**Estimation for Wireless Communications – MIMO/OFDM Cellular and Sensor Networks**
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**Lecture 25**
**Example - Equalization and Approximation Error for Zero Forcing Channel Equalizer**

Hello! Welcome to another module in this massive open online course on estimation for wireless communication. So we are looking at, in the previous module, we have started looking at the approximation error of an equalizer, correct? So we have formulated the system model of the equalizer, derived the least squares equalizer that is C bar, the equalizer vector C bar from the, employing the principle of least squares and for that particular equalizer, we have started computing the approximation error, alright?

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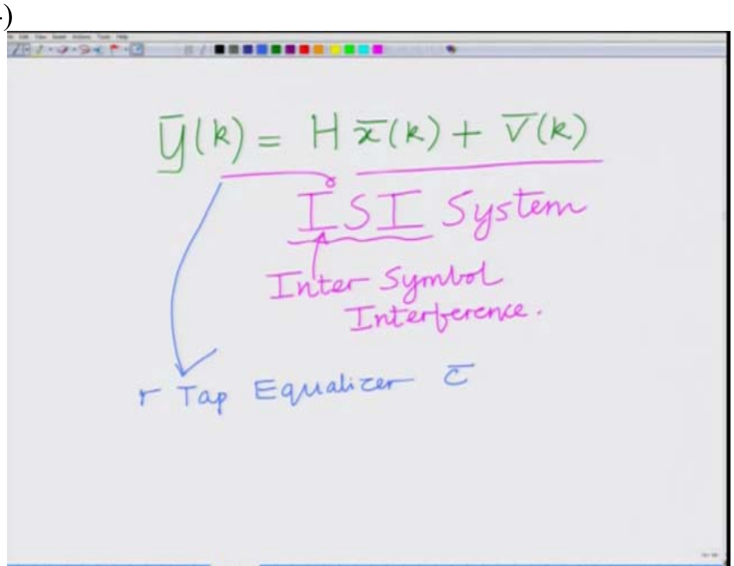


So firstwe have formulated our system model, as Y bar K equals H X bar K plus V bar K, right? This is our ISI system model, system model for, inter symbol interference, where ISI stands for inter symbol, where ISI stands for inter symbol interference and we have demonstrated the R tap equalizer for this is given as, we wanted to construct an R tap equalizer for this and the R tap equalizer C bar for this is given as, C bar equals H, H transpose inverse H times this vector one bar two.

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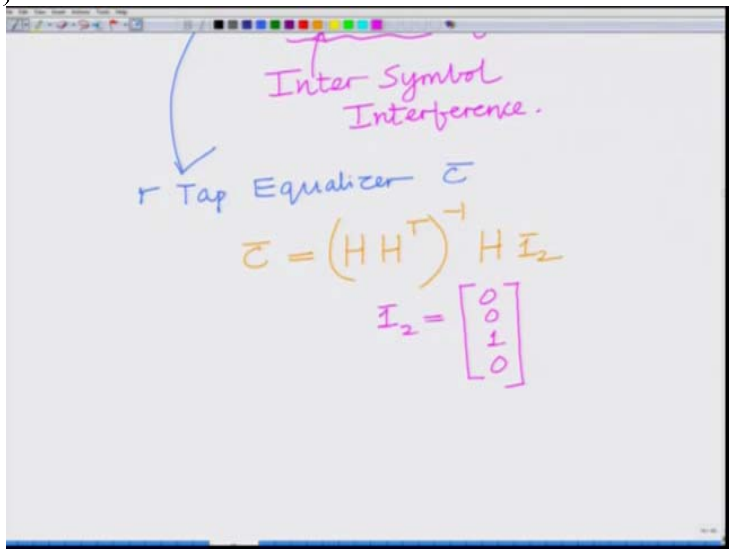


If this vector one bar two is basically equal to the column vector with zeroes everywhere except in the second position counting from, counting from zero, okay? So this is the expression for equalizer vector C bar employed (derive) (employed using) derived employing the least square principle and then the equalizer itself is implemented as C bar transpose Y bar alright, so equalizer is implemented as C bar transpose Y bar and for this we want to find the approximation error, that is how close is this to the vector one bar two.

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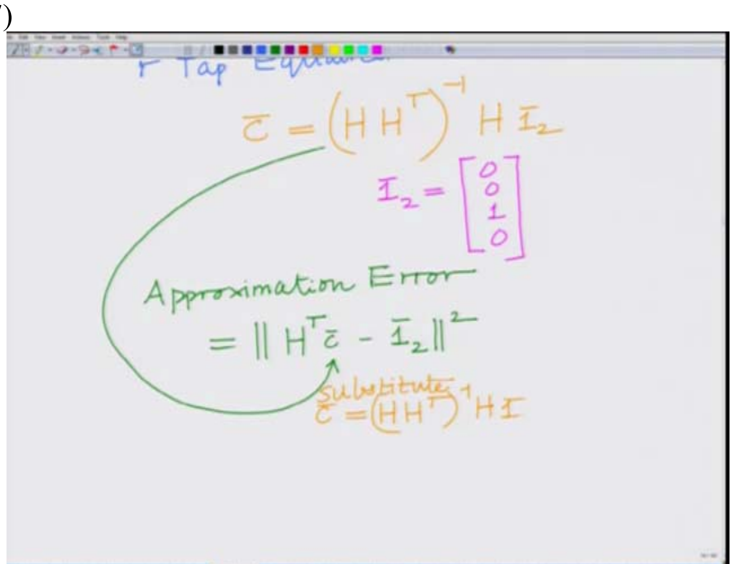


Therefore, we want to find the approximation error, remember this is what we have started computing, the approximation error which is equal to norm of H transpose C bar minus your vector one bar two whole square and here we have substituted the expression for C bar, so this expression for C bar we substitute here, we substitute C bar is in fact equals H transpose inverse H into one bar two over here, and what that gives us basically the approximation error equals that is the error, equals norm of H transpose H H transpose inverse H into one bar two minus one bar two whole square and this matrix that is H transpose H H transpose inverse H.

