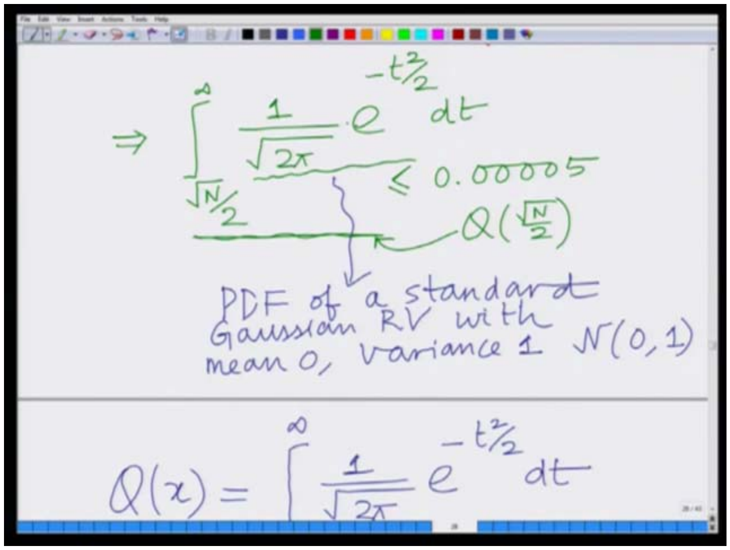

This integral over here is nothing but q of square root of n divided by 2, this is the q function, from here we are replacing x by square root of N by 2.

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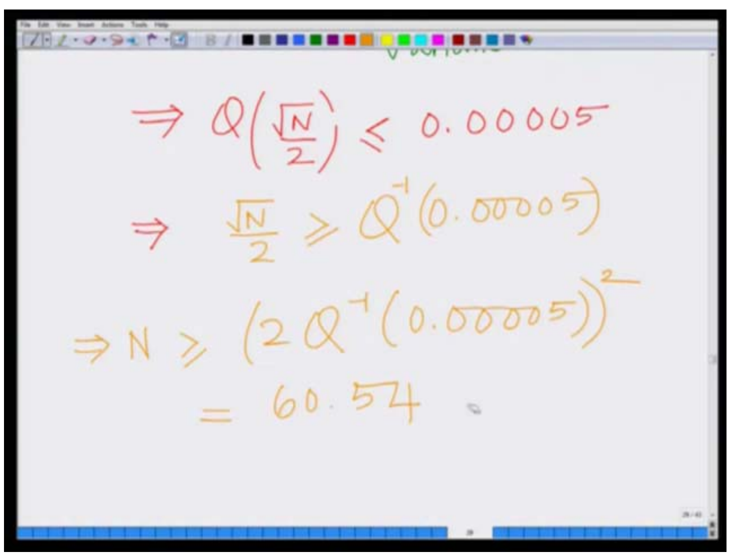

Therefore, we have this reduces to the standard condition that q of square root of n by 2 has to be less than or equal to 0.00005 which implies basically now I simplify it in terms of the Gaussian q function.

Square root of n by 2 is greater than or equal to, q function is a decreasing function is greater than or equal to q inverse of 0.00005 which implies basically N has to be greater than or equal to 2 q inverse 0.0005 the whole square is basically equal to 60 0.54 samples or observations.

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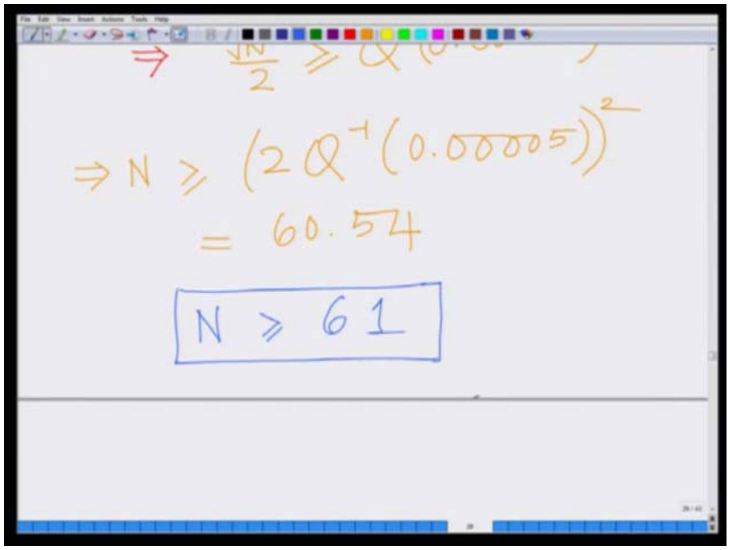

Therefore, what the results that we have is N, since N is an integer, N has to be greater than or equal to 61 observations and this is an interesting result. What we have demonstrated here is basically to have a reliability of 99.99%, right. So what is the definition of reliability, the reliability here is that we want the estimate h hat to lie within a Sigma by 2 radius of the true parameter h.

And what we are saying is if that has to happen with a probability greater than or greater than equal to 99.99% or a probability greater than or equal to 0.9999, then the minimum number of samples capital N that is required is basically your 61 sample. 61 samples are required with is the minimum number of sample that is required.

And now this the q function, this expression for this q function can be simplified a little bit further because there is no close form expression for the q function. One can approximate the q function as follows. So q of x can be approximated which we can will employ frequently, - half e to the power of - half x square, which means, so this is an approximation for your q function.

