Estimation for Wireless Communication –MIMO/OFDM on Cellular and Sensor Networks
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Lecture Number 38
Sequential Estimation of Wireless Channel Coefficient - Estimate and Variance Update Equations

Hello, welcome to another module in this massive open online course on Estimation for Wireless Communications. So we are currently looking at sequential estimation in which we are continuously updating the estimate at every time instant and we are looking at it in the context of wireless channel estimation, correct. So we are looking at sequential estimation, where we are continuously updating the estimate.

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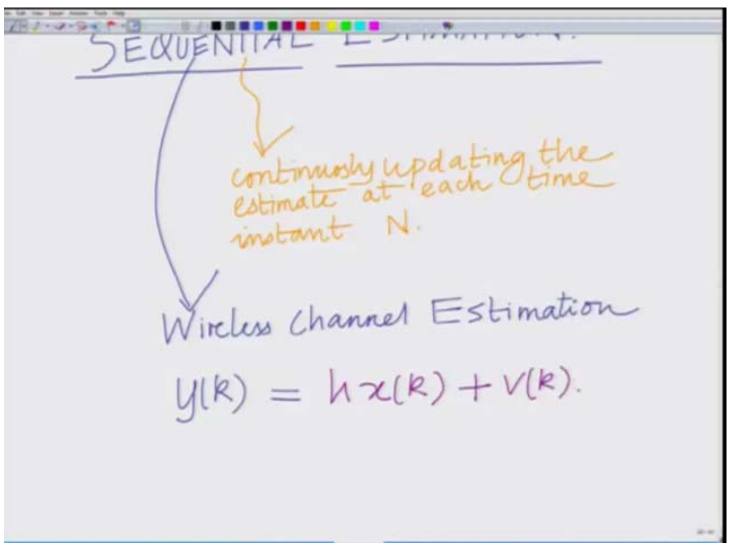

We are continuously updating, not recomputing, we are updating the estimate at every time instant N or at each time instant N, correct. So we are computing the estimate h hat N at time instant N h hat N + 1 at time instant N + 1, h hat N + 2 at time instant N + 2 and so on. And the important thing to realise here is that we are not recomputing the estimate at every time instant.

But using the previous that is the latest estimate that is available and simply updating it at every time instant, okay. That is important as we have said in several practical applications because we cannot recompute the estimate at every time instant, right. And we are looking at this in the context of wireless channel; we are looking at sequential estimation in the context of our wireless channel estimation problem.

Where we have y k the received symbol equals h times x k + v k. And this h is the fading Channel coefficient which has to be estimated, this is your unknown fading Channel this is your unknown fading Channel coefficient which has to be estimated and we are considering that transmission of pilot symbols. Remember, we said x k, this is your pilot symbol.

We are considering the transmission of N pilot symbols that is capital N pilot symbols.

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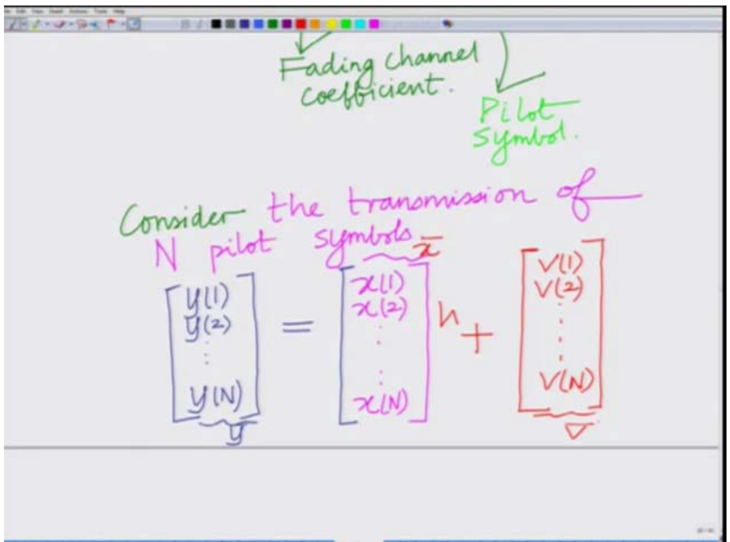

Consider the transmission of N pilot symbols and for this N pilot symbols remember, we said power output symbols y 1, y 2 up to y N can be represented as the vector y bar equals your pilot vector that is x 1, this is the N dimensional pilot vector x1, x 2 up to x N.

This is your pilot vector which is x bar times of course the fading Channel coefficient h + your noise vector v 1 consisting of noise samples v 1, v 2 up to v N, this is your noise vector v bar, as a result we have the vector model which is y bar equals x bar h + v bar.

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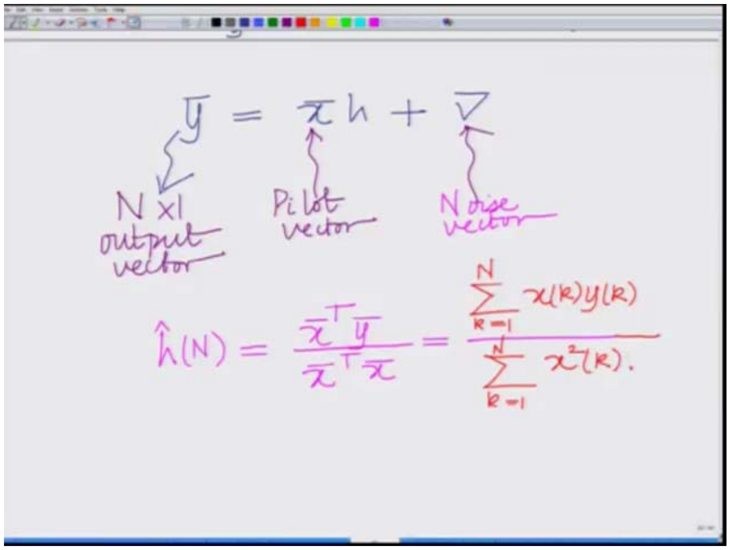

This is your N dimensional output vector to, correct. Y bar is your N cross 1; this is your N cross 1 output vector.

This is the N cross 1 pilot vector, h is the fading Channel coefficient and v bar is your noise vector also your N dimensional noise vector, correct. And now we know that remember we have to compute, we compute the estimate, this is the estimate at time instant N because we have received N pilot symbols x 1, x 2 up to x N and we can compute the estimate h hat of N at time instant N as we already know.