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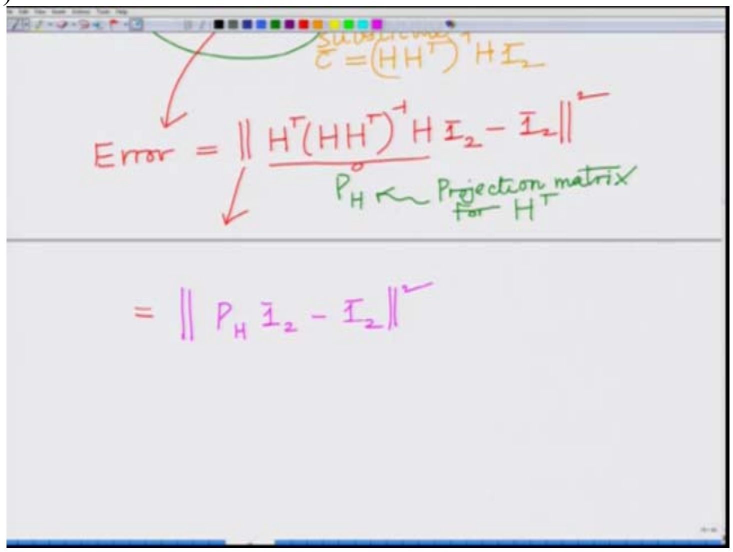


We call this matrix as your matrix PH, remember we said this is the projection matrix for the range for this before H transpose. This is the projection matrix for H transpose and now we can write this as this is equal to, what is this equal to? This is equal to, now denoting this by, so PH into this vector one bar two minus one bar two norm square which is basically equal to norm of PH minus the identity matrix into the vector one bar two whole square.

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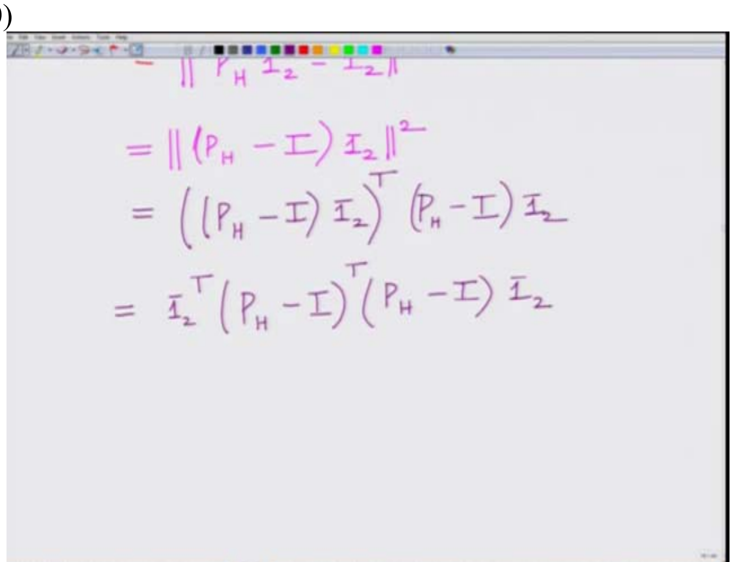

Which can be further simplified asbasically norm V bar square V bar transpose V bar, so this is PH minus identity times one bar two transpose into the vector itself that is PH minus identity into one bar two, which is basically one bar two vector transpose times PH minus identity transpose PH minus identity into this vector one bar two. And at this point what we said is we want to (ex) we want to explore the properties of this projection matrix PH.

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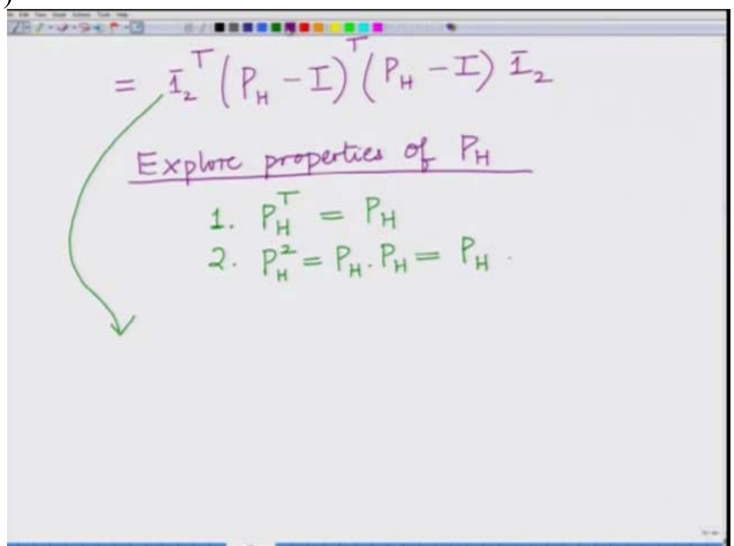


PH, which is the projection matrix of H transpose, we wanted to explore the properties of this matrix PH, okay? So we wanted to explore properties of PH, explore properties of PH and we said that PH satisfies the following two properties, one PH is symmetric, which means PH transpose equals PH, that is the transpose of PH equals itself, two more importantly PH square that is, there is a very interesting property, PH the matrix multiplied by itself PH times PH is indeed equal to PH.

These are the two properties that this matrix PH is satisfy and we now employ these two properties we want to now employ these two properties to simplify the expression for the approximation error that is one bar two transpose into PH minus I transpose into PH minus I times one bar. If I want to simplify this employing these properties and that is what we are going to do now.

