Estimation for Wireless Communication –MIMO/OFDM on Cellular and Sensor Networks
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Lecture Number 03
Properties of Maximum Likelihood (ML) Estimate- Mean and Unbiasedness

Hello, welcome to another module in this massive open online course on estimation for wireless communication. So far we have looked at a simple model of a wireless sensor network, we have formulated the likelihood function and based on the likelihood function we have derived the maximum likelihood estimate of the unknown parameter.

So far what we have done is we have considered a simple wireless sensor network an example of a wireless, a scenario rather of a wireless sensor network.

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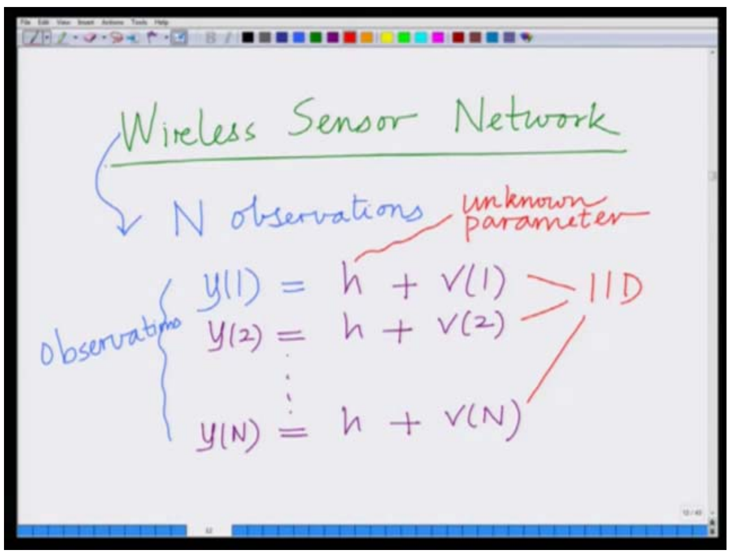

We have looked at N observations in this wireless sensor network, the observations are y 1 equals h + v 1, y 2 equals h + v 2, y N equals h + v N where these y 1, y 2, y N is are our observations, right?

h is the unknown, h is the unknown parameter and these v 1, v 2, v N, these are IID independent, so v 1, v 2, v N, these are IID that is Independent Identically Distributed Gaussian noise samples with means 0 and variance Sigma Square. These are IID Gaussian noise samples, all right.

So these are IID Gaussian noise samples with mean 0 and variance equal to Sigma Square. These are IID Gaussian noise samples with mean 0 and variance equal to Sigma Square and for this scenario we had derived the Likelihood Function. We had derived the Likelihood Function for this scenario. And we have shown that the maximum likelihood estimate is given by the sample mean.

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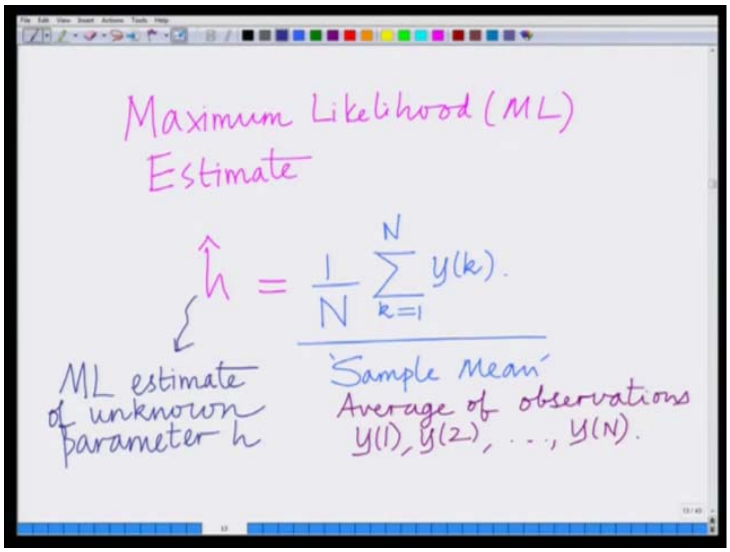

The ML estimate, the Maximum Likelihood Estimate or the ML estimate that is h hat, the Maximum Likelihood Estimate of the unknown parameter is basically one over N submission k equal to 1 to N y of k. That is, this is the also known as the, the sample mean or simply basically your average of observations y 1. This is the sample mean that is h hat which is the Maximum Likelihood Estimate.