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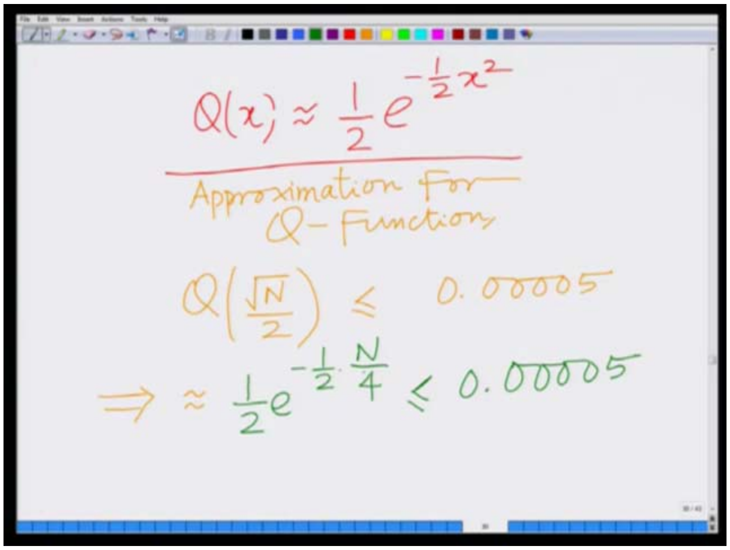

So this is the approximation for the q function. So this is the approximation for the q function. And therefore now, one can also use this approximation, so q or square root of N by 2 less than or equal to 0.00005. Now q of square root of 2, I approximate it, which implies approximation half a power - half square, that is half e raise to - square root of N by to whole square.

That is N by basically 4 must be less than or equal to 0.00005 which basically implies that… Taking log on both sides.

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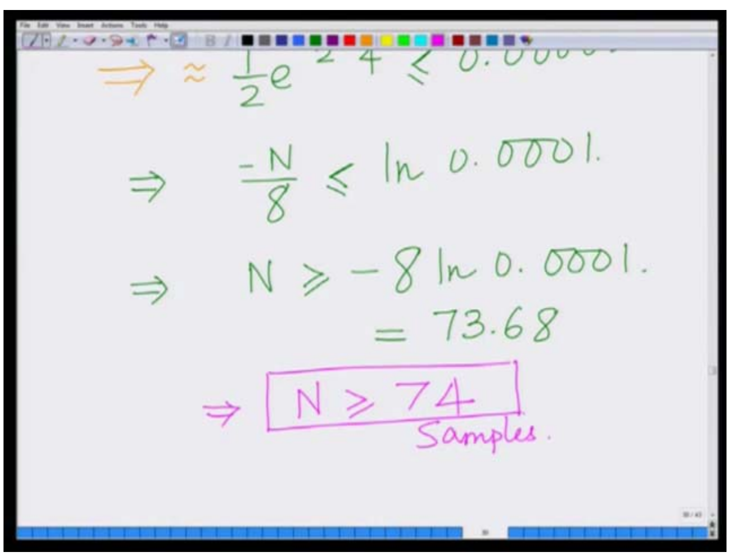

-N divided by 8 less than or equal to log to the base e or log natural of twice 0.00005, which is 0.0001. Which means, it basically implies your N is greater than or equal to - 8 because there is a negative sign, the equality gets reversed.

N is greater or equal to - 8 log natural 0.0001 and this is basically equal to 73.68 and therefore since N has to be an integer, this implies basically that N has to be greater than or equal to 74. N 2 a greater than or equal to 74 samples which is again an interesting result.

So we have used this approximate, we have got this approximate value of N using the approximation for the q function which we have also employed frequently in this course because there is no close form expression for the q function and we will have to use the q function quite frequently to calculate the reliability of the estimate.

So what we have illustrated now is we have illustrated an interesting example. For instance, in the context of wireless sensor network, we have seen that a sensor node can keep receiving observations yeah and based on these observations we compute the likelihood function. From the likelihood function we compute the Maximum Likelihood Estimate. Maximum Likelihood Estimate is unbiased, it is Gaussian, it has variance Sigma Square by N.

Now, one can a now, we also said in the previous module that because the estimate h hat is random, one can never be certain that it is not necessarily equal to h. In fact, it is not equal; because it is Gaussian it is equal. The probability that it is exactly equal to the true parameter h is 0. Yeah, so what we can say is, we have to come up with other notions of reliability and that is what we have tried to define in today's module.

How can we characterise, how accurate this random or this random estimate h hat is. And one measure of reliability is that we said, it has to lie within a Sigma by 2 radius of h with a probability that is greater than or equal to 99.99% or 0.9999. And therefore, what is the sample, number of samples N that is required for that.

We said the number of samples that is required for that is approximately of the order of 60 to 70 samples that you require. So although, h is random we said in the previous module that as the number of samples increases, the variance of the estimate progressively decreases and therefore, the sample keeps, the sample lies closer.

The probability that sample lies close to the true estimate keeps increasing. And now we have used this property to calculate what is to characterise precisely, what is the number of samples required to achieve a certain degree of reliability.

So this is an example, an interesting example which relies which relates this N to the reliability of the Maximum Likelihood Estimate. So let us stop this module here, we will continue with other aspects in subsequent modules. Thank you very much.