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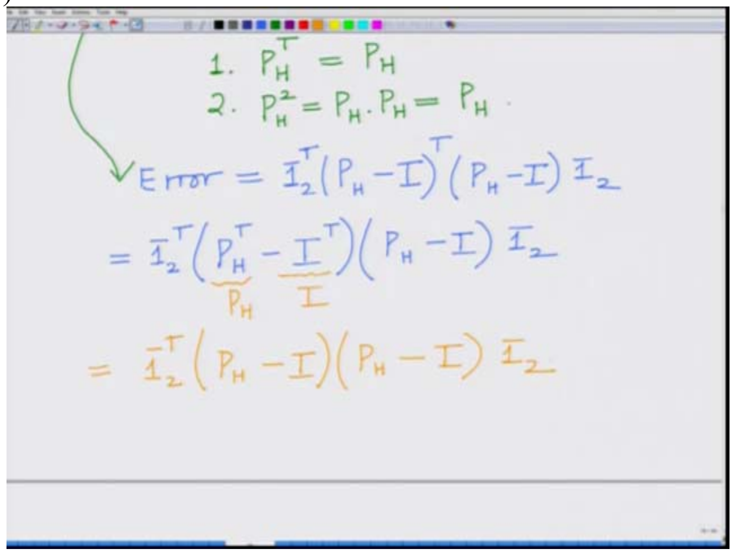


So employing these properties, I have this is equal to, error is equal to, (ch) so we already derived our error is equal to, our error is equal to, this is equal to basically we have one bar two transpose into PH minus I transpose into PH minus I into one bar two which is equal to one bar two transpose into PH transpose minus I transpose, in to PH minus I into one bar two.

Now this is equal to, this is equal to, now, you can see PH transpose, this is equal to PH itself because that is what we have proved earlier and I transpose is the identity matrix itself, so this is one bar two transpose into PH minus identity, into PH minus identity into one bar two, which is equal to, now you can see this is basically equal to, one bar two transpose times multiplying out the quantities in the bracket PH into PH minus I times PH which is PH minus PH times identity matrix which is PH plus identity times identity which is identity itself times one bar two.

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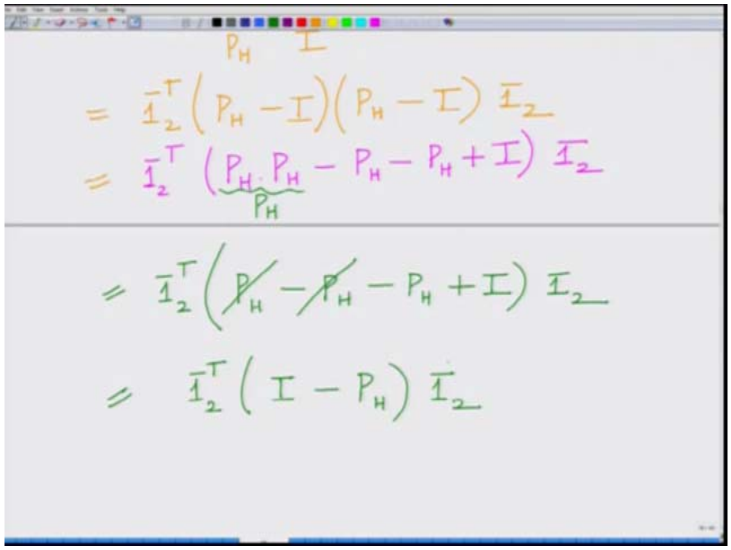


Now you can see PH times PH is equal to PH itself, so this is basically your one bar two transpose. Where we are using the property that PH times PH is equal to PH. So this is PH minus PH, minus PH plus I times one bar two. So these two PH cancel and what we have, this is equal to one bar two transpose I minus the matrix PH times one bar two.

So we can in fact simplify this as one bar two transpose times I minus the matrix PH times one bar two transpose. So this is the expression that we have developed for approximation error so far. We can simplify it even further in fact we can now simplify this as the error is equal to one bar two transpose into identity into one bar two which is one bar two transpose one bar two minus one bar two transpose PH times one bar two. Now let us explore these quantities one bar two transpose one bar two.

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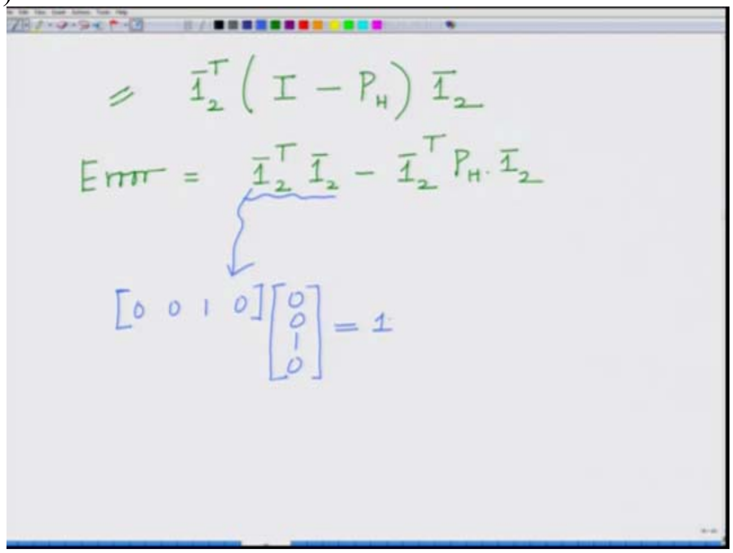


Now this quantity one bar two transpose one bar two, this is the row vector zero zero one zero, times the column vector zero zero one zero which is basically equal to one. Now this vector, this entry one bar two transpose PH one bar two, this is basically one bar two or one bar two transpose PH into one bar two. This is basically zero zero one zero row vector times PH times the column vector zero zero one zero, and you can check that all this does is basically it picks the two comma two entry of the matrix PH.