What is this; this is the ML estimate of the unknown parameter h, which is given by this average that is, 1 over N submission k equals 1 to N y of k. That is the average of the observations y 1, y 2, y N are also basically the sample mean. That is the sample is y 1, y 2, y N. This is the mean or the average of the sample therefore, this is also known as the sample mean.

This is the Maximum Likelihood Estimate for the simple scenario, simple wireless sensor network scenario with N observations that we have considered so far of sensing an unknown parameter h which we said can either be the parameters such as the temperature, pressure, the humidity, et cetera yeah. So now let us examine the behavior or the properties of this maximum likelihood estimate.

So in starting this module, let us explore the properties. Let us explore the properties of the Maximum Likelihood Estimate.

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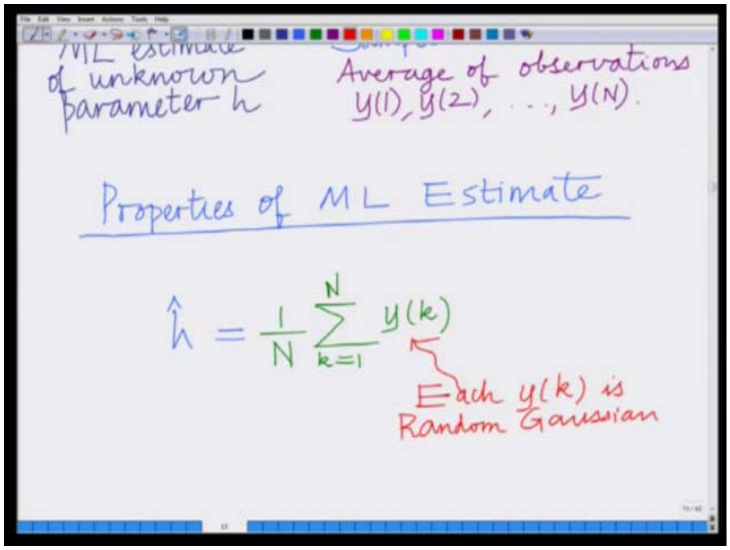

Now remember we have h hat equals 1 over N submission k equals 1 to N y k. And recall that each h k each y k observation y k is random in nature. In fact, each y k is random Gaussian if you remember the very first module, we had said that these ys; y 1, y 2 1 y n are noisy observations with mean given by the unknown parameter h and the variance given by the variance Sigma Square of the noise V k.

Therefore each y k, that is it is important to stress this aspect each y k is random in nature, in fact random Gaussian, which means h hat which is the sample mean or average of these y k is also in nature. h hat in fact h hat equals the average of these different observations y k, your h hat is the average of these different observations Y k, the Y is the observation, the noisy observations are themselves random in nature, therefore h hat is random.

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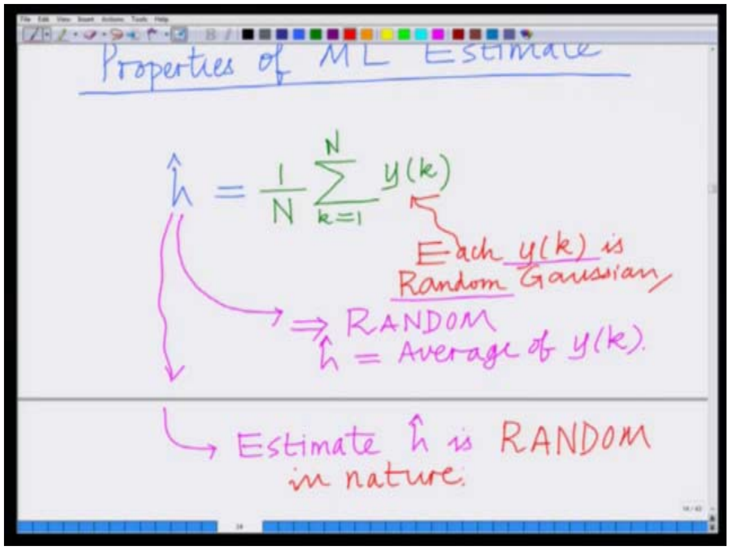

That is the estimate is random in nature, that is the first thing that we have to realise, that is this estimate, the estimate h hat is random that is, it is not a deterministic quantity, it is random in nature. Now since h hat is random in nature, it is not necessary that it is equal to true parameter h. Therefore, h hat is a random quantity therefore it is not always equal to the true parameter h.