That is h hat of N equals x bar transpose y bar divided by x bar transpose x bar, which is equal to basically your submission k equal to 1 to N that is x k y k divided by submission k equal to 1 to N x square of k. This is h hat of N, this is the estimate at time instant N, this is the estimate at time instant N corresponding to N received pilot symbols and also the variance p of N at time instant remember, this is Sigma Square.

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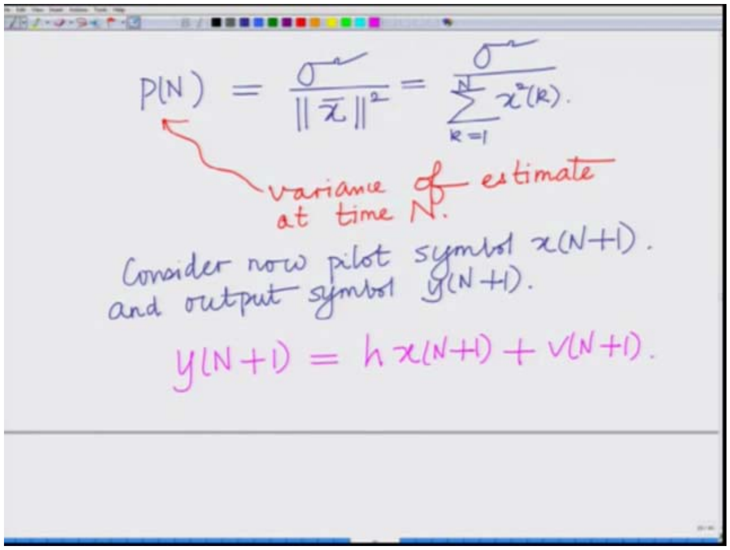

We already derived the expression for it divided by norm x bar square Sigma Square divided by norm x bar square which is Sigma Square divided by submission k equal to 1 to N x square of k. And what is p of N; this is the variance of the estimate that is variance of the estimate or variance of variance of the estimation error basically, we can think of this as the variance of the estimation error or basically, the mean square error.

This is your variance of estimate at time and basically this is also the mean square error at time N okay. Now what we said is we have a new pilot symbol transmitted small x of N - 1 at time instant N small x of N + 1 at time instant N + 1 and corresponding to that we have the observed output symbol small y of N + 1 at time N + 1.

You want to use this new information to update the channel estimate h hat of N at time instant N to h hat of N + 1 at time instant N + 1 okay. So naturally what we have now, consider now the pilot symbol x N + 1 and output symbol y N + 1 and corresponding to that we have y N + 1 equals h times x N + 1 + v N + 1 okay, this is at time instant N + 1 and therefore the estimate h hat N at time instant N + 1.

One way to compute the estimate at time instant N + 1 look at this, I can simply replace this N, one simple way to simply replace this N by N + 1.

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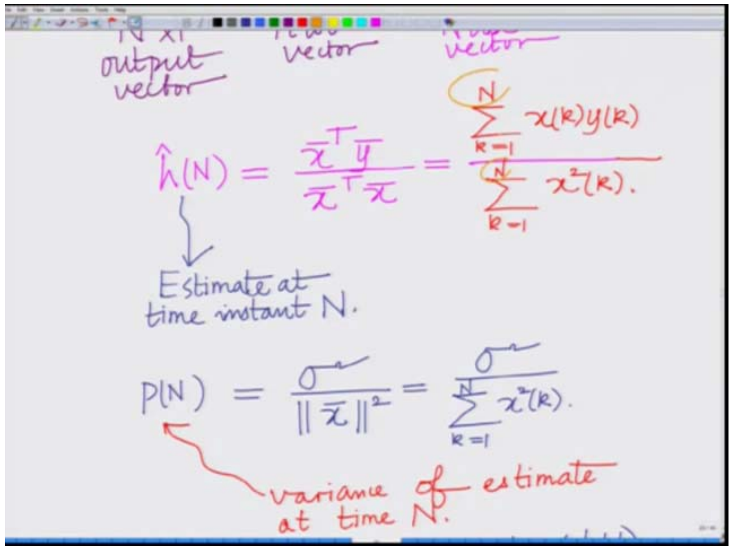

And therefore now I have to recompute the estimate, all I have to do is I have k equal to 1 to N + 1 x k y k divided by submission k equal to 1 to N + 1 x square of k okay, so this is your estimate.

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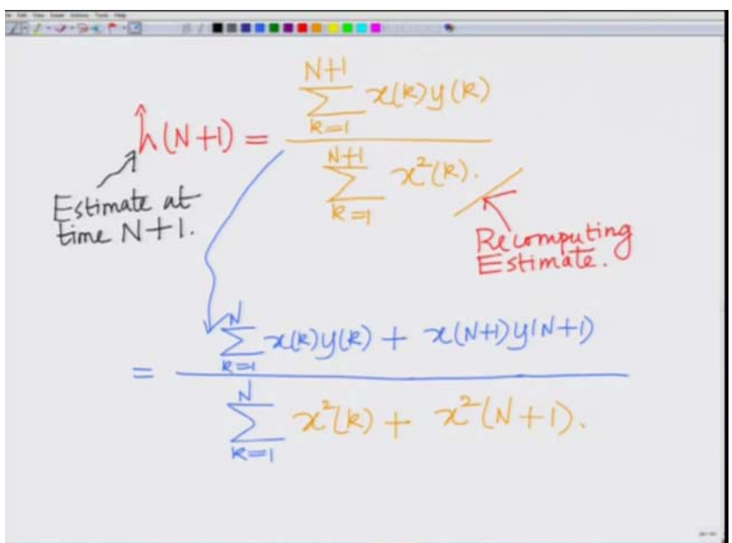

However, remember what we said this is basically recomputing the, this is simply recomputing the estimate, all right. We are not using the previous estimate h hat N and the variance p of N right. We do not want to do it because recomputing the estimate is simply increases the complexity at every time instant physically, you are again starting from the beginning and you are recomputing.

Basically, starting from scratch and recomputing the entire estimate and that is what we do not want to do, we want to simply update the estimate and that is the procedure that we are going to describe next. So now observe that, I can write this estimate at time instant N + 1 as follows. I can write this as submission; now let us write this as submission k equal to 1 to N x k y k + x N + 1.