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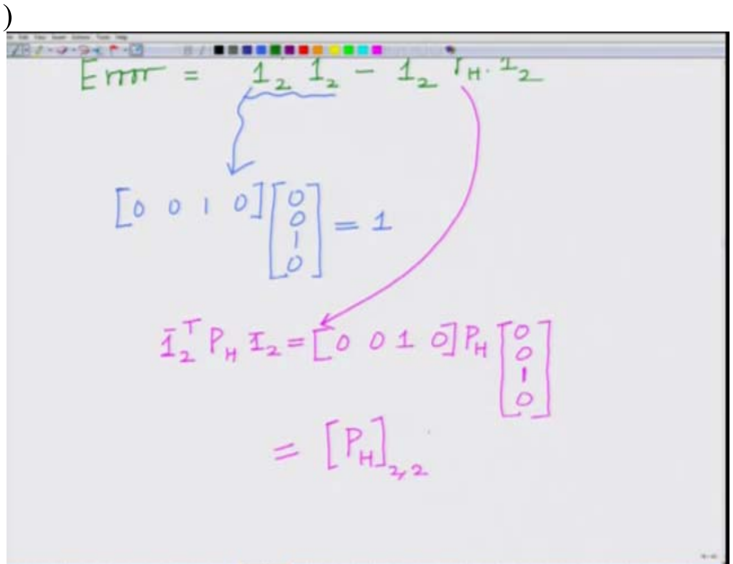


So what this does, is it picks the, so this is PH two comma two what this denotes, is this denotes, this denotes, this notation denotes, two comma two entry of matrix two comma two entry of the matrix PH, PH where the counting is from zero, entries are counted from starting, that is this denotes the two comma two entry of the matrix PH, that is the entry in the second row and second column, but the counting has to start from zero.

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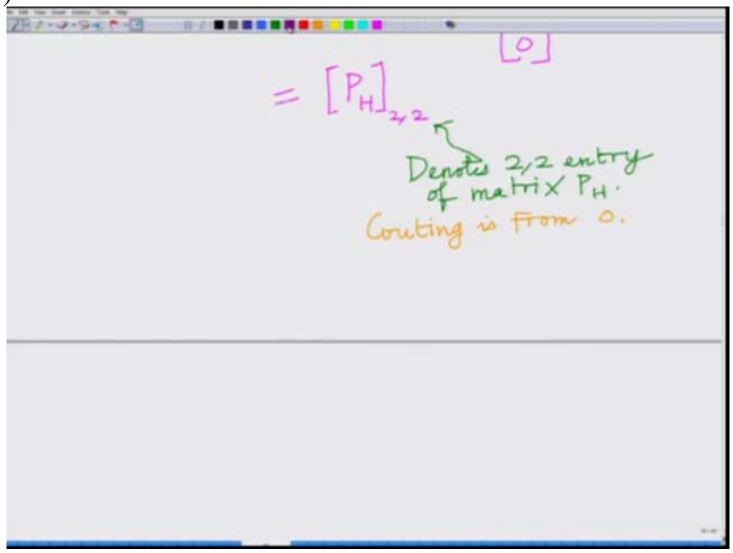


That is start counting from zeroth column, first column, second column so on, zeroth row, first row, second row, so on and you will look at the intersection of the entry at the second row and second column, where the entry is. And now, therefore the error, or the approximation error, is given as, is given as, one minus, well let's write it down completely, one bar two, transpose, one bar two minus one bar two, transpose PH into one bar two which is equal to, one minus PH two cross two, that is the approximation error.

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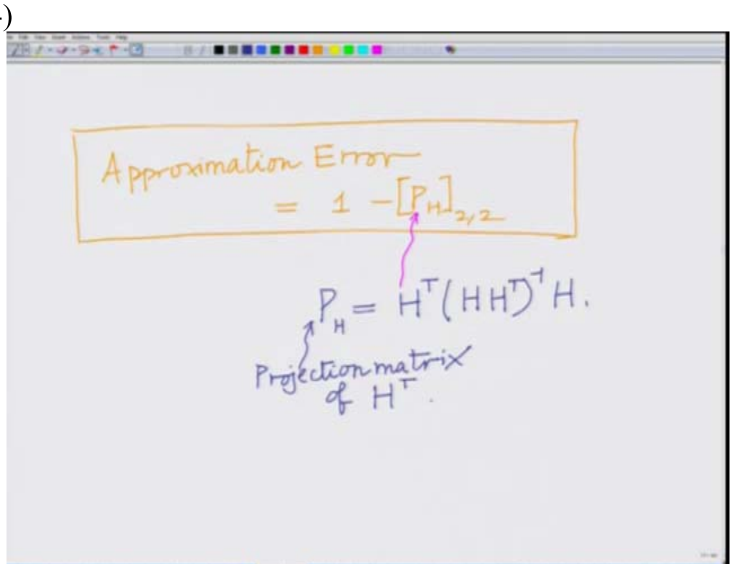


So let me write this down completely, the approximation error, one minus PH two cross, two cross two, this is the expression for the approximation error, where the matrix PH let us mention that again, PH is the projection matrix of H transpose which is H transpose H H transpose inverse times H where PH is the projection matrix, PH is the projection matrix of H transpose where PH is basically projection matrix of H transpose and the approximation error is one minus the two comma two (ep) two comma two-eth entry.

That is the entry in the second row and second column of the matrix PH where the counting starts from zero, okay? Alright? So that is the expression or (()) (14:18) or a simplified expression for the approximation error and denotes how well this equalizer is able to approximate that ideal desired vector one bar two. Remember that is, that is what our initial goal was to try to approximate H transpose C bar as close as possible, to make it as close as possible to this vector one bar two so that you remove the inter symbol interference form the symbol XK that is inter symbol interference on the symbol XK arising from the symbols XK plus two XK plus one and XK minus one, alright?

So to understand this better let us do a simple example, so to understand this better let us do our previous example, so the example for an approximation error, for, so let us do this example for our approximation error. So to understand this better let us do an example and let us go back to the simple, that is the two tap ISI channel that we had considered previously, that is, remember for our equalizer we had considered YK equals XK plus point five XK minus one plus VK. This is the ISI channel, so we have L equal to two taps, L equal to two taps, H zero equal to one and H one is equal to point five.