Therefore, h hat is not necessarily equal to infact it is not necessarily equal to h.

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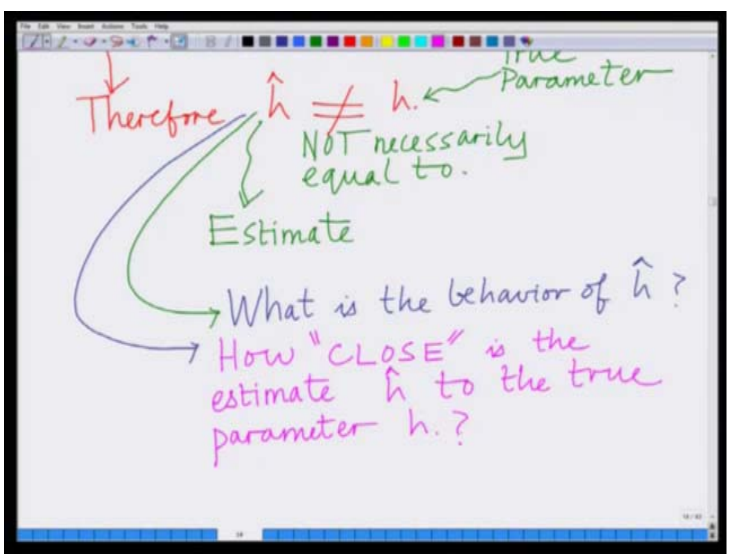

Therefore, the estimate this is the estimate h hat is the estimate h is the true parameter, the underlying the true parameter which is unknown, right. So we have the estimate h hat which is derived from the observations and then we have the true parameter that is the original parameter h which is unknown and which we are trying to measure.

And what we are saying is h hat is random, therefore it is not necessary that hat is actually equal to h. And therefore we want to find how good how good is this estimate h hat. What is the behaviour of this h hat, I mean what kind of properties does this h hat exhibit and how close it is to the true parameter h. Since it is not exactly equal to h, can we at least hope in some sense that it is close to the true parameter h.

So h hat is random, therefore what we want to do if we want to explore what is the behaviour of h or what is the behaviour of h hat. And in some sense we want to explore, we want to address this important question how close is h hat in what sense, how close is the estimate h hat to the true, how close is this estimate h to the true parameter h. Now in that sense let us start by computing the mean.

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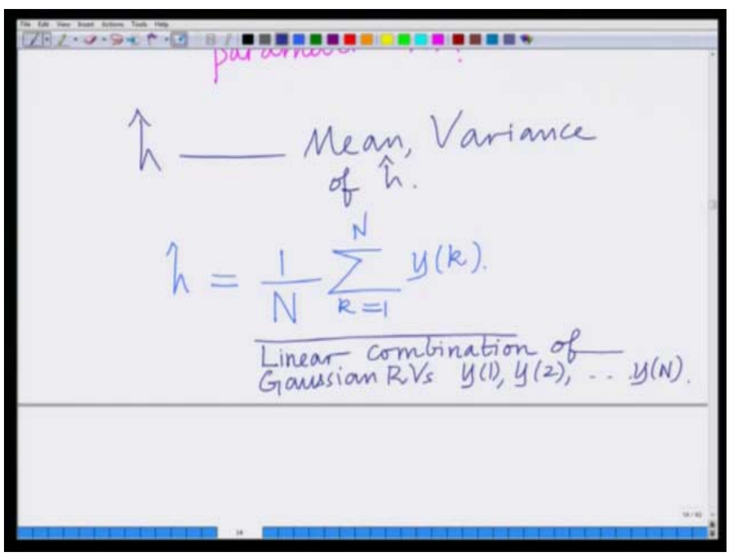

Let us start by computing the mean and variance of this estimate. So to address this quotient, to address what is the behaviour of this estimate h hat and also how close is this estimate h hat to the true value of the underlying parameter h. Let us start, let us start by computing the mean and variance of this random quantity of this estimate h hat which is random in nature. For that once again recall that h hat is the sample mean.

h hat equals 1 over n submission k equals 1 y k, this is the sample mean. Observe that this is the linear combination of Gaussian random variables, linear combination of Gaussian random variables y1, y 2 up to y n. Remember we started in the very first module itself. We said that y k each y k is Gaussian in nature because the noise V k is Gaussian in nature; you are shifting that by the constant mean that is h.