That is separating the contribution of the term N + 1 divided by submission k equal to 1 to N of x square k + again X square of N + 1. By the way this is the estimate at time instant N + 1, h hat of N + 1 lest we not forget this is the estimate at time instant N + 1. And I am writing this as submission k equal to 1 to N x k y k + x N + 1 y N + 1 divided by submission k equal to 1 to N x square k + x square N + 1.

And now if you observe this term over here that a submission k equal to 1 to N x k y k look at this.

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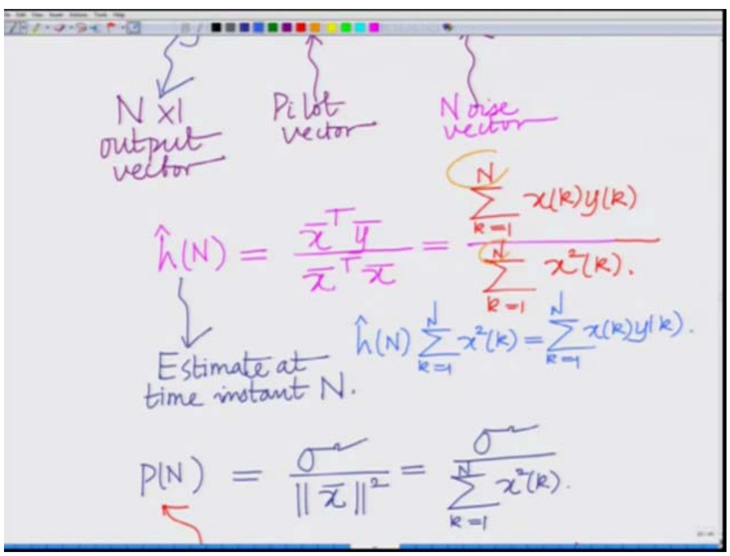

This is something that we already have that is submission k equal to 1 to N x k y k, that is equal to so we have from this relation if you observe what we have here is basically h hat of N times submission k equal to 1 to N x square of k is basically your submission k equal to 1 to N x k y k and this is what I am going to substitute over below.

That is submission k equal to 1 x k y k equals h hat N into submission k equal to 1 to N x square of k, so substituting that what I have over here is basically.

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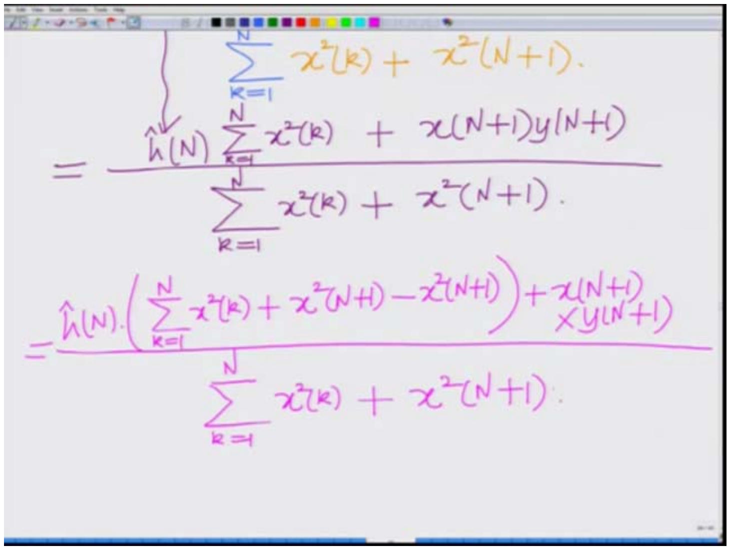

This quantity here is your h hat of N times submission k equal to 1 to N x square of k + the rest stays as it is, x N + 1, y N + 1 divided by k equals 1 to N x square of k + x square of N + 1. And now what I am going to do is from this factor over here.

I am going to add and subtract x square of N + 1, so that basically you can see what I am going to do now, it is very simple, it is not complicated at all. I am going to simply write this as h hat of N times well, submission k equal to 1 to N x square of k + x square of N + 1 - x square of N + 1 that is basically adding and subtracting x square of N + 1 and the rest remains as it is.

That is x N + 1 times y N + 1 divided by submission k equal to 1 to N x square of k + x square of N + 1 that remains the same and now if you look at this, if you look at this part over here, can see that this part is that submission x square of k + x square of N + 1 is same as the denominator, so that cancels and what we have is now if you simplify it a little patiently, what you have is h hat of N + now I am going to write the rest of the term.

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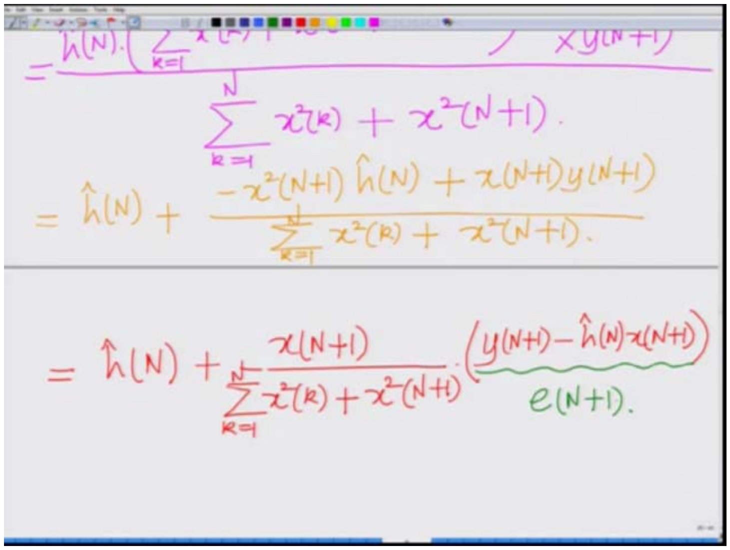

That is, - x square N + 1 into h hat of N + x N + 1 times y N + 1 divided by submission k equal to 1 x square of k + x square of N + 1, okay. So now we have simplified that and now I am going to simplify it more, just write it in particularly, so I have h hat of N remember that is the, so now if you look at this, now I have already h hat of N, this is the estimated time instant N + I can take x N - 1 common.

So I have x N - 1 divided by submission k equal to 1 to N x square of k + x square of N - 1 times. Now look at this, I can write this as y N + 1 - h hat of N times x N + 1. And therefore now if you look at it, this quantity is very interesting, I am going to call this as the error of the prediction error at time instant N + 1. What is this quantity; now look at this e N + 1, let us examine this a little closely.