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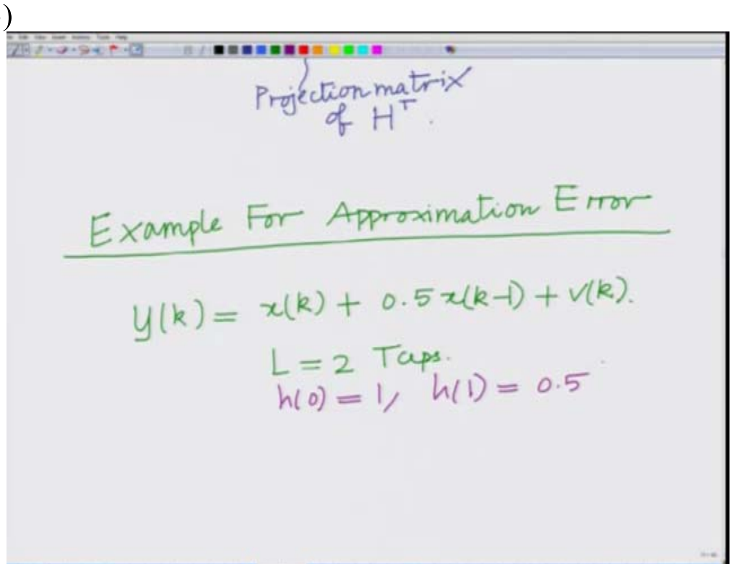


So we're going back, so to look at a simple example for this approximation error, we're going back to a previous example, where YK, where the inter symbol interference channel that we are considered is YK equals XK plus point five times XK minus one plus VK, VK of course is the noise and the channel taps. There is a two tap wireless channel, inter symbol interference wireless channel where H zero equals one and H one is equal to point five, so this is the two tap ISI wireless channel example that we are considering, okay?

And for this let us now derive, we have already derived the equalizer, let us now derive the approximation error for this, so we have L equal to two for R equal to three for a three tap equalizer, for R equal to three, that is our three tap, for R equals three that is our three tap equalizer. We have seen that H, matrix H is basically one point five, zero zerozero one point five zero zerozero one point five, okay, so this is the matrix.

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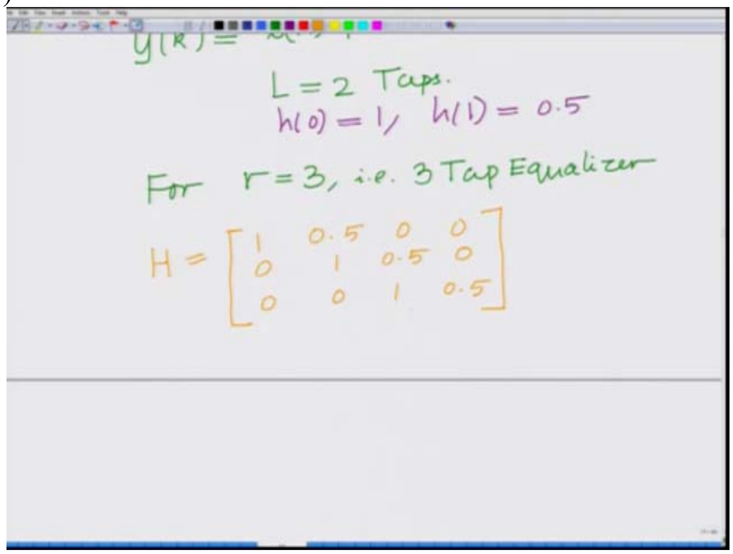


Now we want to compute, now for this naturally, remember the projection matrix of H transpose is a key to evaluate the approximation error. So we want to compute the projection matrix of H, which is H transpose HX transpose inverse times H, okay? We want to compute the projection matrix of H transpose, We want to compute the projection matrix, projection matrix of H transpose which is given as PH equals H transpose, H, H transpose inverse times H.

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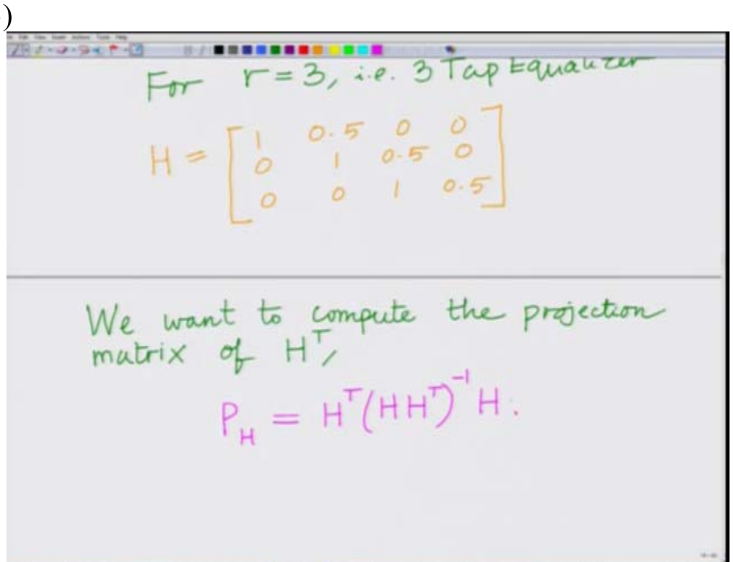


Now let us start by computing HH transpose, although we have already done that before, let us do it again for the sake of completeness, HH transpose we've already written down the expression for H that is your three cross four matrix which is one point five zero zerozero one point five zero zerozero one point five. An H transpose is one point five zero zerozero one point five zero zerozero one point five. This is a four cross three matrix.