Therefore, each y k is also Gaussian in nature. And further what we have is this estimate h hat which is the sample mean h hat is the sample mean, therefore h hat is the linear combination of these observations y1, y 2 upto y k. Now since h hat is the linear combination of these Gaussian random variables y1, y 2 y k, h hat itself is Gaussian in nature because the linear combination of Gaussian random variables is Gaussian itself.

h hat is the linear combination of Gaussian random variables, h hat itself is Gaussian in nature. So this h hat is Gaussian in nature, since it is a linear combination of Gaussian random variables. So the first thing that we have established is that the maximum likelihood estimate h hat itself is a Gaussian random, it is a random variable and in fact it is Gaussian random variables, hence it is a linear combination of Gaussian random variables, okay.

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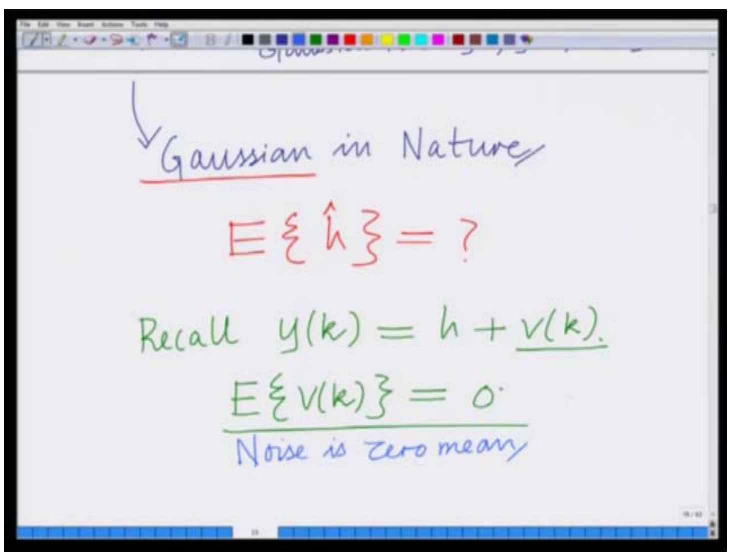

So h hat is Gaussian nature, so h hat is Gaussian in nature. Now what is the mean of h hat, now what is the main or the expected value of, what is the expected value of h hat? Now recall, for this now recall each y k each y k is h + V k, yeah. And further we have remember we assumed the noise to be 0 means. That is we have expected V k for each V k we have expected V k 0, this is because the noise is 0 mean, all right.

So we are going to use this property, remember in each observation V k, in each noisy observation in each our noisy observation y k, the noise V k has mean 0. Now let us look at h hat, h hat is in fact 1 over n submission k equal to 1 to n y k.

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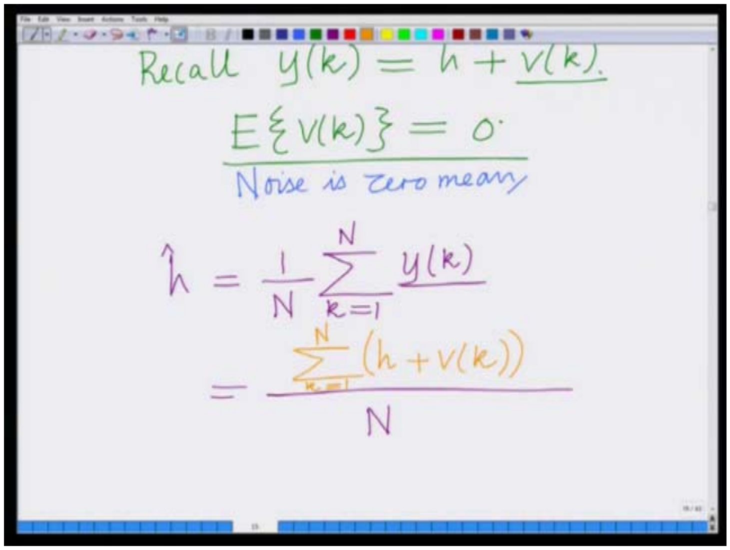

Now substituting this expression for small y k, remember each y k is h + v k, so this is submission k equal to 1 to n, h + v k which is equal to now divided by n.