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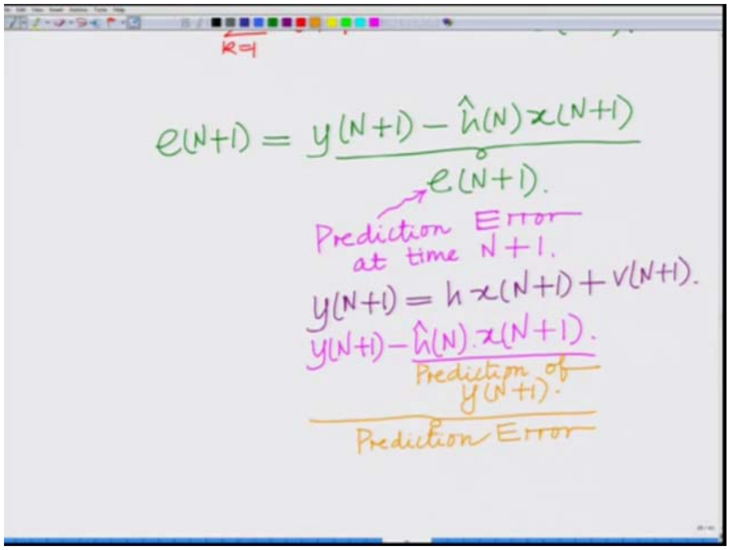

E N + 1 equals y N + 1 - h hat of N times x N + 1 and what I am calling this, I am calling this as your prediction error at time instant N + 1. This is e N + 1, this is your prediction error at time instant N + 1, the prediction error, this is the prediction error e N + 1 which is y of N + 1 - h hat of N times x N + 1, this we are calling the prediction error e of N + 1 at time instant N + 1.

And the reason is very straightforward, now if you look at this, we have at time instant N + 1, y N + 1 equals h times x N + 1 + your which is v N + 1. Now therefore if you look at this quantity y N + 1 - what you look at this is basically your h hat, so basically you have your estimate h hat of N at time instant N. Therefore, h hat N times x + 1, what is this quantity? This quantity is basically what your prediction of y N + 1.

So at time instant N you have your estimate h hat of N correct. So if you do not know y N + 1 that is the symbol the output symbol, what the output symbol y N + 1 is going to be, you simply do h hat of N times x of N + 1 that is going to give you an estimate of y N + 1 at time instant at time instant N + 1, right. So h hat of N times x N + 1 is the prediction of y N + 1 at time instant N + 1.

And therefore y N + 1 - h hat of N times x N + 1 is the prediction error. And that is basically the real reason, correct. So this h hat of N this what is this; this is your prediction of y N + 1 and therefore this is the by naturally the actual symbol - the prediction that is, y N + 1 - h hat of N times x N + 1, this is the prediction error at time instant N + 1, okay.

So this quantity y N + 1 - h hat N times x N + 1, this basically tells you this prediction error tells you how good your estimate h hat of N is at time instant N. Because if h hat of N is very accurate the estimate is very accurate, then you expect y N + 1 - h hat of N times x N + 1 to be very small because h hat of N times x N x N + 1 is a good predictor of y N + 1.

On the other hand, if the estimation error itself is large, then the prediction is going to be off right, so you are going to have a large prediction error and that is the key component in this sequential estimation scheme that is the prediction error and the update naturally depends on the prediction error because if the prediction error is large, that means you have a poor estimate, right.

Therefore you have to update by a larger quantity, but on the other hand if your prediction error is small, then that means you already have a very good estimate, so therefore you do not need to change your estimate by much. So you do not need to update by much or basically the magnitude of the update itself is going to be small.

And that is the important aspect that one has to realise in the sequential estimation, okay. Now let us look at this quantity over here, this other quantity. So we have looked at the prediction error, now let us look at this other quantity, now this other quantity, let us simplify this a little bit, so consider now the other quantity that is your x N + 1 divided by what is that?

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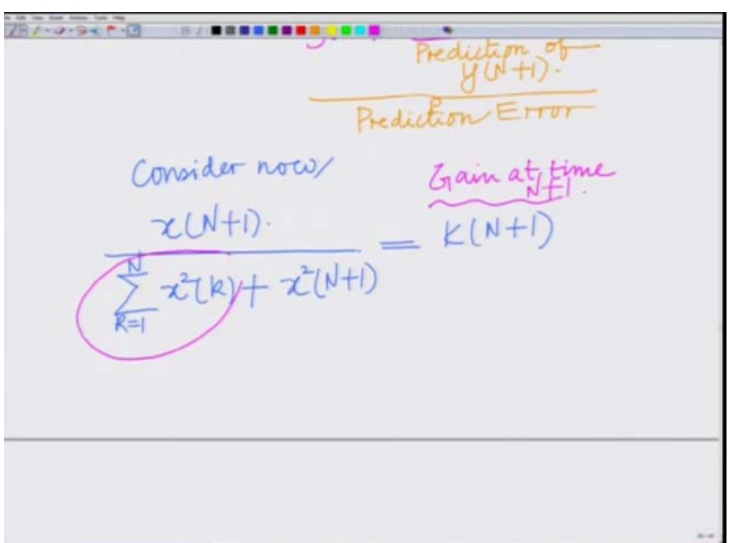

That is submission k equal to 1 to N x square of k + x square of N + 1.Let us call this as k N + 1 what is that? This is your gain, this is termed as the gain, and this is termed as the gain at time N + 1. And now look at this quantity submission k equal to 1 to N x square of k now submission k equal to 1 to N of x square of k if you remember if you go all the way back over here, if you look at the variance, now look at this, from this I can write.

Basically, submission k equal to 1 to N x square of k that is submission let me just write it over here.

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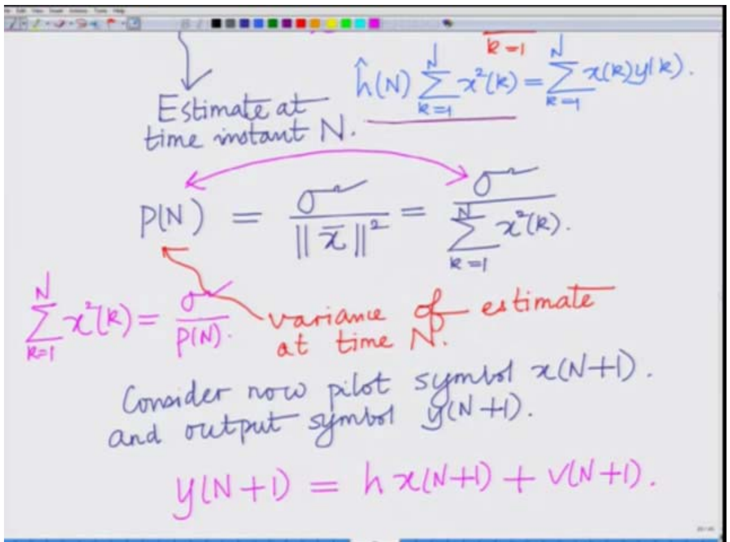

That your submission k equal to 1 to N x square of k is nothing but Sigma square times is nothing but Sigma square divided by p of N. From this relation over here, you can see that submission k equal to 1 to N in the denominator submission k equal to 1 to N x square k.