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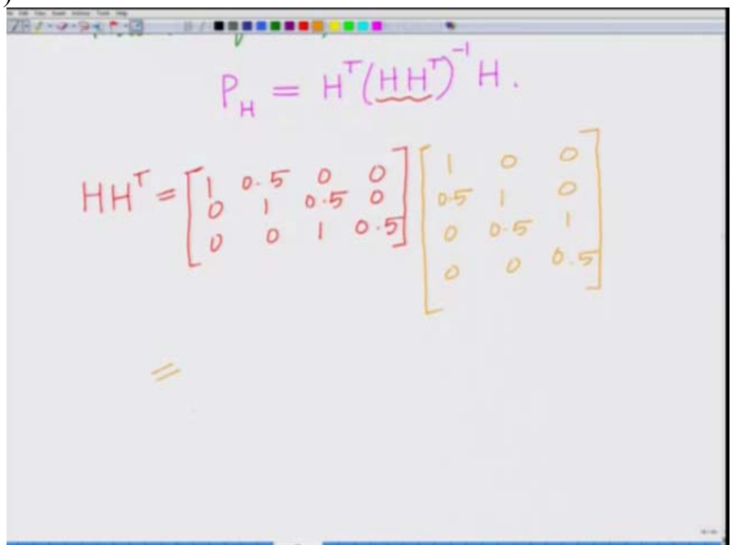


And when I multiply these two matrices together, what I get is basically, I get the matrix the three cross three matrix which is one point two five point five zero point five one point two five comma point (fiv) point five zero point five, one point two five which now writing as fractions we can write H transpose H, which is equal to pi by four half zero half five by four half and the last row is zero half pi by four. So this is basically your three cross three or basically your R cross R matrix H transpose R cross R matrix H transpose H.

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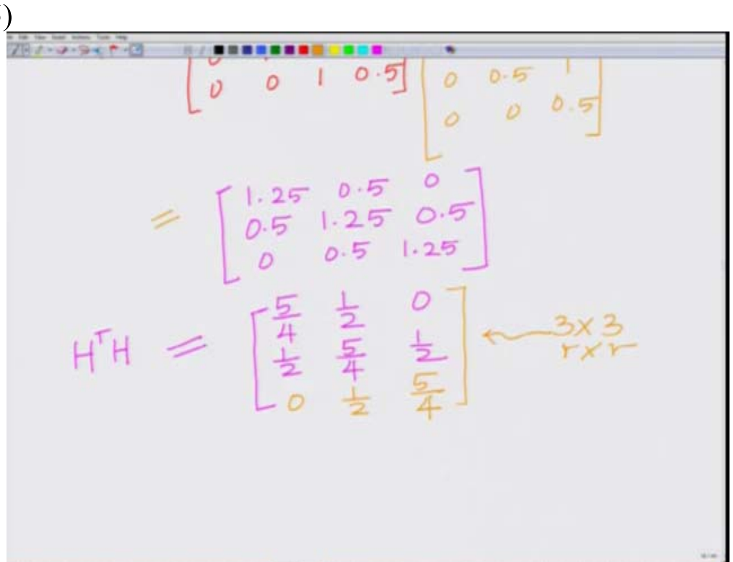

Now what we want to find next is basically to, we have to find H transpose or we want to find HH transpose, I'm sorry, this is HH transpose so this is HH transpose and now what we want to find is basically we want to find HH transpose inverse. Now to find that we (wan) we have to basically start by finding what is the determinant of HH transpose, correct? We have to find first the determinant of HH transpose, we have already found that.

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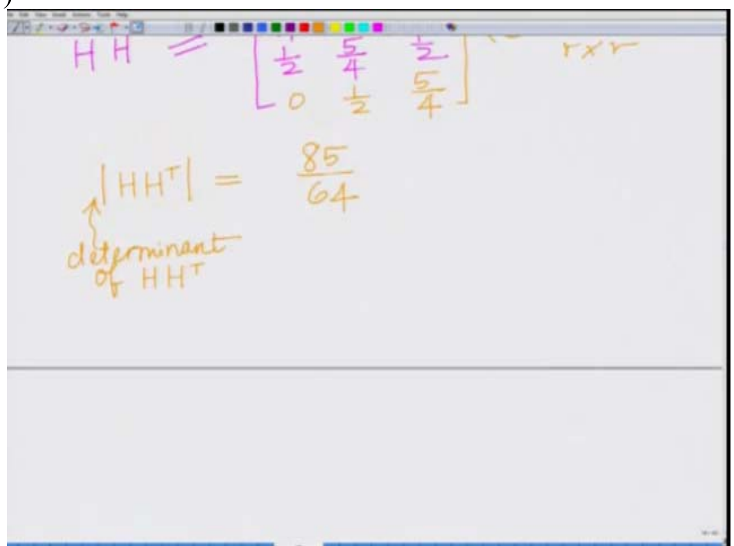


You can refer to the previous modules, this is the determinant of HH transpose, this is the determinant of HH transpose, which is eighty five divided by sixty four, now HH transpose inverse, is equal to one over the determinant times the matrix of cofactors. Remember we computed this also, that is twenty one divided by sixteen minus five by eight one over four minus five over eight twenty five over sixteen minus five over eight, one over four, minus five over eight and twenty one by sixteen, remember this is your matrix of cofactors.

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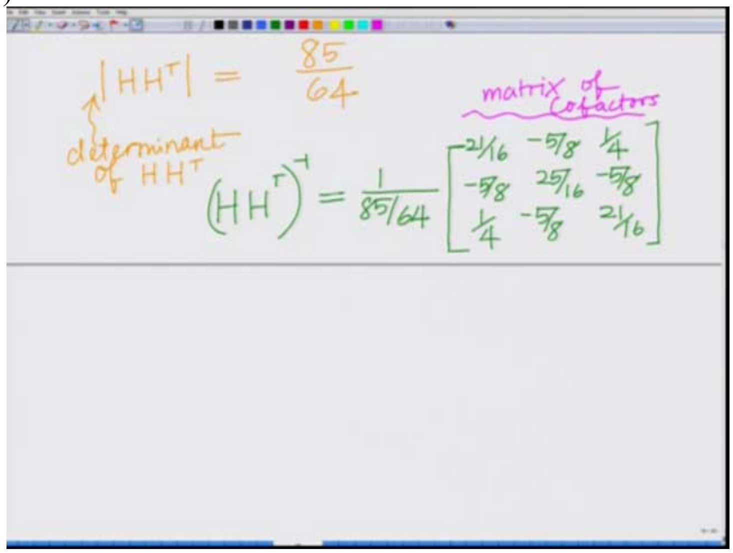


This is the matrix of , this is the matrix of cofactors, now taking the factor of sixty four inside, what we have is H transpose H inverse is one over eight five times eighty four minus forty, and you can check all of these calculations minus forty, hundred minus forty sixteen minus forty and eighty four. And this is basically the expression for HH transpose inverse. And this is something we have already derived that is for this frequency selective channel Y, YKequals XK plus point five, XK minus one plus VK, we have derived.