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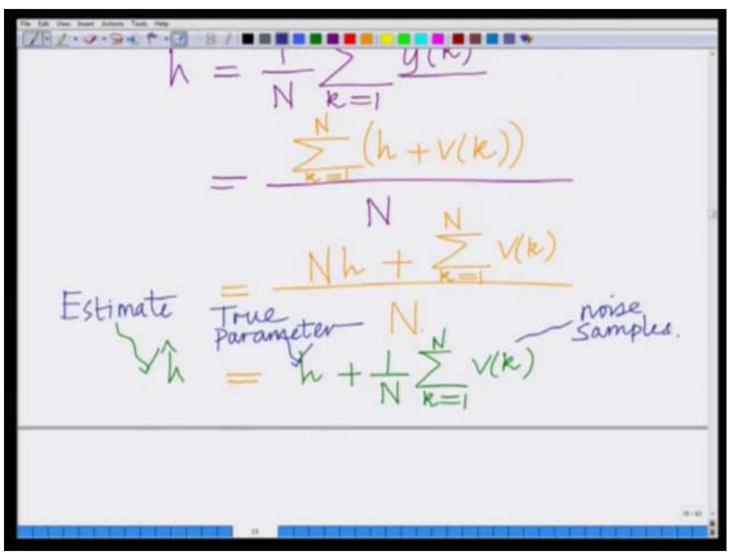

I have submission h + V k which is n times h + submission k equals 1 to n V k which cannot be simplified as h + submission 1 over n k equals 1 to n V k, this is my h hat. So I have simplified this expression for h hat, h hat equals that is the estimate. Remember this is h hat which is; let us not forget what these are. This is the estimate h hat, this is the true parameter h and these V ks these are the noise, the observation noise.

Such that each expected V k equal to 0. Therefore, now if we look at the average value of the estimate h hat or the expected value of the estimate h hat.

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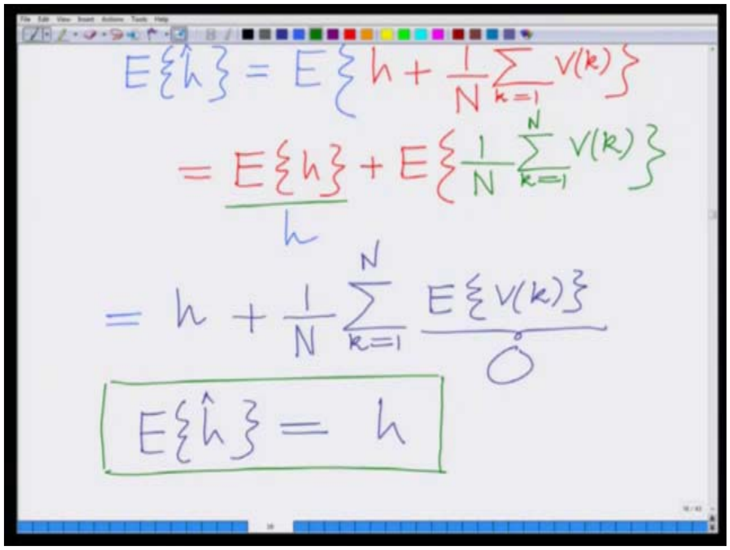

I have the expected value of h hat is given as the expected value of the expression above which is the expected value of h + 1 over n submission k equal to 1 to n V k which is equal to expected value of h + expected value of 1 over n submission k equal to 1 to n V k.

Now look at this, this is a constant, so the expected value of h is basically h, so this is equal to your h + 1 over n submission k equal to 1 to n expected value of each V k. Remember we said the noise is 0 mean, so expected value of each V k equals 0, therefore net what we have is that basically your expected value of h hat, that is your the average value of the estimate the expected value of h hat equals h and this is a very important property.

That is, even though what we are saying is even though your estimate h hat is random in nature that is, h hat is not necessarily always equal to h. However, it exhibits a very important property and the property that it exhibits is that on an average, that is if you look at expected value of h hat, that is the average of h hat is basically equal to the true parameter h and this is a very important property of the estimate.

That is it says that the mean of the estimate is equal to the true parameter h, such an estimator is known such an estimate is known as in “Unbiased estimate”. This is an important property, they says that basically the average value of the estimate. What does this signify; this signifies that average value of estimate equals true value of the parameter h.