If I bring it to the left, then I have submission k equal to 1 to N x square k equals Sigma square over p N. Now I am going to substitute that over here. Remember p N is the variance at time instant N.

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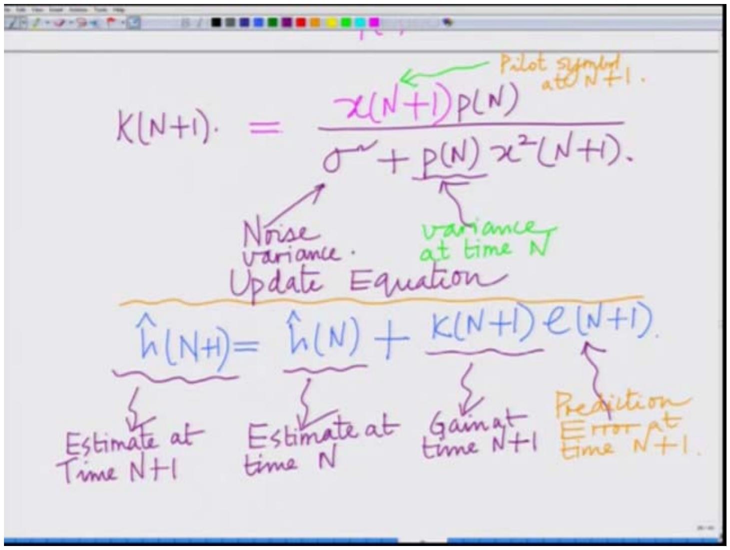

So this implies your k N + 1, the gain at time instant N + 1 is basically x N + 1 divided by Sigma square divided by p N the variance at time instant N + x square N + 1, which is equal to now simplify this as follows.

x N + 1 times your quantity p N divided by Sigma square + p N times x square of N + 1 and what is this? This is your gain k N + 1 at time instant N + 1 and therefore now we have explained expressed the gain as basically look at this, this depends only on what is the Sigma square? This is the noise variance of noise v, yeah. What is p N? This is variance at time instant N.

And of course x N + 1, this is the pilot symbol at N + 1 rather, pilot symbol at time N + 1. And therefore now we have this k N + 1 and now therefore, we have this quantity which is the gain k N + 1 this quantity which is the error e N + 1 and therefore the update remember, this is what is this? This is h hat of N + 1 therefore, the update h hat of N and that is important update h hat of N.

Your update is basically h hat of N equals or h hat of N + 1 equals remember h hat of N + your k N + 1, the gain at time instant N + 1 times e N + 1, the prediction error at time instant N + 1 and that is the important thing to remember. So what do we have? What is interesting about this equation, so this is the estimate at time instant N + 1. What is this? This is basically your estimate at time N.

This is your estimate at time N, this is the gain at time N + 1, this is the gain of your estimating filter at time N + 1 and this is an important quantity, this is what we have already seen e N + 1, this is the prediction error at N + 1 and therefore now what we have derived is basically we have derived an update equation for the estimated N + 1 without recomputing the estimate.

So h hat of N + 1 that is the estimated time N + 1 equals h hat of N which is the latest estimate that is estimate at time N + the gain k N + 1 at time instant N + 1 times e N + 1 which is the prediction error at time instant N + 1. And therefore now what we have is basically we have the update equation and this is important to realise, what we have? We have the update equation and we do not need to recompute.

And therefore it is not necessary to recompute the estimate, it is simply one can simply update the estimate, simply update the estimate update your estimate. Simply update your estimate, it is not necessary to recompute the estimate and if you look at this. You will realise that this depends only on h hat of N and k N + 1 and e N + 1. Now look at k N + 1.

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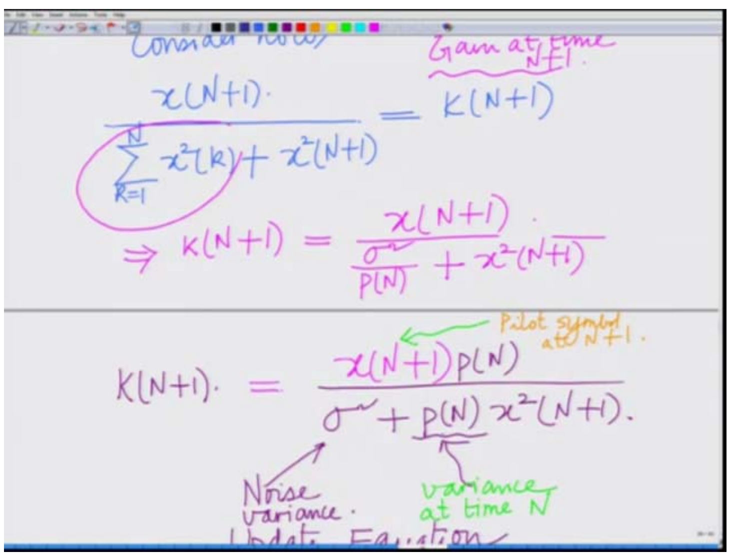

k N + 1 depends only on p N and x N + 1.Similarly, prediction error e N + 1 depends only on y N + 1 x N + 1 and h hat N, so basically what you can realise is that this entire equation, update equation for h hat N + 1 does not anywhere use the previous either pilot symbols small x 1, small x 2 up to small x N or the previous output symbols small y 1, small y 2 up to small y N okay. So all we are using is basically h hat N the estimate at time instant N, right.

P N, the variance at time instant N okay and y N + 1 that is the output at time instant N + 1 and x N + 1 which is the pilot symbol at time instant N + 1. So these are basically the only things that we are using. We are not recomputing the estimate basically, we are only updating the estimate and that is the important point to realise over here that we are simply updating the estimate okay.