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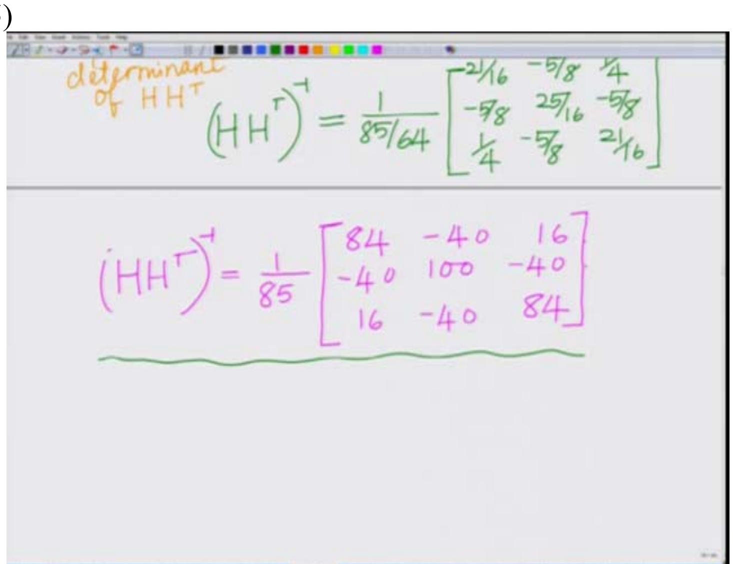


We've formulated the system model to derive expression for the matrix H and from that we have derived HH transpose and we've now computed HH transpose inverse also, okay? So now we have to compute the projection matrix, okay? So to compute the projection matrix let us first start by computing HH transpose inverse times H, yes, just computed HH transpose inverse so that is basically your, eighty four minus forty sixteen minus forty hundred minus forty, sixteen minus forty eighty four times H, times the matrix H.

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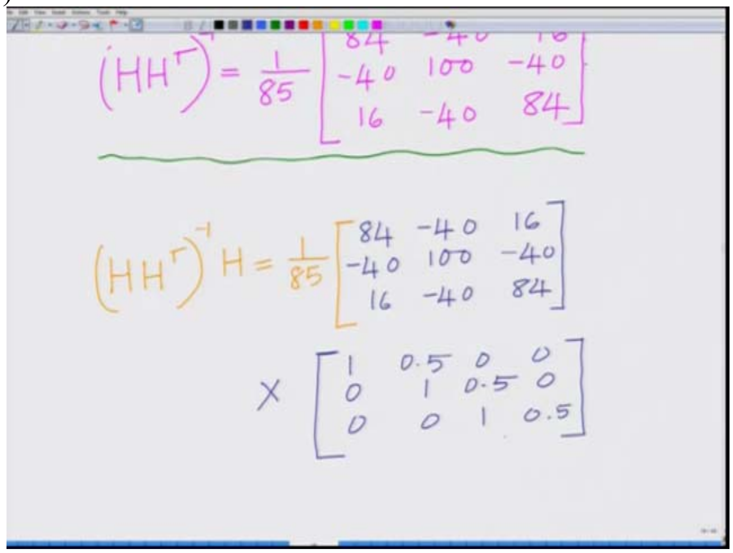


And remember the matrix H is one that is one point five, zero zerozero one point five zero zerozero one one point five, and this is equal to you can multiply that out and now you can see this matrix is given as, basically three cross three times three cross four, so that gives a three cross four matrix, that is one over eighty five times eighty four to minus four eight minus forty eighty ten minus twenty sixteen minus thirty two sixty four forty two. This is the expression for your matrix H, H transpose inverse times H.

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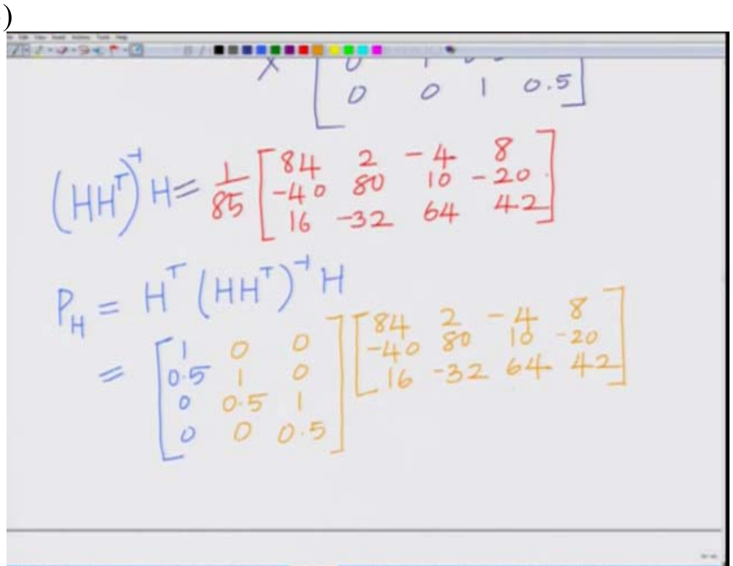


Okay, so we have now computed the matrix H H transpose inverse times H, now I have to multiply on the left by the matrix H transpose to finally compute PH, so PH is equal to, and we have computed that also, PH is equal to H transpose times H H transpose inverse into H, which is equal to basically now H transpose is we've seen this is the four cross three matrix one point five zero of course zero one point five zero, zero zero one point five times, I am going to bring the factor one over eighty five outside.

This is eighty four the (H transpo) HH transpose inverse H, I am substituting that eighty four two minus four eight minus forty, eighty ten minus twenty sixteen minus thirty two, sixty four forty two and of course there is this factor of one over eighty five, the factor which has brought out, and now when I simplify this you get PH finally multiplying this thing out you get PH, the projection matrix of H transpose that is PH, your matrix PH is equal to one over eighty five times eighty four two minus four eight two eighty one eight minus sixteen minus four eight sixty nine thirty two eight minus sixteen thirty two twenty one.