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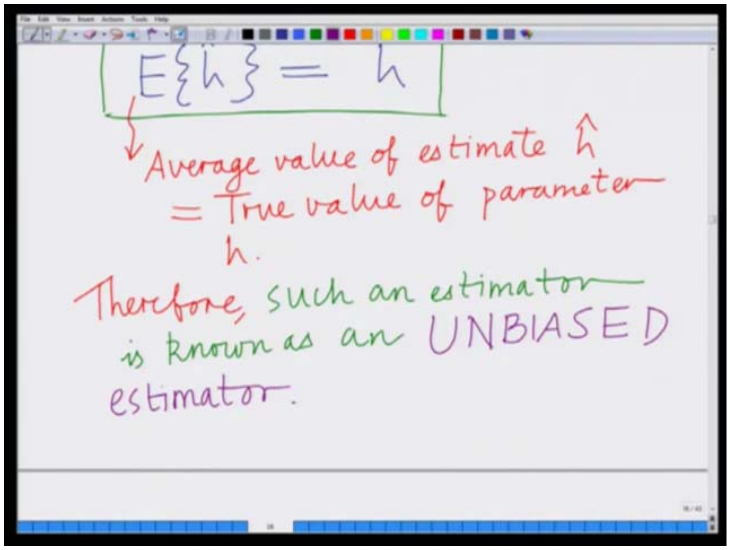

Therefore, such an estimator such an estimator is known as in is known as an Unbiased that is, it has no bias that is, it is no preference, it is an unbiased estimate or an unbiased. It is known as, such an estimator is basically known as basically known as an "Unbiased Estimator" that is, it yields in unbiased estimate.

Which means even though it is random in nature, the average value of the estimate is basically equal to the underlying, the true underlying parameter h. And therefore the ML estimate that we have derived is unbiased in nature. Let us also write that down, let us also note that down. Therefore, the ML estimate or in another words our sample mean that is, the sample mean is unbiased in nature yeah.

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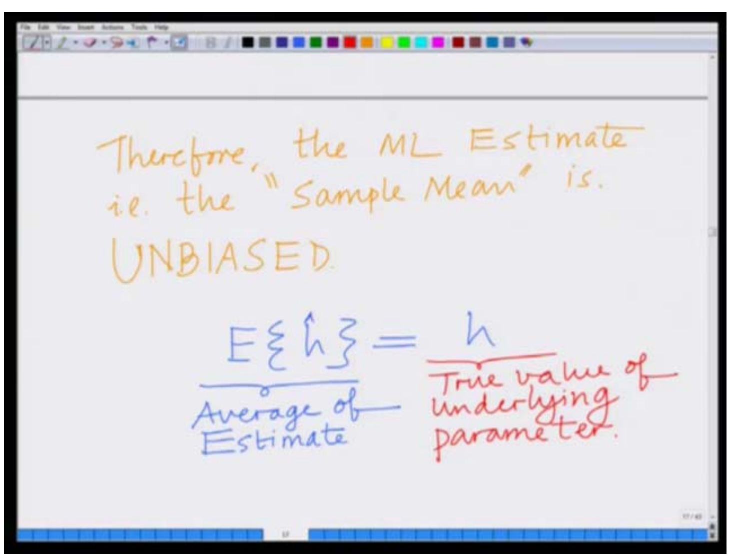

Which means that what do we have, we have expected value of h hat is equal to h, what is this, this is the average value of the estimate and what is this, this is the true value of the underlying parameter. This is the true value of the underlying parameter and what we are saying is, on an average, the average value of the estimate is equal to the true value of the underlying parameter.

Therefore, the estimate or therefore this maximum likelihood estimate is unbiased in nature which is a very important property of the estimate. In fact it is a very desirable property of the estimate yeah. So this unbiased property is a very important and desirable property of the estimate, which says that on an average, the average value or the mean of the estimate is equal to the true value of the underlying parameter, all right.

So in this module what we have looked at is, we have started characterising the behaviour of this ML estimate yeah, which is basically for a simple sensor network with noisy observations of an underlying parameter, we have said that the maximum likelihood estimate is the sample mean of the observation and we have also reduced that even though this is random and, random Gaussian in nature.

That is the estimate h hat is a Gaussian random variable that is, it is not necessarily always equal to the true underlying parameter h. On an average, that is the expected value of this estimate or this estimator is equal to the true value of the underlying parameter h and therefore this maximum likelihood estimate is an unbiased estimate which is an important property.

So we will stop this module here and continue with other properties in the subsequent modules, thank you very much.