And this is the update equation for the estimate, so at this update equation as we have said uses only.

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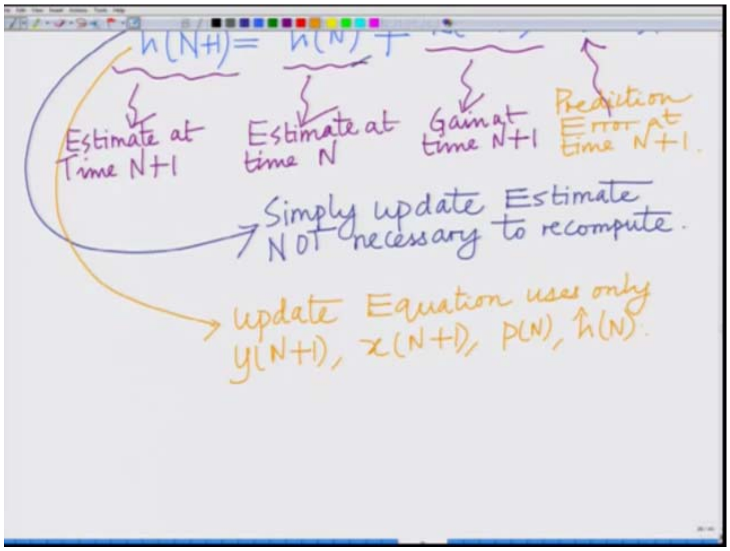

The update equation uses only your y N + 1, x N + 1, p N and h hat of N all right. So nowhere are we using the previous pilot symbols x 1, x 2 up to x N or the previous outputs y 1, y 2 up to y N.

So all the information that corresponds to the previous pilot symbols and the previous output symbols so as to speak is encapsulated, right is present in h hat of N and p of N, right. So in that sense we do not need to use the previous pilot symbols, we can simply use h hat of N and update the estimate to h hat of N + 1. Now other quantity that we obviously have to update is the variance that is p of N + 1 and that can also be simply calculated as follows.

So update for the update for the error variance, now the update for the error variance is basically now you can see p N + 1.

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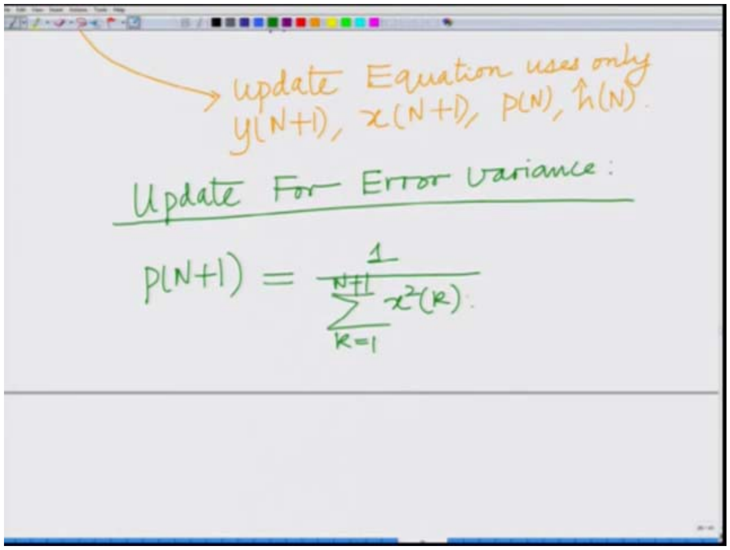

The variance is basically 1 over now let us look at again let us first recompute the estimate and then we will see how it can be updated similar to what we have done before, you can if you go way back you can see p N equals Sigma square divided by submission k equal to 1 to N x square of k.

So this N I can replace by N + 1 and I can get the variance that is the variance at time instant N + 1 that is p N + 1 equals 1 over submission k equal to 1 to N + 1 your x square of k, which I am going to again right it by splitting this into term corresponding to submission up to k.

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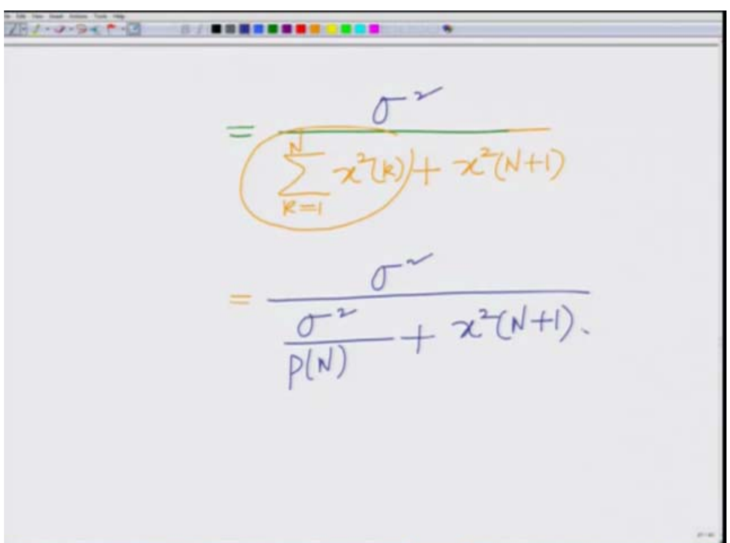

That is submission up to N that is k equal to 1 to N submission x square of k + term corresponding to N + 1 that is x square of N + 1.

In fact, let me just write this little clearly this is x square of k. Now we know that this quantity submission k equal to 1 to N x square of k, this is nothing but Sigma square divided by p N, that is what we have already, submission k equal to 1 to N x square k is Sigma square divided by p N, that is something that we have already seen. Therefore, now this is your 1 over Sigma square, 1 over.

I am sorry, this is not 1 over, this is Sigma square divided by the numerator there is Sigma square, so there is basically Sigma square divided by p N + x square of N + 1 and therefore now I can simplify this as.