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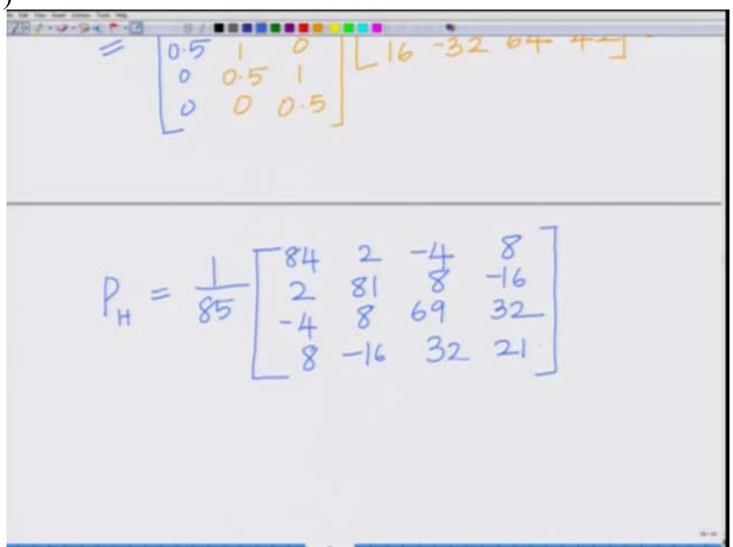


And this you can see, you can now and now so therefore this basically gives the, your projection matrix of H transpose PH which is basically H transpose H H transpose inverse times H and in this we have to take the two cross two element. So the two cross two element counting start counting from zero, the two cross two element you can see is given as basically zero one two three this is the (call) counting of the columns the rows zero one two three.

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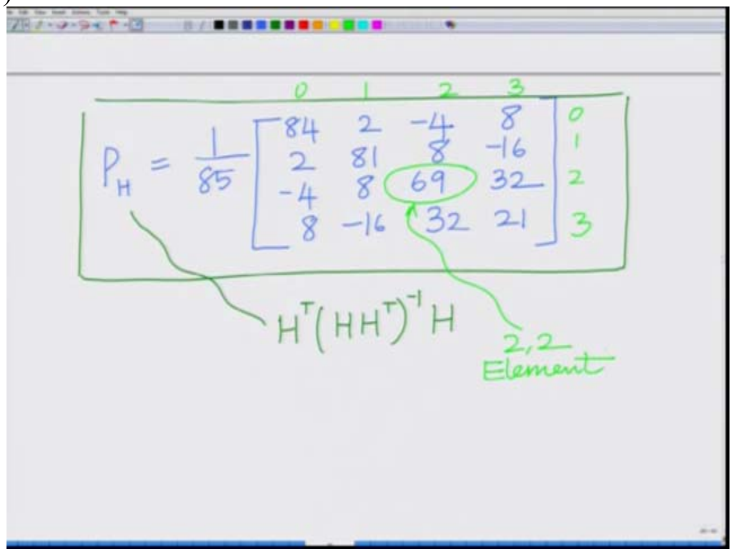


Now you can see the two cross two element basically corresponds to the second column and the second row, which is this element. So this is basically, of course there is a factor of one over eighty five outside, so this is basically your two cross two element, okay? And now using that the approximation error, equals one minus the two cross two element of the projection matrix H that is one minus sixty nine divided by eighty five, sixty nine divided by eighty five, which is basically equal to sixteen divided by eighty five.

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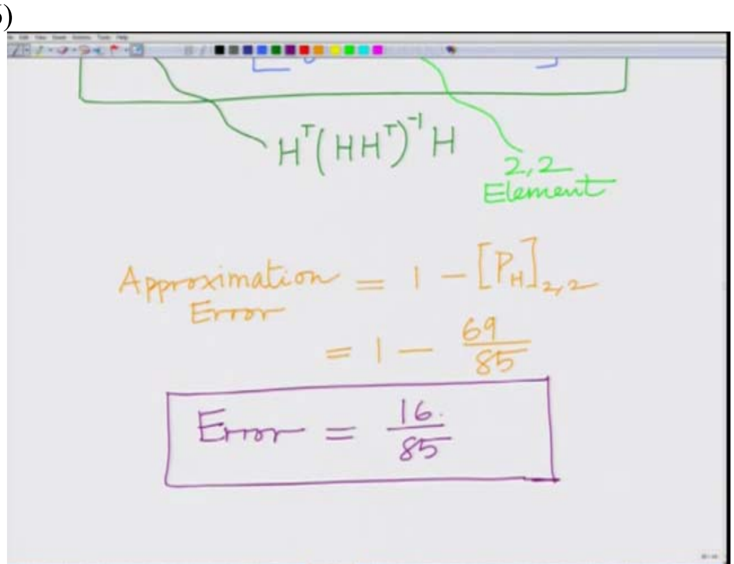


So the approximation error equals sixteen divided by eighty five and that is what we have. And this is the approximation error that (we've have) we've been able to (deliv) (()) (20.19) for this simple example. So what we have done in today's module is basically we have started out with the, from where we left off in the previous module and we looked at the expression for the approximation error for this least square so this zero forcing equalizer, alright?

We’ve simplified that using the properties of the projection matrix of H transpose that is PH which is equal to H transpose times HH transpose inverse times H. And basically the approximation error we have simplified as one minus PH of two comma two that is two comma two’th element of this projection matrix PH where the counting of the columns and rows is done starting from zero.

And then we have done a simple example, considering the ISI, ISI the inter symbol interference channel YK equals XK plus point five XK minus one, plus VK and (we’ve) for this channel and corresponding to a three tapped equalizer we have computed this approximation error and we have shown that this approximation error is basically given by this quantity that is sixteen divided by eighty five. Alright? So with this we will stop this module and we will explore other aspects in subsequent modules. Thank you very much.