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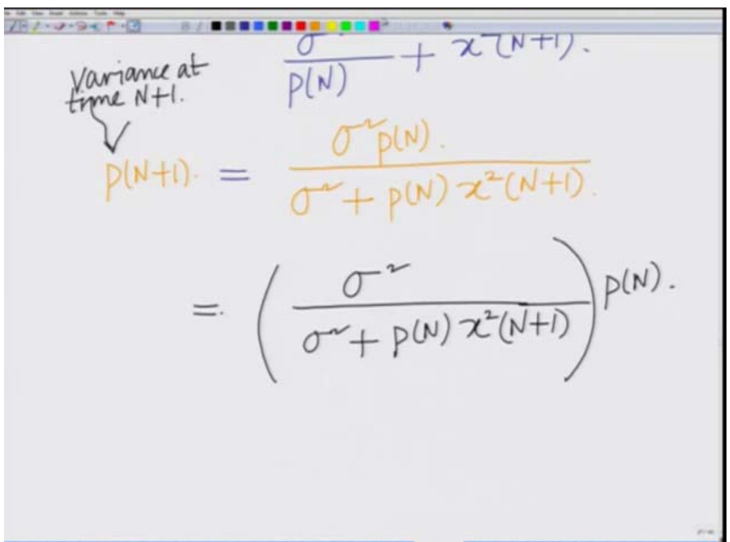

Basically Sigma square times p N divided by Sigma square + p N times x square N + 1, this is your p N + 1 and remember what if p N + 1; this is the variance at time N + 1.And what is this equal to; this is equal to well I can write this as, let me take this p N outside, so I can write this as Sigma square divided by Sigma square + p N into x square N + 1 times p N and what is this; this is basically now I can write this as.

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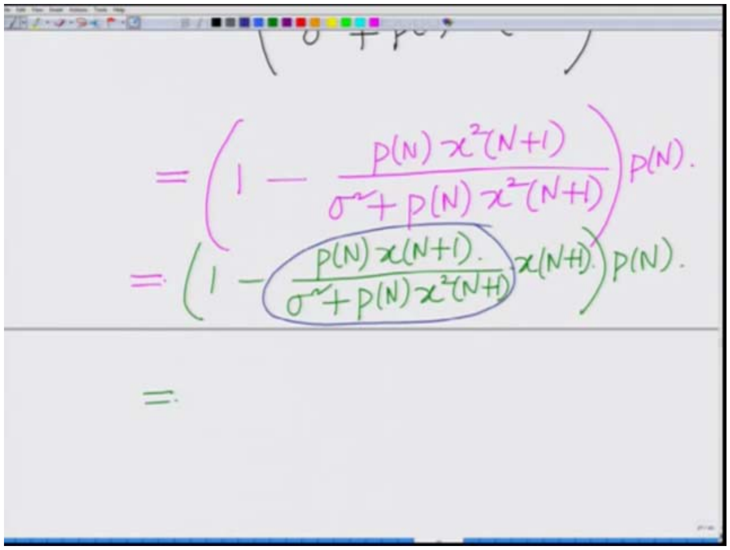

1 - p N into x square N + 1 divided by Sigma square + p N into x square N + 1 into p N, which is equal to, what is this equal to; this is equal to your 1 -, now let us look at this.

I have this quantity p N x N + 1 divided by Sigma square + p N x square N + 1 times x N + 1 times p N and now if you look at this, now look at this quantity p N into x N + 1 divided by p N, if you look at this quantity, this is your p N into x N + 1 divided by Sigma square plus p n into x square N + 1. If you go all the way up over here and you can see this quantity is nothing but your gain k N + 1.

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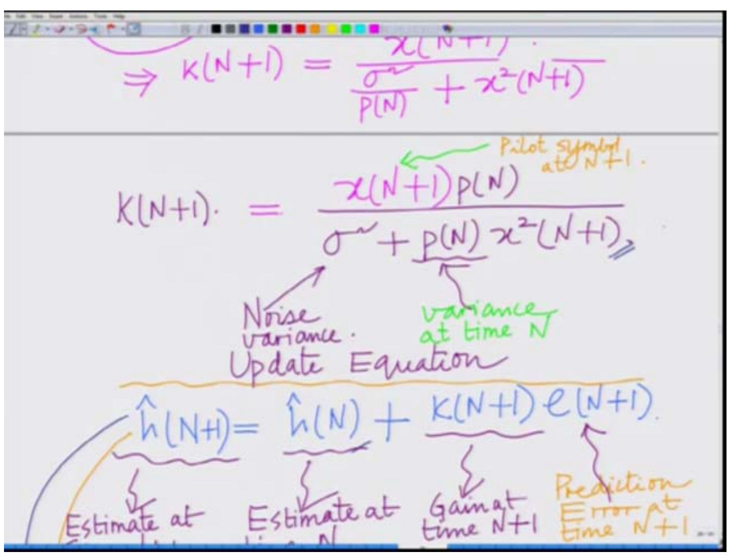

And therefore, now I can get it into a form.

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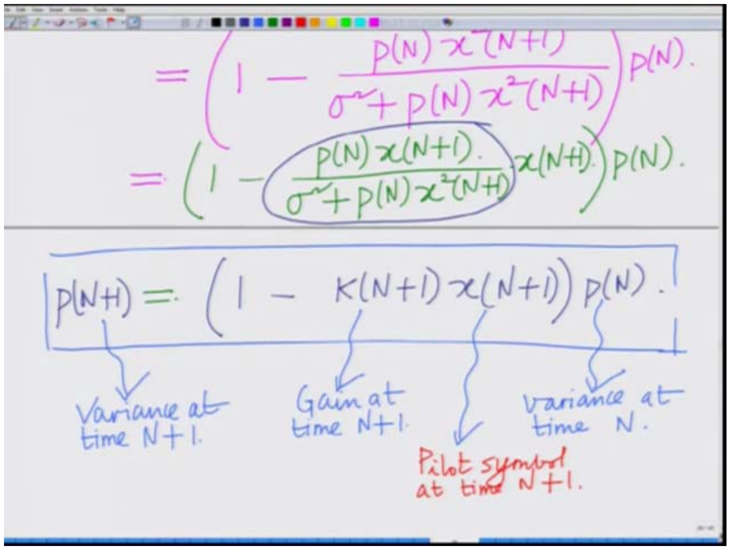

So I have p N + 1, the variance at time instant N + 1 is 1 - k N + 1 into x N + 1 into p N. And that is basically your variance update at, this is your variance at time instant N + 1, p N + 1 is variance at time instant N + 1, p N is variance at time N, k N + 1 is gain at time N + 1 and x N + 1 is of course this is your pilot symbol at time N + 1.

This is your pilot symbol at time N + 1. And therefore now you can once again update the variance p N + 1 corresponding to time instant N + 1, p N + 1 equals 1 - k N + 1 times x N + 1 into p N. Basically, the variance at time instant N. So basically, if you know the variance p N at time instant N and h hat N at time instant N, after observing y N + 1 that is knowing the pilot symbol x N + 1 at time instant N + 1.

And observing the output y N + 1 at time instant y N + 1, I can basically update the estimate from h hat N to h hat N + 1 okay that is what we have seen previously, h hat N + 1 equals h hat N estimated time instant N + the gain k N + 1 at time instant N + 1 times e N + 1, which is the prediction error at time instant e N + 1. And of course the variance at time instant N + 1 p N + 1, we are updating this as 1 - k N + 1 into x N + 1 times p N, all right.

So you update both the estimated time instant N + 1 and the variance at time instant N + 1 and then move forward again to time instant N + 2 using basically the symbol x N + 2 and output y N + 2. And this sequential, now this procedure sequential procedure of course we have illustrated it for N to N + 1, 1 similarly now repeat this at N + 2 and so on naturally, this is the sequential estimation procedure.

So basically what you have to keep in mind is basically the update equations for any sequential estimation procedure.

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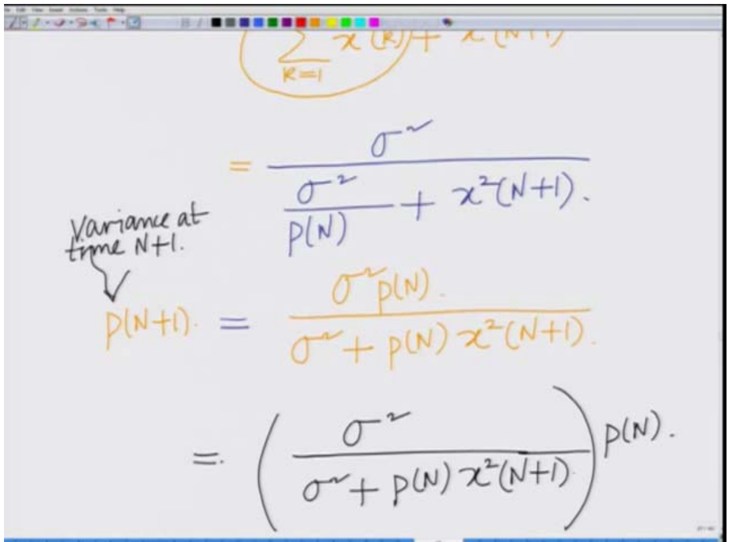

Basically, this is a very big area which ties to sequential estimation for instance for things such as the Kalman filter and so on, which basically estimates a quantity which is continuously varying and the structure is the same. Basically, you have an estimate; basically you have an update equation for the estimate at time N + 1. This is the update equation for the estimate.

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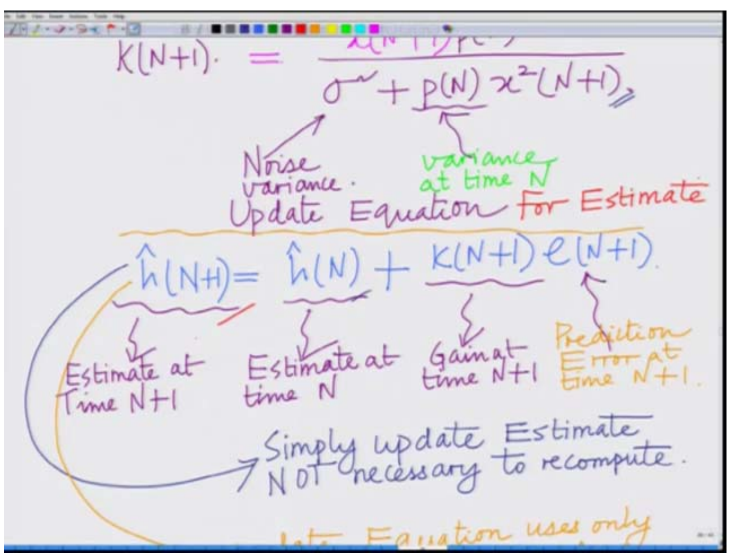

Let me write this update equation for estimate. So you have 2 update equations, you have the update equation for the estimate and then you have this update equation for the variance, so this is your update equation for the variance.

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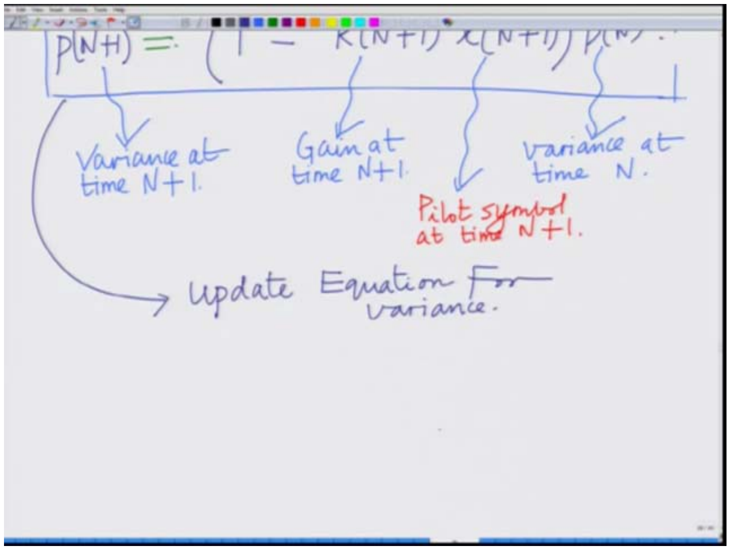

And put together, they constitute basically your sequential estimation procedure, okay. This is the update equation. This is the update equation for the variance and put together, so we have an update equation for the estimate, update equation for the variance and put together basically we have the net sequential estimation procedure that is very interesting.

So so what we have described in this module; basically we have completed the discussion of the sequential estimation procedure where we are not recomputing the estimated time instant, we are rather using the estimated time instant N that a h hat of N and the variance p of N at time instant N and using this and the corresponding pilot symbol and observation x N + 1 and y N + 1 at time instant N + 1, right.

And we are basically updating the estimate from h hat N to h hat N + 1, updating the variance from p of N to p of N + 1 and this is basically a sequential estimation procedure because this can be now sequentially repeated at time instant N + 2, N + 3 and so on without the necessity to recompute the estimate at every time instant, all right. So this module basically comprehensively describes the paradigm of sequential estimation, okay.

So we will stop this module over here and we will continue with an example of the sequential estimation procedure in the subsequent module, thank you very much.